Chapter 3

topology and mappings

We begin this chapter by briefly examining all the major concepts of the metric topology for $\mathbb{R}^n$. Then we discuss limits for functions and mappings from using the rigorous $\epsilon - \delta$ formulation. For this chapter and course a "function" has range which is a subset of $\mathbb{R}$. In contrast, a mapping has a range which is in some subset of $\mathbb{R}^n$ for $n \geq 2$ if we want to make it interesting[1]. Continuity is defined and a number of basic theorems are either proved by me or you. Finally I quote a few important (and less trivial) theorems about topology and mappings in $\mathbb{R}^n$.

3.1 functions and mappings

In this section we disucss basic vocabulary for functions and mappings.

Definition 3.1.1.

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}$ then we say that $f : U \to V$ is a **function** iff $f(x)$ assigns a single value in V for each input $x \in U$. We say a function is **single-valued** from **domain** U to **codomain** V. We denote $dom(f) = U$. The **range** or **image** of the function is defined by:

$$range(f) = f(D) = \{y \in \mathbb{R} \mid \exists x \in U \text{ such that } f(x) = y\}$$

We can also say that "f is a real-valued function of U".

Example 3.1.2. . .

① $f(x) = x^2$ for $x \in \mathbb{R}$, $dom(f) = \mathbb{R}$

② $g(\vec{x}) = \vec{x} \cdot \vec{x} = x_1^2 + x_2^2 + \cdots + x_n^2$ for $\vec{x} \in \mathbb{R}^n$, $dom(g) = \mathbb{R}^n$

[1] I generally prefer the term function for a more abstract concept: I would like to say $f : A \to B$ is an B-valued function of A and I don't make any restriction except that A, B must be sets. Anyhow, I'll try to respect the custom of calculus for this course because it saves us a lot of talking. I will use the term "abstract function" if I don't wish to presuppose the codmain contains only real numbers.

A mapping is an abstract function with codmain in $\mathbb{R}^n$

Definition 3.1.3.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ then we say that $f : U \to V$ is a **mapping** iff $f(x)$ assigns a single value in V for each input $x \in U$. We say a f is a single-value mapping from **domain** U to **codomain** V. We mean for $dom(f) = U$ to be read that the domain of f is U. The **range** or **image** of the mapping is the set of all possible outputs: we denote

$$range(f) = f(D) = \{y \in \mathbb{R}^m \mid \exists x \in U \text{ such that } f(x) = y\}$$

Suppose that $x =\in dom(f)$ and $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ then we say that $f_1, f_2, \ldots, f_m$ are the **component functions** of f and $f = (f_i) = (f_1, f_2, \ldots f_m)$.

In the case $m = 1$ we find that the concept of a mapping reduces to a plain-old-function.

Example 3.1.4. . .

$\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $\Phi(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$

we have component functions

$\Phi_1(r,\theta) = r\cos\theta$ & $\Phi_2(r,\theta) = r\sin\theta$.

Definition 3.1.5.

A mapping $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ is said to be **injective** or **1-1** on $S \subseteq U$ iff $f(x) = f(y)$ implies $x = y$ for all $x, y \in S$. If a mapping is 1-1 on its domain then it is said to be 1-1 or injective. The mapping f is said to be **surjective** or **onto** $T \subseteq V$ iff for each $v \in T$ there exists $u \in U$ such that $f(u) = v$; in set notation we can express: f is onto T iff $f(U) = T$. A mapping is said to be surjective or onto iff it is onto its codomain. A mapping is a **bijection** or **1-1 correspondance** of U and V iff f is injective and surjective.

Example 3.1.6. . .

① $f(x,y) = (e^x, y^2, 2)$ is onto $(0,\infty)\times[0,\infty)\times\{2\}$ given that $dom(f) = \mathbb{R}^2$. To prove this let $(a,b,c) \in (0,\infty)\times[0,\infty)\times\{2\}$ and observe that $\ln(a), \sqrt{b} \in \mathbb{R}$ since $a>0$ and $b \geq 0$ hence

$$f(\ln(a), \sqrt{b}, 2) = (e^{\ln(a)}, (\sqrt{b})^2, 2) = (a, b, 2) = (a,b,c).$$

② $g : \mathbb{R} \longrightarrow (0,\infty)$ defined by $g(x) = e^x$ is a <u>bijection</u>.

③ $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = x^2$ is neither 1-1 nor onto.

We can also adjust the domain of a given mapping by restriction and extension.

Definition 3.1.7.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. If $R \subset U$ then we define the **restriction of f to R** to be the mapping $f|_R : R \to V$ where $f|_R(x) = f(x)$ for all $x \in R$. If $U \subseteq S$ and $V \subset T$ then we say a mapping $g : S \to T$ is an **extension of f** iff $g|_{dom(f)} = f$.

When I say $g|_{dom(f)} = f$ this means that these functions have matching domains and they agree at each point in that domain; $g|_{dom(f)}(x) = f(x)$ for all $x \in dom(f)$. Once a particular subset is chosen the restriction to that subset is a unique function. Of course there are usually many susbets of $dom(f$ so you can imagine many different restictions of a given function. The concept of extension is more vague, once you pick the enlarged domain and codomain it is not even necessarily the case that another extension to that same pair of sets will be the same mapping. To obtain uniqueness for extensions one needs to add more stucture. This is one reason that complex variables are interesting, there are cases where the structure of the complex theory forces the extension of a complex-valued function of a complex variable to be unique. This is very surprising.

Example 3.1.8. . .

① Let $\underset{dom(f) = \mathbb{R}}{f(x) = \sqrt{x^2}}$ then $f|_{[0,\infty)}(x) = x$ whereas $f|_{(-\infty,0]}(x) = -x$.

② Let $f(x) = \ln(x)$ for $x \in (0,\infty)$. If $g(x) = \ln|x|$ for $x \in \mathbb{R} - \{0\}$ then $g|_{(0,\infty)} = f$ so g is an extension of f.

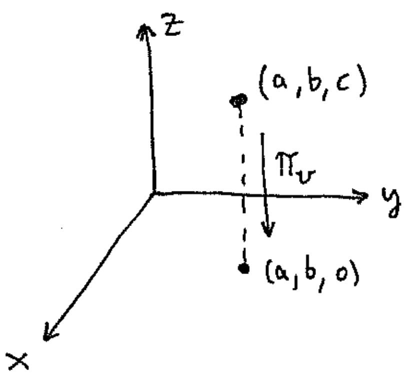
Definition 3.1.9.

Let $\pi_U : \mathbb{R}^n \to U \subseteq \mathbb{R}^n$ be a mapping such that $\pi_U(x) = x$ for all $x \in U$. We say that pi_U is a **projection** onto U. The **identity mapping** on $U \subseteq \mathbb{R}^n$ is defined by $Id_U : U \to U$ with $Id_U(x) = x$ for all $x \in U$. We may also denote $Id_{\mathbb{R}^n} = Id_n = Id$ where convenient. The **j-th projection function** is $\pi_j : \mathbb{R}^n \to \mathbb{R}$ defined by $\pi_j(x_1, x_2, \ldots, x_n) = x_j$

Notice that every identity map is a projection however not every projection is an identity.

Example 3.1.10. . .

Let $U = \mathbb{R}^2 \times \{0\} = \{ (x,y,0) \mid x,y \in \mathbb{R} \}$, this is the xy-plane.



$\pi_U(a,b,c) = (a,b,0)$.

Definition 3.1.11.

Let $f : V \subseteq \mathbb{R}^n \to W \subseteq \mathbb{R}^m$ and $g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ are mappings such that $g(U) \subseteq dom(f)$ then $f \circ g : U \to W$ is a mapping defined by $(f \circ g)(x) = f(g(x))$ for all $x \in U$. We say f is the outside function and g is the inside function.

Notice that the definition of the composite assumes that the range of the inside function fits nicely in the domain of the outside function. If domains are not explicitly given then it is customary to choose the domain of the composite of two functions to be as large as possible. Indeed, the typical pattern in calculus is that the domain is implicitly indicated by some formula. For example, $g(x) = e^x \frac{x-4}{x-4}$ has implied domain $dom(g) = (-\infty, 4) \cup (4, \infty)$ however if we simply the formula to give $g(x) = e^x$ then the implied domain of $\mathbb{R}$ is not correct. Of course we can not make that simplification unless $x \neq 4$. In short, when we do algebra for variables we should be careful to consider the values which the variables may assume. Often one needs to break a calculation into cases to avoid division by zero.

Example 3.1.12. . .

Let $\Xi : (0,\infty) \times (-\pi/2, \pi/2) \to \mathbb{R}^2$ be defined by $\Xi(r,\theta) = (r\cos\theta, r\sin\theta)$.
and $F : (0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}^2$ be defined by $F(x,y) = \left(\sqrt{x^2+y^2},\ \tan^{-1}(y/x)\right)$.
Let $(r,\theta) \in dom(\Xi)$, notice that $\underbrace{\cos(-\pi/2,\pi/2) = (0,1) \text{ whereas } \sin(-\pi/2,\pi/2) = (-1,1)}$,

$$\begin{aligned}(F \circ \Xi)(r,\theta) &= F(\Xi(r,\theta)) \\ &= F(r\cos\theta, r\sin\theta) \\ &= \left(\sqrt{r^2\cos^2\theta + r^2\sin^2\theta},\ \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right)\right) \\ &= \left(\sqrt{r^2},\ \tan^{-1}(\tan\theta)\right) \\ &= (r,\theta)\end{aligned}$$

need these to be sure $\Xi(r,\theta)$ is in dom(F)

Definition 3.1.13.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping, if there exists a mapping $g : f(U) \to U$ such that $f \circ g = Id_{f(U)}$ and $g \circ f = Id_U$ then g is the inverse mapping of f and we denote $g = f^{-1}$.

If a mapping is injective then it can be shown that the inverse mapping is well defined. We define $f^{-1}(y) = x$ iff $f(x) = y$ and the value x must be a single value if the function is one-one. When a function is not one-one then there may be more than one point which maps to a particular point in the range.

Example 3.1.14. . .

$\Xi : (0,\infty) \times (-\pi/2, \pi/2) \longrightarrow \mathbb{R}^2 (0,\infty) \times \mathbb{R}$ be defined by $\Xi(r,\theta) = (r\cos\theta, r\sin\theta)$. We can show Ξ is injective and onto $(0,\infty) \times \mathbb{R}$ thus there exists $\Xi^{-1} : (0,\infty) \times \mathbb{R} \longrightarrow dom(\Xi)$. In particular,

$$\Xi^{-1}(x,y) = \left(\sqrt{x^2+y^2},\ \tan^{-1}(y/x)\right)$$

Note the Ex 3.1.12 shows $\Xi \circ \Xi^{-1} = Id\big|_{(0,\infty)\times\mathbb{R}}$.

Definition 3.1.15.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. We define a **fiber** of f over $y \in range(f)$ as

$$f^{-1}\{y\} = \{x \in U | f(x) = y\}$$

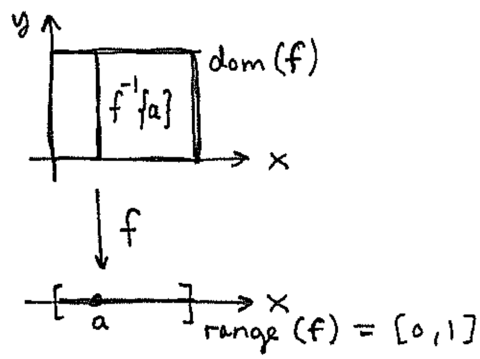
Notice that the inverse image of a set is well-defined even if there is no inverse mapping. Moreover, it can be shown that the fibers of a mapping are disjoint and their union covers the domain of the mapping:

$$f(y) \neq f(z) \quad \Rightarrow \quad f^{-1}\{y\} \cap f^{-1}\{z\} = \emptyset \qquad\qquad \bigcup_{y \,\in\, range(f)} f^{-1}\{y\} = dom(f).$$

This means that the fibers of a mapping *partition* the domain.

Example 3.1.16. . .

Let $f(x,y) = x$ for all $(x,y) \in dom(f) = [0,1] \times [0,1]$



Definition 3.1.17.

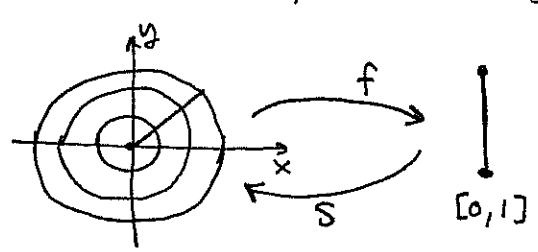
Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. Furthermore, suppose that $s : \mathbf{V} \to U$ is a mapping which is constant on each fiber of f. In other words, for each fiber $f^{-1}\{y\} \subseteq U$ we have some constant $u \in U$ such that $s(f^{-1}\{y\}) = u$. The subset $s^{-1}(U) \subseteq U$ is called a **cross section** of the fiber partition of f.

How do we construct a cross section for a particular mapping? For particular examples the details of the formula for the mapping usually suggests some obvious choice. However, in general if you accept the **axiom of choice** then you can be comforted in the existence of a cross section even in the case that there are infinitely many fibers for the mapping.

Example 3.1.18. . .

Let $f(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$ such that $x^2 + y^2 \leq 1$.

The fibers are circles, and the origin $f^{-1}\{0\} = (0,0)$. Let us



define a <u>section</u> by

$s(r) = (\sqrt{r}/\sqrt{2}, \sqrt{r}/\sqrt{2})$

for each $r \in [0,1]$. Notice

$s[0,1] =$ ray at $\theta = \pi/4$.

Notice $f(s(r)) = f(r/\sqrt{2}, r/\sqrt{2}) = \left(\frac{\sqrt{r}}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{r}}{\sqrt{2}}\right)^2 = \sqrt{r^2} = |r| = r$.

Note $f|_{s[0,1]}$ is injective because each fiber is reduced to a point.

Proposition 3.1.19.

> Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. The restriction of f to a cross section S of U is an injective function. The mapping $\tilde{f} : U \to f(U)$ is a surjection. The mapping $\tilde{f}|_S : S \to f(U)$ is a bijection.

The proposition above tells us that we can take any mapping and cut down the domain and/or codomain to reduce the function to an injection, surjection or bijection. If you look for it you'll see this result behind the scenes in other courses. For example, in linear algebra if we throw out the kernel of a linear mapping then we get an injection. The idea of a local inverse is also important to the study of calculus.

Example 3.1.20. . . Let $f : [0,1]\times[0,1] \longrightarrow \mathbb{R}$ be defined by $f(x,y) = x$ is not onto $\mathbb{R}$, and it's not injective since $f^{-1}\{x\} = \{x\}\times[0,1]$. You can check $s : [0,1] \longrightarrow \text{dom}(f)$ with $s(x) = (x, 1/2)$ is a section of f. Moreover, $\tilde{f} : [0,1]\times[0,1] \longrightarrow [0,1]$ is onto and $\tilde{f}|_{s[0,1]}$ is a bijection.

Definition 3.1.21.

> Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping then we say a mapping g is a **local inverse** of f iff there exits $S \subseteq U$ such that $g = (f|_S)^{-1}$.

Usually we can find local inverses for functions in calculus. For example, $f(x) = \sin(x)$ is not 1-1 therefore it is not invertible. However, it does have a local inverse $g(y) = \sin^{-1}(y)$. If we were more pedantic we wouldn't write $\sin^{-1}(y)$. Instead we would write $g(y) = \left(\sin|_{[\frac{-\pi}{2},\frac{\pi}{2}]}\right)^{-1}(y)$ since the inverse sine is actually just a local inverse. To construct a local inverse for some mapping we must locate some subset of the domain upon which the mapping is injective. Then relative to that subset we can reverse the mapping. The inverse mapping theorem (which we'll study mid-course) will tell us more about the existence of local inverses for a given mapping.

Definition 3.1.22.

> Let $f : U_1 \subseteq \mathbb{R}^n \to V_1 \subseteq \mathbb{R}^p$ and $g : U_1 \subseteq \mathbb{R}^n \to V_2 \subseteq \mathbb{R}^q$ be a mappings then (f,g) is a mapping from U_1 to $V_1 \times V_2$ defined by $(f,g)(x) = (f(x), g(x))$ for all $x \in U_1$.

There's more than meets the eye in the definition above. Let me expand it a bit here:

$$(f,g)(x) = (f_1(x), f_2(x), \dots, f_p(x), g_1(x), g_2(x), \dots, g_q(x)) \quad \text{where} \quad x = (x_1, x_2, \dots, x_n)$$

You might notice that Edwards uses π for the identity mapping whereas I use Id. His notation is quite reasonable given that the identity is the cartesian product of all the projection maps:

$$\pi = (\pi_1, \pi_2, \dots, \pi_n)$$

I've had courses where we simply used the coordinate notation itself for projections, in that notation have formulas such as $x(a,b,c) = a,\ x_j(a) = a_j$ and $x_j(e_i) = \delta_{ji}$.

Example 3.1.23. . . .

① $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $f(x,y,z) = (x, yz^2)$

$g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be defined by $g(x,y,z) = \|(x,y,z)\|$

$(f, g) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ has $(f, g)(\vec{x}) = (f(\vec{x}), g(\vec{x}))$

② $\pi_{xy} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $\pi_{xy}(x,y,z) = (x,y)$

$\pi_z : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be defined by $\pi_z(x,y,z) = z$

You can see $(\pi_{xy}, \pi_z) = Id_{\mathbb{R}^3}$.

The constructions thus far in this section have not relied on the particular properties of real vectors. If you look at the definitions they really only depend on an understanding of sets, points and subsets. In contrast, the definition given below defines the sum of two mappings, the scalar product of a mapping and a constant or a function, and the dot-product of two mappings.

Definition 3.1.24.

Let $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a mappings and $c \in \mathbb{R}$ and $h : U \to \mathbb{R}$ a function. We define:

1. $f + g$ is a mapping from U to $\mathbb{R}^m$ where $(f + g)(x) = f(x) + g(x)$ for all $x \in U$.
2. hf is a mapping from U to $\mathbb{R}^m$ where $(hf)(x) = h(x)f(x)$ for all $x \in U$.
3. cf is a mapping from U to $\mathbb{R}^m$ where $(cf)(x) = cf(x)$ for all $x \in U$.
4. $f \cdot g$ is a function of U where $(f \cdot g)(x) = f(x) \cdot g(x)$ for all $x \in U$.

We cannot hope to define the product and quotient of mappings to be another new mapping because we do not know how to define the product or quotient of vectors for arbitrary dimensions. In contrast, we can define the product of matrix-valued maps or of complex-valued maps because we have a way to multiply matrices and complex numbers. If the range of a function allows for some type of product it generally makes sense to define a corresponding operation on functions which map into that range.

Definition 3.1.25.

Let $f, g : U \subseteq \mathbb{R}^n \to \mathbb{C}$ be complex-valued functions. We define:

1. fg is a complex-valued function defined by $(fg)(x) = f(x)g(x)$ for all $x \in U$.
2. If $0 \notin g(U)$ then f/g is a complex-valued function defined by $(f/g)(x) = f(x)/g(x)$ for all $x \in U$.

Example 3.1.26. . .

$f(\theta) = e^{i\theta} = \cos\theta + i\sin\theta$ defines $f : \mathbb{R} \to \mathbb{C}$

$g(\theta) = 3 + i\theta$

$$\begin{aligned}(fg)(\theta) = f(\theta)g(\theta) &= (\cos\theta + i\sin\theta)(3 + i\theta) \\ &= 3\cos\theta + 3i\sin\theta + i\theta\cos\theta - \theta\sin\theta \\ &= 3\cos\theta - \theta\cos\theta + i(3\sin\theta + \theta\cos\theta).\end{aligned}$$

Definition 3.1.27.

Let $A, B : U \subseteq \mathbb{R} \to \mathbb{R}^{m\times n}$ and $X : U \subseteq \mathbb{R} \to \mathbb{R}^{n\times p}$ be matrix-valued functions and $f : U \subseteq \mathbb{R} \to \mathbb{R}$. We define:

1. $A + B$ is a matrix-valued function defined by $(A + B)(x) = A(x) + B(x)$ for all $x \in U$.
2. AX is a matrix-valued function defined by $(AX)(x) = A(x)B(x)$ for all $x \in U$.
3. fA is a matrix-valued function defined by $(fA)(x) = f(x)A(x)$ for all $x \in U$.

The calculus of matrices is important to physics and differential equations.

Example 3.1.28. . .

Let $A, B : \mathbb{R} \to \mathbb{R}^{2\times 2}$ be defined by $A(t) = \begin{bmatrix} 1 & 2 \\ t & t^2 \end{bmatrix}$ & $B(t) = \begin{bmatrix} e^t & t \\ t^2 & t^3 \end{bmatrix}$

$$(A+B)(t) = A(t) + B(t) = \begin{bmatrix} 1 & 2 \\ t & t^2 \end{bmatrix} + \begin{bmatrix} e^t & t \\ t^2 & t^3 \end{bmatrix} = \begin{bmatrix} 1+e^t & 2+t \\ t+t^2 & t^2+t^3 \end{bmatrix}.$$

$$(AB)(t) = A(t)B(t) = \begin{bmatrix} 1 & 2 \\ t & t^2 \end{bmatrix}\begin{bmatrix} e^t & t \\ t^2 & t^3 \end{bmatrix} = \begin{bmatrix} e^t + 2t^2 & t + 2t^3 \\ te^t + t^4 & t^2 + t^5 \end{bmatrix}.$$

Let $f(t) = \sin t$,

$$(fA)(t) = f(t)A(t) = \sin t \begin{bmatrix} 1 & 2 \\ t & t^2 \end{bmatrix} = \begin{bmatrix} \sin t & 2\sin t \\ t\sin t & t^2\sin t \end{bmatrix}.$$

3.2 elementary topology and limits

In this section we describe the *metric topology* for $\mathbb{R}^n$. In the study of functions of one real variable we often need to refer to open or closed intervals. The definition that follows generalizes those concepts to n-dimensions. I have included a short discussion of general topology in the Appendix if you'd like to learn more about the term.

Definition 3.2.1.

An **open ball** of radius ϵ centered at $a \in \mathbb{R}^n$ is the subset all points in $\mathbb{R}^n$ which are less than ϵ units from a, we denote this open ball by

$$B_\epsilon(a) = \{x \in \mathbb{R}^n \mid ||x - a|| < \epsilon\}$$

The **closed ball** of radius ϵ centered at $a \in \mathbb{R}^n$ is likewise defined

$$\overline{B}_\epsilon(a) = \{x \in \mathbb{R}^n \mid ||x - a|| \le \epsilon\}$$

Notice that in the $n = 1$ case we observe an open ball is an open interval: let $a \in \mathbb{R}$,

$$B_\epsilon(a) = \{x \in \mathbb{R} \mid ||x - a|| < \epsilon\} = \{x \in \mathbb{R} \mid |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

In the $n = 2$ case we observe that an open ball is an open disk: let $(a, b) \in \mathbb{R}^2$,

$$B_\epsilon((a,b)) = \big\{(x,y) \in \mathbb{R}^2 \mid ||\,(x,y) - (a,b)\,|| < \epsilon\big\} = \big\{(x,y) \in \mathbb{R}^2 \mid \sqrt{(x-a)^2 + (y-b)^2} < \epsilon\big\}$$

For $n = 3$ an open-ball is a sphere without the outer shell. In contrast, a closed ball in $n = 3$ is a solid sphere which includes the outer shell of the sphere.

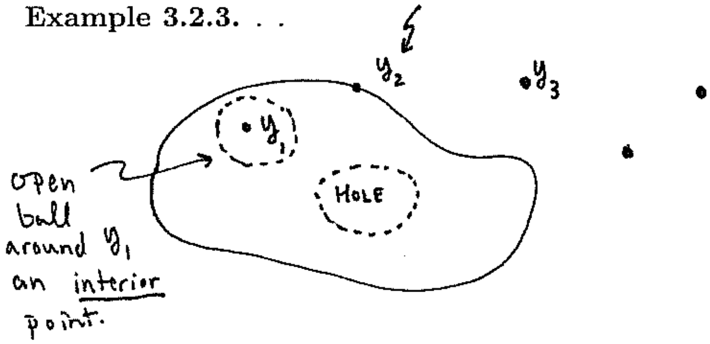
Definition 3.2.2.

Let $D \subseteq \mathbb{R}^n$. We say $y \in D$ is an **interior point** of D iff there exists some open ball centered at y which is completely contained in D. We say $y \in \mathbb{R}^n$ is a **limit point** of D iff every open ball centered at y contains points in $D - \{y\}$. We say $y \in \mathbb{R}^n$ is a **boundary point** of D iff every open ball centered at y contains points not in D and other points which are in $D - \{y\}$. We say $y \in D$ is an **isolated point** of D if there exist open balls about y which do not contain other points in D. The set of all interior points of D is called the **interior of** D. Likewise the set of all boundary points for D is denoted ∂D. The **closure** of D is defined to be $\overline{D} = D \cup \{y \in \mathbb{R}^n \mid y \text{ a limit point}\}$

If you're like me the paragraph above doesn't help much until I see the picture below. All the terms are aptly named. The term "limit point" is given because those points are the ones for which it is natural to define a limit.

y_2 is not an interior pt.

Example 3.2.3. . .



① y_1 & y_2 are limit points of pictured subset, $D \subset \mathbb{R}^2$

② y_3 is isolated point

③ $y_2 \in \partial D$, y_2 is boundary pt.

④ $\overline{D}$ formed by connecting dots in interior hole.

Definition 3.2.4.

Let $A \subseteq \mathbb{R}^n$ is an **open set** iff for each $x \in A$ there exists $\epsilon > 0$ such that $x \in B_\epsilon(x)$ and $B_\epsilon(x) \subseteq A$. Let $B \subseteq \mathbb{R}^n$ is an **closed set** iff its complement $\mathbb{R}^n - B = \{x \in \mathbb{R}^n \mid x \notin B\}$ is an open set.

Notice that $\mathbb{R} - [a, b] = (\infty, a) \cup (b, \infty)$. It is not hard to prove that open intervals are open hence we find that a closed interval is a closed set. Likewise it is not hard to prove that open balls are open sets and closed balls are closed sets. I may ask you to prove the following proposition in the homework.

Proposition 3.2.5.

A closed set contains all its limit points, that is $A \subseteq \mathbb{R}^n$ is closed iff $A = \overline{A}$.

Example 3.2.6. . .

① (a,b) has limit points $x=a$ & $x=b$. However, $a,b \notin (a,b)$ and we note that (a,b) is not a closed set.

② $\overline{(a,b)} = [a,b]$ which is a closed set. Closed intervals are closed sets.

In calculus I the limit of a function is defined in terms of deleted open intervals centered about the limit point. We can define the limit of a mapping in terms of deleted open balls centered at the limit point.

Definition 3.2.7.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. We say that f has limit $b \in \mathbb{R}^m$ at limit point a of U iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $x \in \mathbb{R}^n$ with $0 < ||x - a|| < \delta$ implies $||f(x) - b|| < \epsilon$. In such a case we can denote the above by stating that

$$\lim_{x \to a} f(x) = b$$

In calculus I the limit of a function is defined in terms of deleted open intervals centered about the limit point. We now define the limit of a mapping in terms of deleted open balls centered at the limit point. The term "deleted" refers to the fact that we assume $0 < ||x - a||$ which means we do not consider $x = a$ in the limiting process. In other words, the limit of a mapping considers values close to the limit point but not necessarily the limit point itself. The case that the function is defined at the limit point is special, when the limit and the mapping agree then we say the mapping is continuous at that point.

Example 3.2.8. . .

In calculus I we prove that elementary functions are continuous on the interior of their domains. For example $f(x) = e^x$, $\cos(x)$, $\sin(x)$, $P(x)$ (polynomial) are continuous on $\mathbb{R}$ whereas $r(x) = \frac{p(x)}{q(x)}$ is continuous for $x \in \mathbb{R}$ such that $q(x) \neq 0$.

Oh, to be honest we don't prove all these things, hopefully we at least show the students how they might try to prove these assertions...

Definition 3.2.9.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. If $a \in U$ is a limit point of f then we say that f is **continuous at** a iff

$$\lim_{x \to a} f(x) = f(a)$$

If $a \in U$ is an isolated point then we also say that f is continous at a. The mapping f is **continous on** S iff it is continous at each point in S. The **mapping** f **is continuous** iff it is continuous on its domain.

Notice that in the $m = n = 1$ case we recover the definition of continuous functions from calc. I.

Proposition 3.2.10.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping with component functions $f_1, f_2, \dots, f_m$ hence $f = (f_1, f_2, \dots, f_m)$. If $a \in U$ is a limit point of f then

$$\lim_{x \to a} f(x) = b \qquad \Leftrightarrow \qquad \lim_{x \to a} f_j(x) = b_j \text{ for each } j = 1, 2, \dots, m.$$

.

We can analyze the limit of a mapping by analyzing the limits of the component functions:

Example 3.2.11. . .

Let $f(x) = \left(\sqrt{x^2},\ \sin(x),\ \frac{\sin x}{x}\right)$ thus $f = (f_1, f_2, f_3)$

where $f_1(x) = \sqrt{x^2}$, $f_2(x) = \sin(x)$, $f_3(x) = \frac{\sin x}{x}$ for $x \in \mathbb{R} - \{0\}$.

$$\left.\begin{aligned} &\lim_{x \to 0} f_1(x) = \sqrt{0^2} = 0 \\ &\lim_{x \to 0} (\sin(x)) = 0 \\ &\lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1 \end{aligned}\right\} \quad \lim_{x \to 0} \left(\sqrt{x^2}, \sin x, \frac{\sin x}{x}\right) = (0, 0, 1).$$

The following follows immediately from the preceding proposition.

Proposition 3.2.12.

Suppose that $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ is a mapping with component functions $f_1, f_2, \dots, f_m$. Let $a \in U$ be a limit point of f then f is continous at a iff f_j is continuous at a for $j = 1, 2, \dots, m$. Moreover, f is continuous on S iff all the component functions of f are continuous on S. Finally, a mapping f is continous iff all of its component functions are continuous. .

The proof of the proposition is in Edwards, it's his Theorem 7.2. It's about time I proved something.

Proposition 3.2.13.

The projection functions are continuous. The identity mapping is continuous.

Proof: Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $x \in \mathbb{R}^n$ such that $0 < ||x - a|| < \delta$ then it follows that $||x - a|| < \epsilon$.. Therefore, $\lim_{x \to a} x = a$ which means that $\lim_{x \to a} Id(x) = Id(a)$ for all $a \in \mathbb{R}^n$. Hence Id is continuous on $\mathbb{R}^n$ which means Id is continuous. Since the projection functions are component functions of the identity mapping it follows that the projection functions are also continuous (using the previous proposition). □

Definition 3.2.14.

The **sum** and **product** are functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$s(x, y) = x + y \qquad p(x, y) = xy$$

Proposition 3.2.15.

The sum and product functions are continuous.

Preparing for the proof: Let the limit point be (a, b). Consider what we wish to show: given a point (x, y) such that $0 < ||(x, y) - (a, b)|| < \delta$ we wish to show that

$$|s(x, y) - (a + b)| < \epsilon \qquad \text{or for the product} \qquad |p(x, y) - (ab)| < \epsilon$$

follow for appropriate choices of δ. Think about the sum for a moment,

$$|s(x, y) - (a + b)| = |x + y - a - b| \leq |x - a| + |y - b|$$

I just used the triangle inequality for the absolute value of real numbers. We see that if we could somehow get control of $|x - a|$ and $|y - b|$ then we'd be getting closer to the prize. We have control of $0 < ||(x, y) - (a, b)|| < \delta$ notice this reduces to

$$||(x - a, y - b)|| < \delta \quad \Rightarrow \quad \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

it is clear that $(x - a)^2 < \delta^2$ since if it was otherwise the inequality above would be violated as adding a nonegative quantity $(y - b)^2$ only increases the radicand resulting in the squareroot to be larger than δ. Hence we may assume $(x - a)^2 < \delta^2$ and since $\delta > 0$ it follows $\boxed{|x - a| < \delta}$. Likewise, $\boxed{|y - b| < \delta}$. Thus

$$|s(x, y) - (a + b)| = |x + y - a - b| < |x - a| + |y - b| < 2\delta$$

We see for the sum proof we can choose $\delta = \epsilon/2$ and it will work out nicely.

Proof: Let $\epsilon > 0$ and let $(a, b) \in \mathbb{R}^2$. Choose $\delta = \epsilon/2$ and suppose $(x, y) \in \mathbb{R}^2$ such that $||(x, y) - (a, b)|| < \delta$. Observe that

$$||(x,y) - (a,b)|| < \delta \;\Rightarrow\; ||(x-a, y-b)||^2 < \delta^2 \;\Rightarrow\; |x-a|^2 + |y-b|^2 < \delta^2.$$

It follows $|x - a| < \delta$ and $|y - b| < \delta$. Thus

$$|s(x,y) - (a+b)| = |x + y - a - b| \leq |x-a| + |y-b| < \delta + \delta = 2\delta = \epsilon.$$

Therefore, $\lim_{(x,y)\to(a,b)} s(x, y) = a + b$. and it follows that the sum function if continuous at (a, b). But, (a, b) is an arbitrary point thus s is continuous on $\mathbb{R}^2$ hence the sum function is continuous. $\Box$.

Preparing for the proof of continuity of the product function: I'll continue to use the same notation as above. We need to study $|p(x, y) - (ab)| = |xy - ab| < \epsilon$. Consider that

$$|xy - ab| = |xy - ya + ya - ab| = |y(x-a) + a(y-b)| \leq |y||x-a| + |a||y-b|$$

We know that $|x-a| < \delta$ and $|y-b| < \delta$. There is one less obvious factor to bound in the expression. What should we do about $|y|$?. I leave it to the reader to show that:

$$\boxed{|y-b| < \delta \qquad \Rightarrow \qquad |y| < |b| + \delta}$$

Now put it all together and hopefully we'll be able to "solve" for ϵ.

$$|xy - ab| = \leq |y||x-a| + |a||y-b| < (|b| + \delta)\delta + |a|\delta = \delta^2 + \delta(|a| + |b|) \text{ " = " } \epsilon$$

I put solve in quotes because we have considerably more freedom in our quest for finding δ. We could just as well find δ which makes the " = " become an <. That said let's pursue equality,

$$\delta^2 + \delta(|a| + |b|) - \epsilon = 0 \qquad \delta = \frac{-|a| - |b| \pm \sqrt{(|a|+|b|)^2 + 4\epsilon}}{2}$$

Since $\epsilon, |a|, |b| > 0$ it follows that $\sqrt{(|a|+|b|)^2 + 4\epsilon} < \sqrt{(|a|+|b|)^2} = |a| + |b|$ hence the (+) solution to the quadratic equation yields a positive δ namely:

$$\boxed{\delta = \frac{-|a| - |b| + \sqrt{(|a|+|b|)^2 + 4\epsilon}}{2}}$$

Yowsers, I almost made this a homework. There may be an easier route. You might notice we have run across a few little lemmas (I've boxed the punch lines for the lemmas) which are doubtless useful in other $\epsilon - \delta$ proofs. We should collect those once we're finished with this proof.

Proof: Let $\epsilon > 0$ and let $(a, b) \in \mathbb{R}^2$. By the calculations that prepared for the proof we know that the following quantity is positive, hence choose

$$\delta = \frac{-|a| - |b| + \sqrt{(|a|+|b|)^2 + 4\epsilon}}{2} > 0.$$

Note that[2],

$$
\begin{aligned}
|xy - ab| = |xy - ya + ya - ab| &= |y(x-a) + a(y-b)| && \text{algebra} \\
&\leq |y||x-a| + |a||y-b| && \text{triangle inequality} \\
&< (|b|+\delta)\delta + |a|\delta && \text{by the boxed lemmas} \\
&= \delta^2 + \delta(|a|+|b|) && \text{algebra} \\
&= \epsilon
\end{aligned}
$$

where we know that last step follows due to the steps leading to the boxed equation in the proof preparation. Therefore, $\lim_{(x,y)\to(a,b)} p(x,y) = ab$. and it follows that the product function if continuous at (a,b). But, (a,b) is an arbitrary point thus p is continuous on $\mathbb{R}^2$ hence the product function is continuous. □.

Lemma 3.2.16.

Assume $\delta > 0$.

1. If $a, x \in \mathbb{R}$ then $|x-a| < \delta \;\Rightarrow\; |x| < |a| + \delta$.
2. If $x, a \in \mathbb{R}^n$ then $||x-a|| < \delta \;\Rightarrow\; |x_j - a_j| < \delta$ for $j = 1, 2, \ldots n$.

The proof of the proposition above is mostly contained in the remarks of the preceding two pages.

Example 3.2.17. . .

Let $f(x,y) = x^2 + y^2$. We seek to show $\lim_{(x,y)\to(a,b)} (x^2+y^2) = a^2 + b^2$.

Notice $f(\vec{x}) = \vec{x}\cdot\vec{x} = ||\vec{x}||^2$. Recall the Cauchy Schwartz inequality $|\vec{v}\cdot\vec{w}| \leq ||\vec{v}||\,||\vec{w}||$. Calculate,

$$|f(\vec{x}) - f(\vec{A})| = |\vec{x}\cdot\vec{x} - \vec{A}\cdot\vec{A}| = |(\vec{x}-\vec{A})\cdot(\vec{x}+\vec{A})|$$

You can check, $(\vec{x}-\vec{A})\cdot(\vec{x}+\vec{A}) = \vec{x}\cdot\vec{x} - \vec{A}\cdot\vec{x} + \vec{x}\cdot\vec{A} - \vec{A}\cdot\vec{A} = \vec{x}\cdot\vec{x} - \vec{A}\cdot\vec{A}$.

Let $\epsilon > 0$ and choose $\delta = \dfrac{A + \sqrt{A^2+4\epsilon}}{2}$ where $A = |\vec{A}|$.

Suppose $\vec{x} \in \mathbb{R}^2$ such that $|\vec{x} - \vec{A}| < \delta$ then

$$|f(\vec{x}) - f(\vec{A})| \leq ||\vec{x}-\vec{A}||\,||\vec{x}+\vec{A}|| \leq \delta(\delta + A) = \epsilon.$$

$$\therefore \lim_{\vec{x}\to\vec{A}} f(\vec{x}) = f(\vec{A}).$$

[2] my notation is that when we stack inequalities the inequality in a particular line refers only to the immediate vertical successor.

Note: $||\vec{x}+\vec{A}|| \leq ||\vec{x}|| + ||\vec{A}||$

Proposition 3.2.18.

Let $f : V \subseteq \mathbb{R}^p \to \mathbb{R}^m$ and $g : U \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be mappings. Suppose that $\lim_{x\to a} g(x) = b$ and suppose that f is continuous at b then

$$\lim_{x\to a}(f \circ g)(x) = f\big(\lim_{x\to a} g(x)\big).$$

The proof is in Edwards, see pages 46-47. Notice that the proposition above immediately gives us the important result below:

Proposition 3.2.19.

Let f and g be mappings such that $f \circ g$ is well-defined. The composite function $f \circ g$ is continuous for points $a \in dom(f \circ g)$ such that the following two conditions hold:

1. g is continuous at a
2. f is continuous at $g(a)$.

I make use of the earlier proposition that a mapping is continuous iff its component functions are continuous throughout the examples that follow. For example, I know (Id, Id) is continuous since Id was previously proved continuous.

Example 3.2.20. *Note that if $f = p \circ (Id, Id)$ then $f(x) = \big(p \circ (Id, Id)\big)(x) = p\big((Id, Id)(x)\big) = p(x, x) = x^2$. Therefore, the quadratic function $f(x) = x^2$ is continuous on $\mathbb{R}$ as it is the composite of continuous functions.*

Example 3.2.21. *Note that if $f = p \circ (p \circ (Id, Id), Id)$ then $f(x) = p(x^2, x) = x^3$. Therefore, the cubic function $f(x) = x^3$ is continuous on $\mathbb{R}$ as it is the composite of continuous functions.*

Example 3.2.22. *The power function is inductively defined by $x^1 = x$ and $x^n = xx^{n-1}$ for all $n \in \mathbb{N}$. We can prove $f(x) = x^n$ is continous by induction on n. We proved the $n = 1$ case previously. Assume inductively that $f(x) = x^{n-1}$ is continuous. Notice that*

$$x^n = xx^{n-1} = xf(x) = p(x, f(x)) = (p \circ (Id, f))(x).$$

Therefore, using the induction hypothesis, we see that $g(x) = x^n$ is the composite of continuous functions thus it is continuous. We conclude that $f(x) = x^n$ is continuous for all $n \in \mathbb{N}$.

We can play similar games with the sum function to prove that sums of power functions are continuous. In your homework you will prove constant functions are continuous. Putting all of these things together gives us the well-known result that polynomials are continuous on $\mathbb{R}$.

Proposition 3.2.23.

Let a be a limit point of mappings $f, g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}$ and suppose $c \in \mathbb{R}$. If $\lim_{x\to a} f(x) = b_1 \in \mathbb{R}$ and $\lim_{x\to a} g(x) = b_2 \in \mathbb{R}$ then

1. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}(f(x)g(x)) = \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$.
3. $\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$.

Moreover, if f, g are continuous then $f + g, fg$ and cf are continuous.

Proof: Edwards proves (1.) carefully on pg. 48. I'll do (2.) here: we are given that If $\lim_{x\to a} f(x) = b_1 \in \mathbb{R}$ and $\lim_{x\to a} g(x) = b_2 \in \mathbb{R}$ thus by Proposition 3.2.11 we find $\lim_{x\to a}(f, g)(x) = (b_1, b_2)$. Consider then,

$$\begin{aligned}
\lim_{x\to a}(f(x)g(x)) &= \lim_{x\to a}\big(p(f,g)\big) && \text{defn. of product function} \\
&= p\big(\lim_{x\to a}(f,g)\big) && \text{since } p \text{ is continuous} \\
&= p(b_1, b_2) && \text{by Proposition 3.2.11.} \\
&= b_1 b_2 && \text{definition of product function} \\
&= \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big).
\end{aligned}$$

In your homework you proved that $\lim_{x\to a} c = c$ thus item (3.) follows from (2.). $\square$.

The proposition that follows does follow immediately from the proposition above, however I give a proof that again illustrates the idea we used in the examples. Reinterpreting a given function as a composite of more basic functions is a useful theoretical and calculational technique.

Proposition 3.2.24.

Assume $f, g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}$ are continuous functions at $a \in U$ and suppose $c \in \mathbb{R}$.

1. $f + g$ is continuous at a.
2. fg is continuous at a
3. cf is continuous at a.

Moreover, if f, g are continuous then $f + g, fg$ and cf are continuous.

Proof: Observe that $(f + g)(x) = (s \circ (f, g))(x)$ and $(fg)(x) = (p \circ (f, g))(x)$. We're given that f, g are continuous at a and we know s, p are continuous on all of $\mathbb{R}^2$ thus the composite functions $s \circ (f, g)$ and $p \circ (f, g)$ are continuous at a and the proof of items (1.) and (2.) is complete. To prove (3.) I refer the reader to their homework where it was shown that $h(x) = c$ for all $x \in U$ is a continuous function. We then find (3.) follows from (2.) by setting $g = h$ (function multiplication commutes for real-valued functions). $\square$.

We can use induction arguments to extend these results to arbitrarily many products and sums of power functions.To prove continuity of algebraic functions we'd need to do some more work with quotient and root functions. I'll stop here for the moment, perhaps I'll ask you to prove a few more fundamentals from calculus I. I haven't delved into the definition of exponential or log functions not to mention sine or cosine. We will assume that the basic functions of calculus are continuous on the interior of their respective domains. Basically if the formula for a function can be evaluated at the limit point then the function is continuous.

It's not hard to see that the comments above extend to functions of several variables and mappings. If the formula for a mapping is comprised of finite sums and products of power functions then we can prove such a mapping is continuous using the techniques developed in this section. If we have a mapping with a more complicated formula built from elementary functions then that mapping will be continuous provided its component functions have formulas which are sensibly calculated at the limit point. In other words, if you are willing to believe me that $\sin(x), \cos(x), e^x, \ln(x), \cosh(x), \sinh(x), \sqrt{x}, \frac{1}{x^n}, \dots$ are continuous on the interior of their domains then it's not hard to prove:

$$f(x, y, z) = \left(\sin(x) + e^x + \sqrt{\cosh(x^2) + \sqrt{y + e^x}},\ \cosh(xyz),\ xe^{\sqrt{x+\frac{1}{yz}})}\right)$$

is a continuous mapping at points where the radicands of the square root functions are nonnegative. It wouldn't be very fun to write explicitly but it is clear that this mapping is the Cartesian product of functions which are the sum, product and composite of continuous functions.

Definition 3.2.25.

A **polynomial in** n**-variables** has the form:

$$f(x_1, x_2, \dots, x_n) = \sum_{i_1, i_2, \dots, i_k = 0}^{\infty} c_{i_1, i_2, \dots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_k}$$

where only finitely many coefficients $c_{i_1, i_2, \dots, i_n} \neq 0$. We denote the set of multinomials in n-variables as $\mathbb{R}(x_1, x_2, \dots, x_n)$.

Polynomials are $\mathbb{R}(x)$. Polynomials in two variables are $\mathbb{R}(x, y)$, for example,

$$\begin{array}{lll} f(x,y) & = ax + by & deg(f) = 1, \text{ linear function} \\ f(x,y) & = ax + by + c & deg(f) = 1, \text{ affine function} \\ f(x,y) & = ax^2 + bxy + cy^2 & \text{deg(f)=2, quadratic form} \\ f(x,y) & = ax^2 + bxy + cy^2 + dx + ey + g & \text{deg(f)=2} \end{array}$$

If all the terms in the polynomial have the same number of variables then it is said to be **homogeneous**. In the list above only the linear function and the quadratic form were homogeneous. Returning to the topic of the previous chapter for a moment we should note that a linear transformation has component functions which are homogeneous linear polynomials: suppose that

$L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with matrix $A \in \mathbb{R}^{m \times n}$ then in the notation of this chapter we have $L = (L_1, L_2, \dots, L_m)$ where

$$L_j(x) = (Ax) \cdot e_j = A_{j1}x_1 + A_{j2}x_2 + \cdots + A_{jn}x_n$$

It is clear that such functions are continuous since they are the sum of products of continuous functions. Therefore, linear transformations are continuous with respect to the usual metric topology on $\mathbb{R}^n$.

Remark 3.2.26.

There are other topologies possible for $\mathbb{R}^n$. For example, one can prove that

$$||v||_1 = |v_1| + |v_2| + \cdots + |v_n|$$

gives a norm on $\mathbb{R}^n$ and the theorems we proved transfer over almost without change by just trading $||\cdot||$ for $||\cdot||_1$. The unit "ball" becomes a diamond for the 1-norm. There are many other norms which can be constructed, infinitely many it turns out. However, it has been shown that the topology of all these different norms is equivalent. This means that open sets generated from different norms will be the same class of sets. For example, if you can fit an open disk around every point in a set then it's clear you can just as well fit an open diamond and vice-versa. One of the things that makes infinite dimensional linear algebra more fun is the fact that the topology generated by distinct norms need not be equivalent for infinite dimensions. There is a difference between the open sets generated by the Euclidean norm verses those generated by the 1-norm. Incidentally, my thesis work is mostly built over the 1-norm. It makes the supernumbers happy.

3.3 compact sets and continuous images

It should be noted that the sets $\mathbb{R}^n$ and the empty set $\emptyset$ are both open and closed (these are the only such sets in the metric topology, other sets are either open, closed or neither open nor closed).

Theorem 3.3.1.

The mapping $f : dom(f) \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous iff $f^{-1}(U)$ is open in $dom(f)$ for all open sets $U \subset \mathbb{R}^m$. Additionally, f is continuous iff $f^{-1}(U)$ is closed for each closed set U in $\mathbb{R}^m$.

Notice this theorem makes no explicit reference to the norm. It turns out this theorem is used as the very definition of continuity in more abstract topological settings.

I leave the proof of the closed case to the reader. I tackel the open case here:
Proof: ($\Rightarrow$) Suppose f is continuous and U is open in $\mathbb{R}^m$ then for each $z \in U$ there exists an open ball $B_\epsilon(z) \subset U$. If $x \in f^{-1}(U)$ then there exists $y \in U$ such that $f(x) = y$ and hence there exists an open ball about $B_\epsilon(y) \subset U$. I propose that $f^{-1}(B_\epsilon(y))$ is an open subset of $f^{-1}(U)$ which contains

x. Note that $y \in B_\epsilon(y)$ thus $f(x) = y$ implies $x \in f^{-1}(B_\epsilon(y))$ as according to the definition of inverse image. We seek to show $f^{-1}(B_\epsilon(y)) \subset f^{-1}(U)$. Suppose $v \in f^{-1}(B_\epsilon(y))$. It follows that there exists $w \in B_\epsilon(y)$ such that $f(w) = v$. Note that $B_\epsilon(y) \subset U$ therefore $w \in B_\epsilon(y)$ implies $w \in U$ and so $v \in f^{-1}(U)$ as $w \in U$ has $f(w) = v$. We have shown that an arbitrary element in $f^{-1}(B_\epsilon(y))$ is also in $f^{-1}(U)$ hence $f^{-1}(B_\epsilon(y)) \subseteq f^{-1}(U)$.

($\Leftarrow$) Assume that $f^{-1}(U)$ is open in $dom(f)$ for each open set $U \subset \mathbb{R}^m$. Let $a \in dom(f)$. Assume $\epsilon > 0$ and note that $B_\epsilon(f(a))$ is an open set in $\mathbb{R}^m$ therefore $f^{-1}(B_\epsilon(f(a)))$ is open in $dom(f)$. Note $a \in f^{-1}(B_\epsilon(f(a)))$ since $f(a) \in B_\epsilon(f(a))$. Thus a is a point in the open set $f^{-1}(B_\epsilon(f(a)))$ so there exists a $\delta > 0$ such that $B_\delta(a) \subset f^{-1}(B_\epsilon(f(a))) \subset dom(f)$. Suppose that $x \in B_\delta(a)$ note that $B_\delta(a) \subset f^{-1}(B_\epsilon(f(a)))$ hence $x \in f^{-1}(B_\epsilon(f(a)))$. It follows that there exists $y \in B_\epsilon(f(a))$ such that $f(x) = y$ thus $||f(x) - f(a)|| < \epsilon$. Thus, $\lim_{x \to a} f(x) = f(a)$ for each $a \in dom(f)$ and we conclude that f is continuous. $\square$

Definition 3.3.2.

A mapping S from $\mathbb{N}$ to $\mathbb{R}^n$ is called a **sequence** and we usually denote $S(n) = S_n$ for all $n \in \mathbb{N}$. If $\{a_n\}_{n=1}^{\infty}$ is a sequence then we say $\lim_{n\to\infty} a_n = L$ iff for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $||a_n - L|| < \epsilon$.

A sequence of vectors is not so different than a sequence of numbers. A sequence in $\mathbb{R}^n$ is just a list of vectors instead of a list of numbers and our concept of distance is provided by the norm rather than the absolute value function.

Example 3.3.3. . .

$$a_n = \langle \tfrac{1}{n}, \tan^{-1}(n), 3 \rangle$$

$$a_n \longrightarrow \langle 0, \tfrac{\pi}{2}, 3 \rangle \quad \text{as} \quad n \longrightarrow \infty$$

You can calculate the limit of a vector-valued sequence by taking limits of the component sequences.

Definition 3.3.4.

A set $C \subset \mathbb{R}^n$ is said to be **compact** iff every sequence of points in C contains a convergent subseqence in C which converges to a point in C

The Bolzano-Weiierstrauss theorem says that every closed interval is compact. It's not hard to see that every closed ball in $\mathbb{R}^n$ is compact. I now collect the interesting results from pg. 52 of Edwards' text: note that to say a set is **bounded** simply means that it is possible to surround the whole set with some sufficiently large open ball.

Proposition 3.3.5.

1. Compact subsets of $\mathbb{R}^n$ are closed and bounded.
2. Closed subsets of a compact set are compact.
3. The cartesian product of compact sets gives a compact set.
4. A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

The proof in Edwards is very understandable and the idea of a compact set is really encapsulated by item (4.).

Proposition 3.3.6.

Let C be a compact subset of $\mathbb{R}^n$ and $f : dom(f) \to \mathbb{R}^m$ a continuous mapping with $C \subset dom(f)$, it follows that $f(C)$ is a compact subset of $\mathbb{R}^m$.

The proposition above simply says that the continuous image of compact sets is compact. We finally come to the real reason I am mentioning these topological theorems in this course.

Proposition 3.3.7.

If D is a compact set in $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ is a continuous function then f attains a minimum and maximum value on D. In other words, there exist at least two points $a, b \in D$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in D$.

Since a closed ball is bounded we have that it is compact and the theorem above tells us that if we take any continuous function then the image of a closed ball under the continuous function will have absolute extreme values relative to the closed ball. This result is important to our later efforts to locate min/max values for functions of several variables. The idea will be that we can approximate the function locally by a quadratic form and the local extreme values will be found by evaluating the quadratic form over the unit-n-sphere.

Definition 3.3.8.

Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. We say f is **uniformly continuous** iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $x, y \in U$ with $||x - y|| < \delta$ we find $||f(x) - f(y)|| < \epsilon$.

Proposition 3.3.9.

If $f : C \to \mathbb{R}$ is a continuous mapping and C is compact then f is uniformly continuous.

The Heine-Borel theorem gives a topological refinement of the definition of compactness we gave earlier in this section. Our definition is equivalent to the following: **a compact set is a set for which every open cover has a finite subcover.** An *open cover* of a set is simply a family of open sets whose unions cover the given set. Theorem 8.10 in Edwards states that if we have a sequence of nested subset in $\mathbb{R}^n$ which contains a compact set:

$$V_1 \subset V_2 \subset V_3 \subset \dots \qquad \text{where} \qquad C \subset \bigcup_{n=1}^{\infty} V_n$$

then if we go far enough out in the sequence we'll be able to find V_N such that $C \subset V_N$. In other words, we can find a finite cover for C. The finite-cover definition is prefered in the abstract setting because it makes no reference to the norm or distance function. In graduate topology you'll learn how to think about open sets and continuity without reference to a norm or distance function. Of course it's better to use the norm and distance function in this course because not using it would just result in a silly needless abstraction which made all the geometry opaque. We have an idea of distance and we're going to use it in this course.

3.4 continuous surfaces

We are often interested in a subset of $\mathbb{R}^m$. A particular subset may be a set of points, a curve, a two-dimensional surface, or generally a p-dimensional surface for $p \leq m$. There are more pathological subsets in general, you might have a subset which is one-dimensional in one sector and two-dimensional in another; for example, $S = (\{0\} \times \mathbb{R}) \cup B_1(0) \subset \mathbb{R}^2$. What dimensionality would you ascribe to S? I give the following defintion to help refine our idea of a p-dimensional continuous surface inside $\mathbb{R}^m$.

Definition 3.4.1.

Let $S \subseteq \mathbb{R}^m$. We say S is **continuous surface of dimension** p iff there exists a finite covering of S say $\bigcup_{i=1}^{k} V_i = S$ such that $V_i = \Phi_i(U_i)$ for a continuous bijection $\Phi_i : U_i \rightarrow V_i$ with continuous inverse and U_i homeomorphic to $\mathbb{R}^p$ for all $i = 1, 2, \dots, k$. We define **homeomorphic to** $\mathbb{R}^p$ to mean that there exists a continuous bijection with continuous inverse from U_i to $\mathbb{R}^p$. In addition, we insist that on the intersections $V_j \cap V_k \neq \emptyset$ the mappings Φ_j, Φ_k are **continuously compatible**. If $V_j \cap V_k \neq \emptyset$ then the mappings Φ_j, Φ_k are said to be **continuously compatible** iff $\Phi_j^{-1} \circ \Phi_k$ is continuous when restricted to $\Phi_k^{-1}(V_j \cap V_k)$. Finally we say two subsets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ are **homeomorphic** iff there exists a continuous bijection from V to W and we write $V \approx W$ in this case.

You might expect we could just use bijectivity to define dimension of a subset but there are some very strange constructions that forbid such simple thinking. For example, Cantor showed that there is one-one mapping of $\mathbb{R}$ onto $[0, 1] \times [0, 1]$-the unit square. The existence of such a mapping prompts us to state that $\mathbb{R}$ and $\mathbb{R}^2$ share the same *cardnality*. The concept of cardnality ignores dimensionality, it purely focuses on the more basic set-theoretic nature of a given set. Cardnality[3] ignores the difference between $\mathbb{R}$ and $\mathbb{R}^n$. Later Netto showed that such mappings were not continuous. So, you might be tempted to say that a p-dimensional surface is a continuous

[3] I have an introductory chapter on this topic in my math 200 notes

image of $\mathbb{R}^p$. However, in 1890 Peano was able to construct a (!!!) continuous mapping of the unit-interval $[0,1]$ onto the unit square $[0,1] \times [0,1]$. Peano's construction was not a one-one mapping. You can gather from these results that we need both bijectivity and continuity to capture our usual idea of dimensionality. The curves that Cantor and Peano constructed are called **space filling curves**. You might look in Han Sagan's text *Space Filling Curves* if you'd like to see more on this topic.

Example 3.4.2. *Lines are one-dimensional surfaces. A line in $\mathbb{R}^m$ with direction $v \neq 0 \in \mathbb{R}^m$ passing through $a \in \mathbb{R}^m$ has the form $L_v = \{a + tv \mid t \in \mathbb{R}\}$. Note $F(t) = a + tv$ is a continuous mapping from $\mathbb{R}$ into $\mathbb{R}^m$. In this silly case we have $U_1 = \mathbb{R}$ and $\Phi_1 = Id$ so clearly Φ_1 is a continuous bijection and the image $F(\mathbb{R}) = L_v$ is a continuous one-dimensional surface.*

Example 3.4.3. *A plane Pin $\mathbb{R}^m$ with point $a \in \mathbb{R}^m$ containing linearly independent vectors $\vec{u}, \vec{v} \in \mathbb{R}^m$ has the form $P = \{a + s\vec{u} + t\vec{v} \mid (s,t) \in \mathbb{R}^2\}$. Notice that $F(s,t) = a + s\vec{u} + t\vec{v}$ provides a continuous bijection from $\mathbb{R}^2$ to P hence P is a two-dimensional continuous surface in $\mathbb{R}^m$.*

Example 3.4.4. *Suppose that $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. I claim that $range(L) \leq \mathbb{R}^m$ is a continuous surface of dimension $rank(L)$. If the matrix of L is A then the dimension of the surface $L(\mathbb{R})$ is precisely the number of linearly independent column vectors.*

All the examples thus far were examples of *flat surfaces*. Usually *curved surfaces* require more attention.

Example 3.4.5. *The open ball of radius one in $\mathbb{R}^n$ centered at the origin is homeomorphic to $\mathbb{R}^n$. To prove this assertion we need to provide a continuous bijection with continuous inverse from $B_1(0)$ to $\mathbb{R}^n$. A moments thought suggests*

$$\Phi(x) = \begin{cases} \frac{x}{||x||} \tan \frac{\pi||x||}{2} & x \in B_1(0) \text{ such that } x \neq 0 \\ 0 & x = 0 \end{cases}$$

might work. The idea is that the point $x \in B_1(0)$ maps to the point which lies along the same ray eminating from the origin but a distance $\tan \frac{\pi||x||}{2}$ along the ray. Note that as $||x|| \to 1$ we find $\tan \frac{\pi||x||}{2} \to \infty$. This map takes the unit-ball and stretches it to cover $\mathbb{R}^n$. It is clear that Φ is continuous since each component function of Φ is the product and composite of continuous functions. It is clear that $\Phi(x) = 0$ iff $x = 0$. Thus, to prove $1-1$ suppose that $\Phi(x) = \Phi(y)$ for $x, y \in B_1(0)$ such that $x, y \neq 0$. It follows that $||\Phi(x)|| = ||\Phi(y)||$. Hence,

$$\tan \frac{\pi||x||}{2} = \tan \frac{\pi||y||}{2}.$$

But $x, y \in B_1(0)$ thus $||x||, ||y|| < 1$ so we find $0 < \frac{\pi||x||}{2}, \frac{\pi||y||}{2} < \frac{\pi}{2}$. Tangent is one-one on the open interval $(0, \pi/2)$ hence $\frac{\pi||x||}{2} = \frac{\pi||y||}{2}$ therefore $||x|| = ||y||$. Consider the vector equation $\Phi(x) = \Phi(y)$, replace $||y||$ with $||x||$ since we proved they're equal,

$$\frac{x}{||x||} \tan \frac{\pi||x||}{2} = \frac{y}{||x||} \tan \frac{\pi||x||}{2}$$

multiply both sides by the nonzero quantity $||x||/\tan\frac{\pi||x||}{2}$ *to find* $x = y$. *We have shown that* Φ *is injective. The inverse mapping is given by*

$$\Phi^{-1}(v) = \frac{2\tan^{-1}(||v||)}{\pi}\frac{v}{||v||}$$

for $v \in \mathbb{R}^n$ *such that* $v \neq 0$ *and* $\Phi^{-1}(0) = 0$. *This map takes the vector* v *and compresses it into the unit-ball. Notice that the vector length approaches infinitity the inverse maps closer and closer to the boundary of the ball as the inverse tangent tends to* $\pi/2$ *as its input tends to infinity. I invite the reader to verify that this is indeed the inverse of* Φ, *you need to show that* $\Phi(\Phi^{-1}(v) = v$ *for all* $v \in \mathbb{R}^n$ *and* $\Phi^{-1}(\Phi(x)) = x$ *for all* $x \in B_1(0)$.

Remark 3.4.6.

The example above gives us license to use open balls as the domains for the mappings which define a continuous surface. You could call the Φ_i **continuous patches** if you wish. A smooth surface will be defined in terms of smooth patches or perhaps in terms of the inverse maps Φ_i^{-1}:which are called **coordinate maps.** We need to define a few ideas about differentiability before we can give the definition for a smooth surface. In fact the concept of a surface and the definition of the derivative in some sense are inseparable. For this reason I have begun the discussion of surfaces in this chapter.

I assume that the closed unit-ball $\overline{B_1(0)}$ is homeomorphic to $\mathbb{R}^2$ in the example below. I leave it to the reader supply proof of that claim.

Example 3.4.7. *I claim that the unit-two-sphere* S^2 *is a two-dimensional continuous surface in* $\mathbb{R}^3$. *We define*

$$S^2 = \partial B_1(0) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

We can write $S^2 = S^+ \cup S^-$ *where we define the upper hemisphere* S^+ *and the lower hemisphere* S^- *in the usual manner:*

$$S^+ = \{(x, y, z) \in S^2 \mid z \geq 0\} \qquad S^- = \{(x, y, z) \in S^2 \mid z \leq 0\}$$

The equator is at the intersection,

$$E = S^+ \cap S^- = \{(x, y, z) \in S^2 | z = 0\} = S^1 \times \{0\}$$

Define mappings $\Phi_\pm : \overline{B_1(0)} \subset \mathbb{R}^2 \to S^\pm$ *as follows:*

$$\Phi_\pm(x, y) = (\ x,\ y,\ \pm\sqrt{1 - x^2 - y^2}\ )$$

where $(x, y) \in \mathbb{R}^2$ *such that* $x^2 + y^2 \leq 1$. *I claim the inverse mappings are*

$$\Phi_\pm^{-1}(x, y, z) = (\ x,\ y\ ).$$

for all $(x, y, z) \in S^{\pm}$. *Let's check to see if my claim is correct in the (+) case. Let* $(x, y, z) \in S^+$,

$$\Phi_+(\Phi_+^{-1}(x, y, z)) = \Phi_+(x, y) = (\ x,\ y,\ \sqrt{1 - x^2 - y^2}\ ) = (x, y, z)$$

since $(x, y, z) \in S^+$ *implies* $z = \sqrt{1 - x^2 - y^2}$. *The (-) case is similar. Likewise let* $(x, y) \in B_1(0) \subset \mathbb{R}^2$ *and calculate*

$$\Phi_-^{-1}(\Phi_-(x, y)) = \Phi_-^{-1}(\ x,\ y,\ -\sqrt{1 - x^2 - y^2}\ ) = (x, y).$$

It follows that $\Phi_{\pm}$ *are bijections and it is clear from their formulas that they are continuous mappings. We should check if these are compatible patches. Consider the mapping* $\Phi_+^{-1} \circ \Phi_-$. *A typical point in the* $\Phi_-^{-1}(E)$ *should have the form* $(x, y) \in S^1$ *which means* $x^2 + y^2 = 1$, *consider then*

$$(\Phi_+^{-1} \circ \Phi_-)(x, y) = \Phi_+^{-1}(x, y, -\sqrt{1 - x^2 - y^2}) = (x, y)$$

thus $\Phi_+^{-1} \circ \Phi_-$ *is the identity mapping which is continuous. We find that the two-sphere is a continuous two-dimensional surface.*

Example 3.4.8. *Let* U *be homeomorphic to* $\mathbb{R}^p$. *The image of a continuous mapping* $F : U \to \mathbb{R}^m$ *is a p-dimensional continuous surface in* $\mathbb{R}^m$. *In this case compatibility is trivially satisfied.*

Remark 3.4.9.

A p-dimensional surface is locally modeled by $\mathbb{R}^p$. You can imagine pasting p-space over the surface. Bijectivity and continuity insure that the pasting is not pathological as in Cantors' bijective mapping of $[0, 1]$ onto $\mathbb{R}^n$ or Peano's continuous mapping of $[0, 1]$ onto $[0, 1] \times [0, 1]$. In a later chapter we'll add the criteria of differentiability of the mapping. This will make the pasting keep from getting crinkled up at a point. For example, a cone is a continuous surface however it is not a smooth surface due to the point of the cone