LAGRANGIAN MECHANICS WITH UNDETERMINED MULTIPLIERS

PURPOSE: GET THE CONSTRAINT FORCES

n particles $3n$ coor
m constraints
$s = 3n - m$ dof

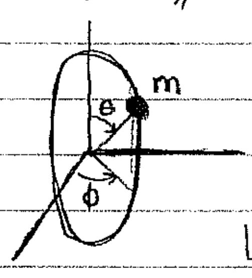
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{k=1}^{m} \lambda_k(t)\frac{\partial f_k}{\partial q_j} = Q_j \quad , j = 1, 2, 3, \dots 3n$$

Unknowns are $q_j(t)$, $j = 1, 2, \dots 3n$ GENER. COOR.
$\lambda_k(t)$, $k = 1, 2, \dots, m$ Undetermined multipliers

n eqs $f_k(q_1, \dots, q_{3n}, t) = 0$; $k = 1, 2, \dots m$

Example // BEAD ON A ROTATING HOOP

straints
$= a$
$= \omega t$



SPHERICAL COOR. (r, θ, ϕ) to locate mass m, Ignore Constraints.

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - mgr\cos\theta$$

(4) $f_1(r, \theta, \phi) = r - a = 0$ unknowns, $r, \theta, \varphi, \lambda_1, \lambda_2$

(5) $f_2(r, \theta, \varphi) = \varphi - \omega t = 0$

(1) $\frac{d}{dt}(m\dot{r}) - (mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - mg\cos\theta) = \lambda_1\frac{\partial f_1}{\partial r} + \lambda_2\frac{\partial f_2}{\partial r} = \lambda_1 = Q_r$

(2) $\frac{d}{dt}(mr^2\dot{\theta}) - (\frac{1}{2}mr^2\sin\theta\cos\theta\,\dot{\phi}^2 + mgr\sin\theta) = \lambda_1\frac{\partial f_1}{\partial \theta} + \lambda_2\frac{\partial f_2}{\partial \theta} = 0 = Q_\theta$

(3) $\frac{d}{dt}(mr^2\sin^2\theta\,\dot{\phi}) - 0 = \lambda_1\frac{\partial f_1}{\partial \varphi} + \lambda_2\frac{\partial f_2}{\partial \varphi} = \lambda_2 = Q_\varphi$

Q_j = generalized force in j direction

(STEP A) • Now we sub ④, ⑤ into ①,②,③

① $-ma\dot{\theta}^2 - ma\,\omega^2 \sin^2\theta + mg\cos\theta$

② $\frac{d}{dt}\left(ma^2\dot{\theta}\right) - ma^2\omega^2 \sin\theta\cos\theta - mga\sin\theta = 0$

③ $\frac{d}{dt}\left(ma^2\omega\sin^2\theta\right) = \lambda_2$

• Now we just have θ, λ_1, λ_2 left to solve.

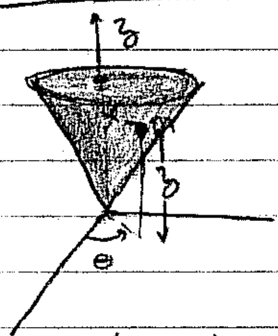
(B) Solve Eq. ② for θ as function of time, $\theta = \theta(t)$

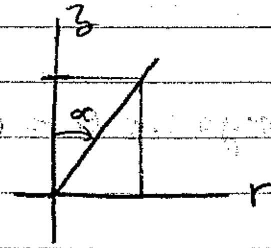
(C) Get $\lambda_1(t) = -ma\dot{\theta}^2 - ma^2\omega^2\sin^2\theta + mg\cos\theta = Q_r$

$\lambda_2(t) = 2ma^2\omega\dot{\theta}\sin\theta\cos\theta = Q_\varphi$

this is the general procedure to find $Q_j(t)$.

EXAMPLE 7.4 Choose Cylindrical Coor. (r, θ, z)





$$\frac{r}{z} = \tan\alpha$$

choose (r, θ) as final independent variable, intuitively the cone places a radial force upon m when viewed from above.

$$f_1(r,\theta,z) = z - r\cot(\alpha) = 0$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz$$

① $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda_1 \frac{\partial f_1}{\partial r} = Q_r$

② $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda_1 \frac{\partial f_1}{\partial \theta} = Q_\theta$

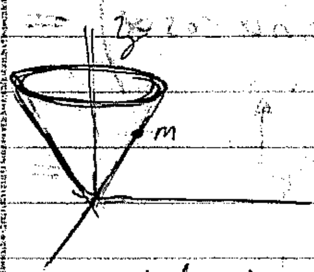
③ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = \lambda_1 \frac{\partial f}{\partial z} = Q_z$

④ $f_1 = z - r\cot(\alpha) = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{k=1}^{m} \lambda_k(t)\, \frac{\partial f_k}{\partial q_j} = Q_j \,, \quad j = 1, 2, \ldots$$

Example 7.4

(r, θ, z)



$$f(r,\theta,z) = z - r\cot\alpha = 0$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - mgz$$

① $\dfrac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = \lambda_1 \dfrac{\partial f_1}{\partial r} = -\lambda_1 \cot\alpha$

② $\dfrac{d}{dt}(mr^2\dot{\theta}) = \lambda_1 \dfrac{\partial f_1}{\partial \theta} = 0$

③ $\dfrac{d}{dt}(m\dot{z}) + mg = \lambda_1 \dfrac{\partial f_1}{\partial z} = \lambda_1$

④ $z = r\cot\alpha$

4 equations with λ_1, r, θ, z all dependent on time

①' $m\ddot{r} - mr\dot{\theta}^2 = -\lambda_1 \cot\alpha$

②' $\dfrac{d}{dt}(mr^2\dot{\theta}) = 0$

③ $m\ddot{r}\cot\alpha + mg = \lambda_1$

① $m\ddot{r} - mr\dot{\theta}^2 = -(m\ddot{r}\cot\alpha + mg)\cot\alpha$

② $mr^2\dot{\theta} = l$ (angular momentum is constant)

① $\left\{ m\ddot{r}(1 + \cot^2\alpha) - mr\left(\dfrac{l^2}{m^2 r^4}\right) + mg\cot\alpha = 0 \right\} \Big/ M, \sin^2\alpha$

①' $m\ddot{r} - mr\sin^2\alpha\left(\dfrac{l^2}{m^2 r^4}\right) + mg\sin\alpha\cos\alpha = 0$

$\Rightarrow$ we can find $r(t) \Rightarrow \theta(t) \Rightarrow \lambda_1(t) \Rightarrow z(t)$
(after much work)

$$\frac{1}{2}m\ddot{r} - \frac{l^2\sin^2\alpha}{mr^3} + mg\sin\alpha\cos\alpha = 0$$

$$m\ddot{r}\dot{r} - \frac{l^2\sin^2\alpha}{mr^3}\dot{r} + mg\sin\alpha\cos\alpha\,\dot{r} = 0$$

$$\underbrace{\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right)}_{\text{RADIAL KINETIC ENERGY}} + \underbrace{\frac{d}{dt}\left(\frac{l^2\sin^2\alpha}{2mr^2}\right)}_{\text{TANGENTIAL KINETIC ENERGY}} + \underbrace{\frac{d}{dt}\left(mgr\sin\alpha\cos\alpha\right)}_{PE} = 0$$

$$= 0.$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{l^2\sin^2\alpha}{2mr^2} + mgr\sin\alpha\cos\alpha\right) = 0$$

$$\frac{1}{2}m\dot{r}^2 + \frac{l^2\sin^2\alpha}{2mr^2} + mgr\sin\alpha\cos\alpha = E \text{ (const)}$$

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}}\left(E - \frac{l^2\sin^2\alpha}{2mr^2} - mgr\sin\alpha\cos\alpha\right)^{1/2}$$

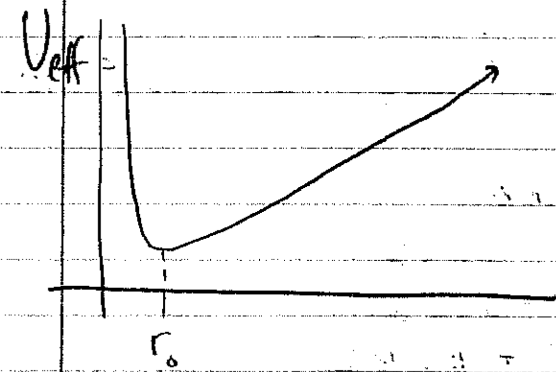
$$\boxed{\int_{r_0}^{r(t)} \frac{dr'}{\left(E - \frac{l^2\sin^2\alpha}{2mr'^2} - mgr'\sin\alpha\cos\alpha\right)^{1/2}} = \sqrt{\frac{2}{m}}\int_0^t dt' = \sqrt{\frac{2}{m}}(t)}$$

this is called reducing it to quadrature
we have reduced the problem to area
this problem is formally solved

$$T + U = E$$

$$\frac{1}{2} m\dot{r}^2 + U_{eff}(r) = E$$

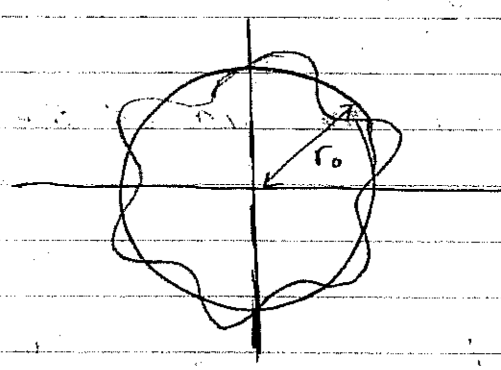
$$U_{eff} = \frac{l^2 \sin^2\alpha}{2mr^2} + mgr\sin\alpha\cos\alpha$$



$$\frac{dU_{eff}}{dr} = 0$$

$$= -\frac{l^2 \sin^2\alpha}{mr^3} + mg\sin\alpha\cos\alpha = 0$$

$$\Rightarrow r_0 = \left(\frac{l^2 \tan\alpha}{m^2 g}\right)^{1/3}$$



intial angular momentum l started at some r other than r_0 results in an oscillation about r_0

$$Q = \sqrt{Q_r^2 + Q_z^2}$$
$$= \lambda_1 \sqrt{1 + \cot\alpha}$$
$$= \frac{\lambda_1}{\sin\alpha}$$

$$Q_r = \lambda_1 \cot\alpha = -Q\cos\alpha$$
$$Q_z = \lambda_1 = Q\sin\alpha$$