Chapter 4

geometry of curves

If the curve is assigned a sense of direction then we call it an **oriented curve**. A particular curve can be parametrized by many different paths. You can think of a parametrization of a curve as a process of pasting a flexible numberline onto the curve.

Definition 4.0.10.

Let $C \subseteq \mathbb{R}^n$ be an oriented curve which starts at P and ends at Q. We say that $\gamma : [a, b] \rightarrow \mathbb{R}^n$ is a **smooth non-stop parametrization** of C if $\gamma([a, b]) = C$, $\gamma(a) = P$, $\gamma(b) = Q$, and γ is smooth with $\gamma'(t) \neq 0$ for all $t \in [a, b]$. We will typically call γ a **path** from P to Q which covers the curve C.

I have limited the definition to curves with endpoints however the definition for curves which go on without end is very similar. You can just drop one or both of the endpoint conditions.

4.1 arclength

Let's begin by analyzing the tangent vector to a path in three dimensional space. Denote $\gamma = (x, y, z)$ where $x, y, z \in C^\infty([a, b], \mathbb{R})$ and calculate that

$$\gamma'(t) = \tfrac{d\gamma}{dt} = < \tfrac{dx}{dt}, \tfrac{dy}{dt}, \tfrac{dz}{dt} > .$$

Multiplying by dt yields

$$\gamma'(t)dt = \tfrac{d\gamma}{dt}dt = < \tfrac{dx}{dt}, \tfrac{dy}{dt}, \tfrac{dz}{dt} > dt.$$

The arclength ds subtended from time t to time $t + dt$ is simply the length of the vector $\gamma'(t)dt$ which yields,

$$ds = ||\gamma'(t)dt|| = \sqrt{\tfrac{dx}{dt}^2 + \tfrac{dy}{dt}^2 + \tfrac{dz}{dt}^2}\,dt$$

You can think of this as the length of a tiny bit of string that is laid out along the curve from the point $\gamma(t)$ to the point $\gamma(t + dt)$. Of course this infinitesimal notation is just shorthand for an explicit limiting processes. If we sum together all the little bits of arclength we will arrive at the total arclength of the curve. In fact, this is how we define the arclength of a curve. The preceding discussion was in 3 dimensions but the formulas stated in terms of the norm generalizes naturally to $\mathbb{R}^n$.

Definition 4.1.1.

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a smooth, non-stop path which covers the oriented curve C. The **arclength function** of γ is a function $s_\gamma : [a,b] \to \mathbb{R}$ where

$$s_\gamma = \int_a^t ||\gamma'(u)||\, du$$

for each $t \in [a,b]$. If $\tilde{\gamma}$ is a smooth non-stop path such that $||\tilde{\gamma}'(t)|| = 1$ then we say that $\tilde{\gamma}$ is a unit-speed curve. Moreover, we say $\tilde{\gamma}$ is parametrized with respect to arclength.

The arclength function has many special properties. Notice that item (1.) below is actually just the statement that the speed is the magnitude of the velocity vector.

Proposition 4.1.2.

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a smooth, non-stop path which covers the oriented curve C. The **arclength function** of γ denoted by $s_\gamma : [a,b] \to \mathbb{R}$ has the following properties:

1. $\frac{d}{dt}(s_\gamma(w)) = ||\gamma'(w)||\frac{dw}{dt}$,
2. $\frac{ds_\gamma}{dt} > 0$ for all $t \in (a,b)$,
3. s_γ is a 1-1 function,
4. s_γ has inverse $s_\gamma^{-1} : s_\gamma([a,b]) \to [a,b]$.

Proof: We begin with (1.). We apply the fundamental theorem of calculus:

$$\frac{d}{dt}(s_\gamma(w)) = \frac{d}{dt}\int_a^w ||\gamma'(u)||\, du = ||\gamma'(w)||\frac{dw}{dt}$$

for all $w \in (a,b)$. For (2.), set $w = t$ and recall that $||\gamma'(t)|| = 0$ iff $\gamma'(t) = 0$ however we were given that γ is non-stop so $\gamma'(t) \neq 0$. We find $\frac{ds_\gamma}{dt} > 0$ for all $t \in (a,b)$ and consequently the arclength function is an increasing function on (a,b). For (3.), suppose (towards a contradiction) that $s_\gamma(x) = s_\gamma(y)$ where $a < x < y < b$. Note that γ smooth implies s_γ is differentiable with continuous derivative on (a,b) therefore the mean value theorem applies and we can deduce that there is some point on $c \in (x,y)$ such that $s'_\gamma(c) = 0$, which is impossible, therefore (3.) follows. If a function is 1-1 then we can construct the inverse pointwise by simply going backwards for each point mapped to in the range; $s_\gamma^{-1}(x) = y$ iff $s_\gamma(y) = x$. The fact that s_γ is single-valued follows from (3.). $\square$

If we are given a curve C covered by a path γ (which is smooth and non-stop but may not be unit-speed) then we can reparametrize the curve C with a unit-speed path $\tilde{\gamma}$ as follows:

$$\boxed{\tilde{\gamma}(s) = \gamma(s_\gamma^{-1}(s))}$$

where s_γ^{-1} is the inverse of the arclength function.

Proposition 4.1.3.

If γ is a smooth non-stop path then the path $\tilde{\gamma}$ defined by $\tilde{\gamma}(s) = \gamma(s_\gamma^{-1}(s))$ is unit-speed.

Proof: Differentiate $\tilde{\gamma}(t)$ with respect to t, we use the chain-rule,

$$\tilde{\gamma}'(t) = \tfrac{d}{dt}(\gamma(s_\gamma^{-1}(t))) = \gamma'(s_\gamma^{-1}(t))\tfrac{d}{dt}(s_\gamma^{-1}(t)).$$

Hence $\tilde{\gamma}'(t) = \gamma'(s_\gamma^{-1}(t))\frac{d}{dt}(s_\gamma^{-1}(t))$. Recall that if a function is increasing on an interval then its inverse is likewise increasing hence, by (2.) of the previous proposition, we can pull the positive constant $\frac{d}{dt}(s_\gamma^{-1}(t))$ out of the norm. We find, using item (1.) in the previous proposition,

$$||\tilde{\gamma}'(t)|| = ||\gamma'(s_\gamma^{-1}(t))||\tfrac{d}{dt}(s_\gamma^{-1}(t)) = \tfrac{d}{dt}(s_\gamma(s_\gamma^{-1}(t))) = \tfrac{d}{dt}(t) = 1.$$

Therefore, the curve $\tilde{\gamma}$ is unit-speed. We have $ds/dt = 1$ when $t = s$ (this last sentence is simply a summary of the careful argument we just concluded). $\square$

Remark 4.1.4.

While there are many paths which cover a particular oriented curve the unit-speed path is unique and we'll see that formulas for unit-speed curves are particularly simple.

Example 4.1.5. .

$\gamma(t) = \vec{r}(t) = \langle R\cos t, 3, R\sin t\rangle$ for $t \geq 0$, $R > 0$ fixed constant.

$$\frac{d\vec{r}}{dt} = \langle -R\sin t, 0, R\cos t\rangle$$

$$\Rightarrow \frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt} = R^2 \quad \& \quad \left\|\frac{d\vec{r}}{dt}\right\| = R$$

$$s(t) = \int_0^t \left\|\frac{d\vec{r}}{du}\right\| du = \int_0^t R\,du = Ru\Big|_0^t = Rt = s.$$

For example, $s(2\pi) = 2\pi R$ (make sense?)

Note $t = s/R$ hence we can reparametrize via s,

$$\tilde{\gamma}(s) = \vec{r}(s/R) = \langle R\cos(s/R), 3, R\sin(s/R)\rangle$$

unit-speed parametrization of curve.

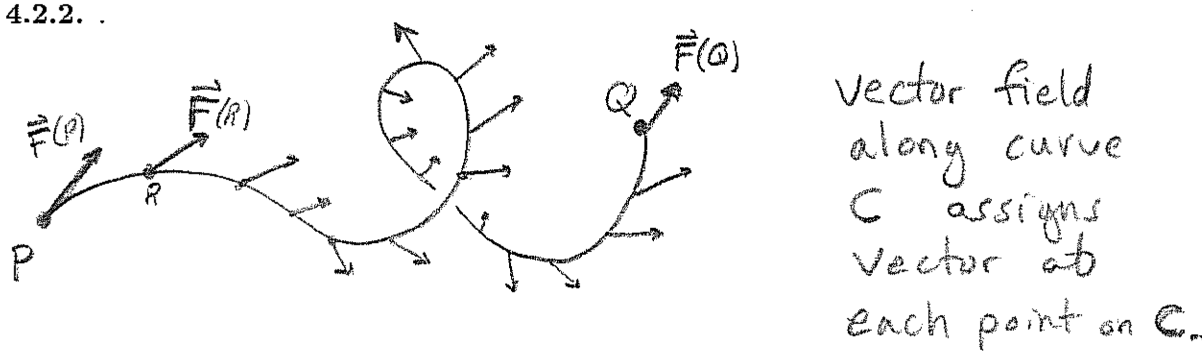
4.2 vector fields along a path

Definition 4.2.1.

Let $C \subseteq \mathbb{R}^3$ be an oriented curve which starts at P and ends at Q. A **vector field along the curve** C is a function from $C \to V^3$. You can visualize this as attaching a vector to each point on C.

The tangent (T), normal(N) and binormal (B) vector fields defined below will allow us to identify when two oriented curves have the same shape.

Example 4.2.2. .



Definition 4.2.3.

Let $\gamma : [a, b] \to \mathbb{R}^3$ be a path from P to Q in $\mathbb{R}^3$. The **tangent vector field** of γ is a mapping $T : [a, b] \to V^3$ defined by

$$T(t) = \frac{1}{||\gamma'(t)||}\gamma'(t)$$

for each $t \in [a, b]$. Likewise, if $T'(t) \neq 0$ for all $t \in [a, b]$ then the **normal vector field** of γ is a mapping $N : [a, b] \to V^3$ defined by

$$N(t) = \frac{1}{||T'(t)||}T'(t)$$

for each $t \in [a, b]$. Finally, if $T'(t) \neq 0$ for all $t \in [a, b]$ then the **binormal vector field** of γ is defined by $B(t) = T(t) \times N(t)$ for all $t \in [a, b]$

Example 4.2.4. *Let $R > 0$ and suppose $\gamma(t) = (R\cos(t), R\sin(t), 0)$ for $0 \leq t \leq 2\pi$. We can calculate*

$$\gamma'(t) =< -R\sin(t), R\cos(t), 0 > \ \Rightarrow \ ||\gamma'(t)|| = R.$$

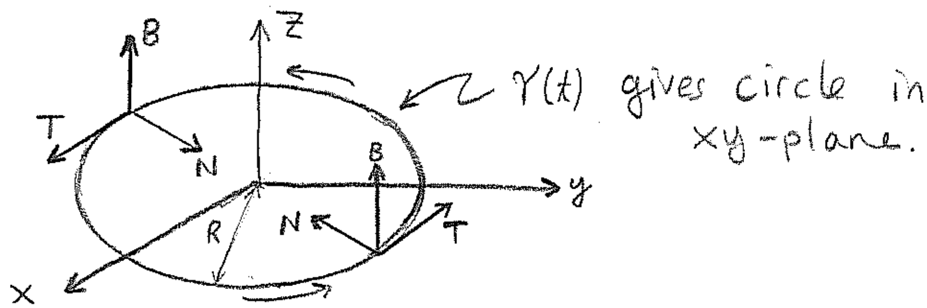
Hence $T(t) =< -\sin(t), \cos(t), 0 >$ and we can calculate,

$$T'(t) =< -\cos(t), -\sin(t), 0 > \ \Rightarrow \ ||T'(t)|| = 1.$$

Thus $N(t) = <-\cos(t), -\sin(t), 0>$. *Finally we calculate the binormal vector field,*

$$\begin{aligned} B(t) = T(t) \times N(t) &= [-\sin(t)e_1 + \cos(t)e_2] \times [-\cos(t)e_1 - \sin(t)e_2] \\ &= [\sin^2(t)e_1 \times e_2 - \cos^2(t)e_2 \times e_1 \\ &= [\sin^2(t) + \cos^2(t)]e_1 \times e_2 \\ &= e_3 = <0,0,1> \end{aligned}$$

Notice that $T \cdot N = N \cdot B = T \cdot B = 0$. *For a particular value of t the vectors* $\{T(t), N(t), B(t)\}$ *give an orthogonal set of unit vectors, they provide a comoving frame for* γ. *It can be shown that the tangent and normal vectors span the plane in which the path travels for times infintesimally close to t. This plane is called the* **osculating plane**. *The binormal vector gives the normal to the osculating plane. The curve considered in this example has a rather boring osculating plane since B is constant. This curve is just a circle in the xy-plane which is traversed at constant speed.*



Example 4.2.5. *Notice that* $s_\gamma(t) = Rt$ *in the preceding example. It follows that* $\tilde{\gamma}(s) = (R\cos(s/R), R\sin(s/R), 0)$ *for* $0 \leq s \leq 2\pi R$ *is the unit-speed path for curve. We can calculate*

$$\tilde{\gamma}'(s) = <-\sin(s/R), \cos(s/R), 0> \ \Rightarrow \ ||\tilde{\gamma}'(s)|| = 1.$$

Hence $\tilde{T}(s) = <-\sin(s/R), \cos(s/R), 0>$ *and we can also calculate,*

$$\tilde{T}'(s) = \tfrac{1}{R} <-\cos(s/R), -\sin(s/R), 0> \ \Rightarrow \ ||\tilde{T}'(t)|| = 1/R.$$

Thus $\tilde{N}(s) = <-\cos(s/R), -\sin(s/R), 0>$. *Note* $\tilde{B} = \tilde{T} \times \tilde{N} = <0,0,1>$ *as before.*

Example 4.2.6. *Let* $m, R > 0$ *and suppose* $\gamma(t) = (R\cos(t), R\sin(t), mt)$ *for* $0 \leq t \leq 2\pi$. *We can calculate*

$$\gamma'(t) = <-R\sin(t), R\cos(t), m> \ \Rightarrow \ ||\gamma'(t)|| = \sqrt{R^2 + m^2}.$$

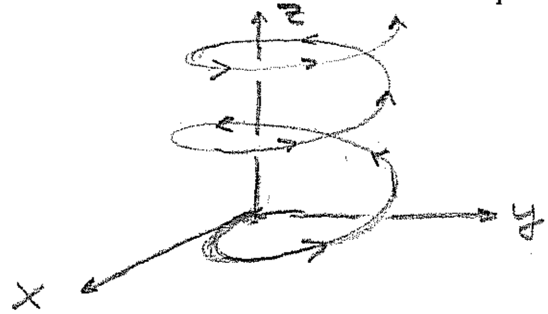
Hence $T(t) = \frac{1}{\sqrt{R^2+m^2}} <-R\sin(t), R\cos(t), m>$ *and we can calculate,*

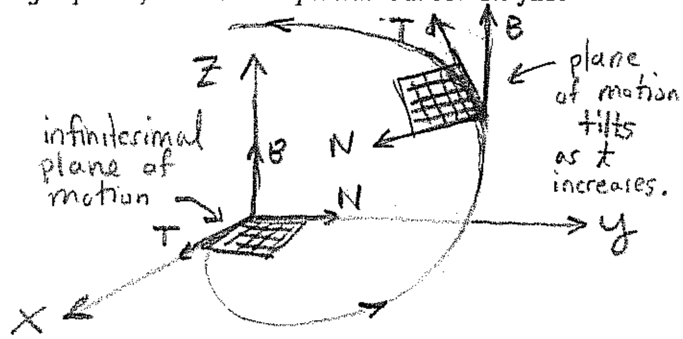
$$T'(t) = \tfrac{1}{\sqrt{R^2+m^2}} <-R\cos(t), -R\sin(t), 0> \ \Rightarrow \ ||T'(t)|| = \tfrac{R}{\sqrt{R^2+m^2}}.$$

Thus $N(t) = <-\cos(t), -\sin(t), 0>$. *Finally we calculate the binormal vector field,*

$$\begin{aligned} B(t) = T(t) \times N(t) &= \tfrac{1}{\sqrt{R^2+m^2}}[-R\sin(t)e_1 + R\cos(t)e_2 + me_3] \times [-\cos(t)e_1 - \sin(t)e_2] \\ &= \tfrac{1}{\sqrt{R^2+m^2}} < m\sin(t), -m\cos(t), R > \end{aligned}$$

We again observe that $T \cdot N = N \cdot B = T \cdot B = 0$. The **osculating plane** *is moving for this curve, note the t-dependence. This curve does not stay in a single plane, it is not a planar curve. In fact this is a circular helix with radius R and slope m. .*





Example 4.2.7. *Lets reparametrize the helix as a unit-speed path. Notice that $s_\gamma(t) = t\sqrt{R^2+m^2}$ thus we should replace t with $s/\sqrt{R^2+m^2}$ to obtain $\tilde{\gamma}(s)$. Let $a = 1/\sqrt{R^2+m^2}$ and $\tilde{\gamma}(s) = (R\cos(as), R\sin(as), ams)$ for $0 \leq s \leq 2\pi\sqrt{R^2+m^2}$. We can calculate*

$$\tilde{\gamma}'(s) = < -Ra\sin(as), Ra\cos(as), am > \;\Rightarrow\; ||\tilde{\gamma}'(s)|| = a\sqrt{R^2+m^2} = 1.$$

Hence $\tilde{T}(s) = a < -R\sin(as), R\cos(as), m >$ and we can calculate,

$$\tilde{T}'(s) = Ra^2 < -\cos(as), -\sin(as), 0 > \;\Rightarrow\; ||\tilde{T}'(s)|| = Ra^2 = \tfrac{R}{R^2+m^2}.$$

Thus $\tilde{N}(s) = < -\cos(as), -\sin(as), 0 >$. Next, calculate the binormal vector field,

$$\begin{aligned}\tilde{B}(s) = \tilde{T}(s) \times \tilde{N}(s) &= a < -R\sin(as), R\cos(as), m > \times < -\cos(as), -\sin(as), 0 > \\ &= \tfrac{1}{\sqrt{R^2+m^2}} < m\sin(as), -m\cos(as), R >\end{aligned}$$

Hopefully you can start to see that the unit-speed path shares the same T, N, B frame at arclength s as the previous example with $t = s/\sqrt{R^2+m^2}$.

4.3 Frenet Serret equations

We now prepare to prove the Frenet Serret formulas for the T, N, B frame fields. It turns out that for nonlinear curves the T, N, B vector fields always provide an orthonormal frame. Moreover, for nonlinear curves, we'll see that the **torsion** and **curvature** capture the geometry of the curve.

Proposition 4.3.1.

If γ is a path with tangent, normal and binormal vector fields T, N and B then $\{T(t), N(t), B(t)\}$ is an orthonormal set of vectors for each $t \in dom(\gamma)$.

Proof: It is clear from $B(t) = T(t) \times N(t)$ that $T(t) \cdot B(t) = N(t) \cdot B(t) = 0$. Furthermore, it is also clear that these vectors have length one due to their construction as unit vectors. In particular this means that $T(t) \cdot T(t) = 1$. We can differentiate this to obtain (by the product rule for dot-products)

$$T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0 \quad \Rightarrow \quad 2T(t) \cdot T'(t) = 0$$

Divide by $||T'(t)||$ to obtain $T(t) \cdot N(t) = 0$. □

We omit the explicit t-dependence for the dicussion to follow here, also you should assume the vector fields are all derived from a particular path γ. Since T, N, B are nonzero and point in three mutually distinct directions it follows that any other vector can be written as a linear combination of T, N, B. This means[1] if $v \in V^3$ then there exist c_1, c_2, c_3 such that $v = c_1 T + c_2 N + c_3 B$. The orthonormality is very nice because it tells us we can calculate the coefficients in terms of dot-products with T, N and B:

$$\boxed{v = c_1 T + c_2 N + c_3 B \quad \Rightarrow \quad c_1 = v \cdot T, \; c_2 = v \cdot N, \; c_3 = v \cdot B}$$

We will make much use of the observations above in the calculations that follow. Suppose that

$$\begin{aligned} T' &= c_{11} T + c_{12} N + c_{13} B \\ N' &= c_{21} T + c_{22} N + c_{23} B \\ B' &= c_{31} T + c_{32} N + c_{33} B. \end{aligned}$$

We observed previously that $T' \cdot T = 0$ thus $c_{11} = 0$. It is easy to show $N' \cdot N = 0$ and $B' \cdot B = 0$ thus $c_{22} = 0$ and c_{33}. Furthermore, we defined $N = \frac{1}{||T'||} T'$ hence $c_{13} = 0$. Note that

$$T' = c_{12} N = \tfrac{c_{12}}{||T'||} T' \quad \Rightarrow \quad c_{12} = ||T'||.$$

To summarize what we've learned so far:

$$\begin{aligned} T' &= c_{12} N \\ N' &= c_{21} T + c_{23} B \\ B' &= c_{31} T + c_{32} N. \end{aligned}$$

We'd like to find some condition on the remaining coefficients. Consider that:

$$\begin{array}{llll} B = T \times N & \Rightarrow & B' = T' \times N + T \times N' & \text{a product rule} \\ & \Rightarrow & B' = [c_{12} N] \times N + T \times [c_{21} T + c_{23} B] & \text{using previous eqn.} \\ & \Rightarrow & B' = c_{23} T \times B & \text{noted } N \times N = T \times T = 0 \\ & \Rightarrow & B' = -c_{23} N & \text{you can show } N = B \times T. \\ & \Rightarrow & c_{31} T + c_{32} N = -c_{23} N & \text{refer to previous eqn.} \\ & \Rightarrow & c_{31} = 0 \text{ and } c_{32} = -c_{23}. & \text{using LI of } \{T, N\} \end{array}$$

[1] You might recognize $[v]_\beta = [c_1, c_2, c_3]^T$ as the coordinate vector with respect to the basis $\beta = \{T, N, B\}$

We have reduced the initial set of equations to the following:

$$\begin{aligned} T' &= c_{12}N \\ N' &= c_{21}T + c_{23}B \\ B' &= -c_{23}N. \end{aligned}$$

The equations above encourage us to define the **curvature** and **torsion** as follows:

Definition 4.3.2.

Let C be a curve which is covered by the unit-speed path $\tilde{\gamma}$ then we define the curvature κ and torsion τ as follows:

$$\kappa(s) = \left|\left|\frac{d\widetilde{T}}{ds}\right|\right| \qquad \tau(s) = -\frac{d\widetilde{B}}{ds}\cdot\widetilde{N}(s)$$

One of your homework questions is to show that $c_{21} = -c_{12}$. Given the result you will prove in the homework we find the famous **Frenet-Serret** equations:

$$\frac{d\widetilde{T}}{ds} = \kappa\widetilde{N} \qquad \frac{d\widetilde{N}}{ds} = -\kappa\widetilde{T} + \tau\widetilde{B} \qquad \frac{d\widetilde{B}}{ds} = -\tau\widetilde{N}.$$

We had to use the arclength parameterization to insure that the formulas above unambiguously define the curvature and the torsion. In fact, if we take a particular (unoriented) curve then there are two choices for orienting the curve. You can show that that the torsion and curvature are independent of the choice of orientation. Naturally the total arclength is also independent of the orientation of a given curve.

Curvature, torsion can also be calculated in terms of a path which is not unit speed. We simply replace s with the arclength function $s_\gamma(t)$ and make use of the chain rule. Notice that $dF/dt = (ds/dt)(d\widetilde{F}/ds)$ hence,

$$\frac{dT}{dt} = \frac{ds}{dt}\frac{d\widetilde{T}}{ds}, \quad \frac{dN}{dt} = \frac{ds}{dt}\frac{d\widetilde{N}}{ds}, \quad \frac{dB}{dt} = \frac{ds}{dt}\frac{d\widetilde{B}}{ds}$$

Or if you prefer, use the dot-notation $ds/dt = \dot{s}$ to write:

$$\frac{1}{\dot{s}}\frac{dT}{dt} = \frac{d\widetilde{T}}{ds}, \quad \frac{1}{\dot{s}}\frac{dN}{dt} = \frac{d\widetilde{N}}{ds}, \quad \frac{1}{\dot{s}}\frac{dB}{dt} = \frac{d\widetilde{B}}{ds}$$

Substituting these into the unit-speed Frenet Serret formulas yield:

$$\frac{dT}{dt} = \dot{s}\kappa N \qquad \frac{dN}{dt} = -\dot{s}\kappa T + \dot{s}\tau B \qquad \frac{dB}{dt} = -\dot{s}\tau N.$$

where $\widetilde{T}(s_\gamma(t)) = T(t)$, $\widetilde{N}(s_\gamma(t)) = N(t)$ and $\widetilde{B}(s_\gamma(t)) = B(t)$. Likewise deduce[2] that

$$\kappa(t) = \frac{1}{\dot{s}}\left|\left|\frac{dT}{dt}\right|\right| \qquad \tau(t) = -\frac{1}{\dot{s}}\left(\frac{dB}{dt}\cdot N(t)\right)$$

[2]I'm using the somewhat ambiguous notation $\kappa(t) = \kappa(s_\gamma(t))$ and $\tau(t) = \tau(s_\gamma(t))$. We do this often in applications of calculus. Ask me if you'd like further clarification on this point.

4.4 curvature, torsion and the osculating plane

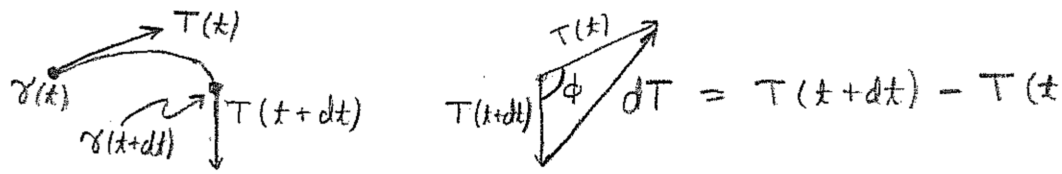
In the preceding section we saw how the calculus and linear algebra suggest we define curvature and torsion. We now stop to analyze the geometric meaning of those definitions.

4.4.1 curvature

Let use begin with the curvature. Assume γ is a non-stop smooth path,

$$\kappa = \frac{1}{\dot{s}}\left|\left|\frac{dT}{dt}\right|\right|$$

Infinitesimally this equation gives $||dT|| = \kappa\dot{s}dt = \kappa\frac{ds}{dt}dt = \kappa\, ds$. But this is a strange equation since $||T|| = 1$. So what does this mean? Perhaps we should add some more detail to resolve this puzzle; let $dT = T(t+dt) - T(t)$.



Notice that

$$\begin{aligned} ||dT||^2 &= [T(t+dt) - T(t)] \cdot [T(t+dt) - T(t)] \\ &= T(t+dt) \cdot T(t+dt) + T(t) \cdot T(t) - 2T(t) \cdot T(t+dt) \\ &= T(t+dt) \cdot T(t+dt) + T(t) \cdot T(t) - 2T(t) \cdot T(t+dt) \\ &= 2(1 - \cos(\phi))) \end{aligned}$$

where we define ϕ to be the angle between $T(t)$ and $T(t+dt)$. This angle measures the change in direction of the tangent vector at t goes to $t+dt$. Since this is a small change in time it is reasonable to expect the angle ϕ is small thus $\cos(\phi) \approx 1 - \frac{1}{2}\phi^2$ and we find that

$$||dT|| = \sqrt{2(1-\cos(\phi)} = \sqrt{2(1-1+\tfrac{1}{2}\phi^2)} = \sqrt{\phi^2} = |\phi|$$

Therefore, $||dT|| = \kappa\, ds = |\phi|$ and we find $\boxed{\kappa = \pm\frac{ds}{d\phi}}$.

Remark 4.4.1.

The curvature measures the infinitesimal change in the direction of the unit-tangent vector to the curve. We say the the reciprocal of the curvature is the **radius of curvature** $r = \frac{1}{\kappa}$. This makes sense as $ds = |1/\kappa| d\phi$ suggests that a circle of radius $1/\kappa$ fits snuggly against the path at time t. We form the **osculating circle at each point along the path by placing a circle of radius** $1/\kappa$ **tangent to the unit-tangent vector in the plane with normal** $B(t)$. We probably should draw a picture of this.

4.4.2 osculating plane and circle

It was claimed that the "infinitesimal" motion of the path resides in a plane with normal B. Suppose that at some time t_o the path reaches the point $\gamma(t_o) = P_o$. Infinitesimally the tangent line matches the path and we can write the parametric equation for the tangent line as follows:

$$l(t) = \gamma(t_o) + t\gamma'(t_o) = P_o + tv_oT_o$$

where we used that $\gamma'(t) = \dot{s}T(t)$ and we evaluated at $t = t_o$ to define $\dot{s}(t_o) = v_o$ and $T(t_o) = T_o$. The normal line through P_o has parametric equations (using $N_o = N(t_o)$):

$$n(\lambda) = P_o + \lambda N_o$$

We learned in the last section that the path bends away from the tangent line along a circle whose radius is $1/\kappa_o$. We find the infinitesimal motion resides in the plane spanned by T_o and N_o which has normal $T_o \times N_o = B(t_o)$. The tangent line and the normal line are perpendicular and could be thought of as a xy-coordinate axes in the osculating plane. The osculating circle is found with its center on the normal line a distance of $1/\kappa_o$ from P_o. Thus the center of the circle is at:

$$Q_o = P_o - \tfrac{1}{\kappa_o}N_o$$

I'll think of constructing x, y, z coordinates based at P_o with respect to the T_o, N_o, B_o frame. We suppose $\vec{r}$ be a point on the osculating circle and x, y, z to be the coefficients in $\vec{r} = P_o + xT_o + yN_o + zB_o$. Since the circle is in the plane based at P_o with normal B_o we should set $z = 0$ for our circle thus $\vec{r} = xT + yN$.

$$||\vec{r} - Q_o||^2 = \tfrac{1}{\kappa_o^2} \quad \Rightarrow \quad ||xT_o + (y + \tfrac{1}{\kappa_o})N_o]||^2 = \tfrac{1}{\kappa_o^2}.$$

Therefore, by the pythagorean theorem for orthogonal vectors, the x, y, z equations for the osculating circle are simply[3] :

$$\boxed{x^2 + (y + \tfrac{1}{\kappa_o})^2 = \tfrac{1}{\kappa_o^2}, \quad z = 0.}$$

[3]Of course if we already use x, y, z in a different context then we should use other symbols for the equation of the osculating circle.

Finally, notice that if the torsion is zero then the Frenet Serret formulas simplify to:

$$\boxed{\frac{dT}{dt} = \dot{s}\kappa N \qquad \frac{dN}{dt} = -\dot{s}\kappa T \qquad \frac{dB}{dt} = 0.}$$

we see that B is a constant vector field and motion will remain in the osculating plane. The change in the normal vector causes a change in the tangent vector and vice-versa however the binormal vector is not coupled to T or N.

Remark 4.4.2.

The torsion measures the infinitesimal change in the direction of the binormal vector relative to the normal vector of the curve. Because the normal vector is in the plane of infinitesimal motion and the binormal is perpendicular to that plane we can say that the torsion measures how the path lifts or twists up off the plane of infinitesimal motion. Furthermore, we can expect path which is trapped in a particular plane (these are called **planar** curves) will have torsion which is identically zero. We should also expect that the torsion for something like a helix will be nonzero everywhere since the motion is always twisting up off the plane of infinitesimal motion. It is probable you will examine these questions in your homework.

4.5 acceleration and velocity

Let's see how the preceding section is useful in the analysis of the motion of physical objects. In the study of dynamics or the physics of motion the critical objects of interest are the position, velocity and acceleration vectors. Once a force is supplied we can in principle solve Newton's Second Law $\vec{F} = m\vec{A}$ and find the equation of motion $\vec{r} = \vec{r}(t)$. Moreover, since the map $t \mapsto \vec{r}(t)$ is a path we can analyze the velocity and acceleration in terms of the **Frenet Frame** $\{T, N, B\}$. To keep it interesting we'll assume the motion is non-stop and smooth so that the analysis of the last section applies.

(for now the next two pages are stolen from a course I took from Dr. R.O. Fulp some years back)

Let $\alpha : [a,b] \longrightarrow \mathbb{R}^3$ be any curve. Def$^{\underline{n}}$ $v(t) = \|\alpha'(t)\|$

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}$$

$$\alpha'(t) = \|\alpha'(t)\| \, T(t) = v(t)\, T(t)$$

$$\alpha''(t) = v(t)\, T'(t) + v'(t)\, T(t)$$

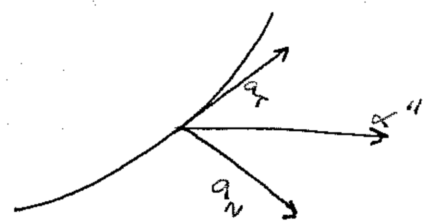
$$= v(t) \|T'(t)\| N(t) + v'(t)\, T(t)$$

<u>Notation</u>

$a_T(t) = v'(t)$ tangential acceleration

$a_N(t) = v(t) \|T'(t)\|$ normal or radial acceleration

$$\alpha''(t) = a_T(t)\, T(t) + a_N(t)\, N(t)$$



$$\alpha' \times \alpha'' = (vT) \times (a_T T + a_N N)$$

$$= v a_N (T \times N)$$

$$= \boxed{v a_N B = \alpha' \times \alpha''}$$

$\|\alpha' \times \alpha''\| = \|v a_N B\| = |v a_N| = v a_N$ by def$^{\underline{n}}$'s of v and a_N.

$$\boxed{\frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} = B}$$

$$\boxed{\frac{\alpha'}{\|\alpha'\|} = T}$$

$$\boxed{B \times T = \frac{1}{\|\alpha' \times \alpha''\|} \frac{1}{\|\alpha'\|}\left[(\alpha' \times \alpha'') \times \alpha'\right] = N}$$

Dot previous relation with B which kills N and T parts as N, T ⊥ B.

$$\alpha''' \cdot B = a_N v \tau$$

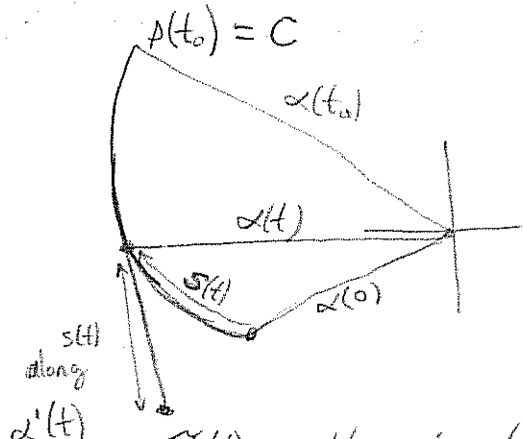
$$a_N \tau = \alpha''' \cdot \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}$$

$$\|\alpha' \times \alpha''\| = a_N V$$

$$a_N \tau = \frac{1}{\|\alpha' \times \alpha''\|} \alpha''' \cdot (\alpha' \times \alpha'')$$

$$\tau = \frac{1}{V a_N \|\alpha' \times \alpha''\|} \alpha''' \cdot (\alpha' \times \alpha'')$$

$$\tau = \frac{\alpha''' \cdot (\alpha' \times \alpha'')}{\|\alpha' \times \alpha''\|^2}$$



$0 \le t \le t_0$

$I(t)$ = the involute of $\alpha(t)$.

4.6 Keplers' laws of planetary motion

KEPLER'S LAWS OF PLANETARY MOTION (284)

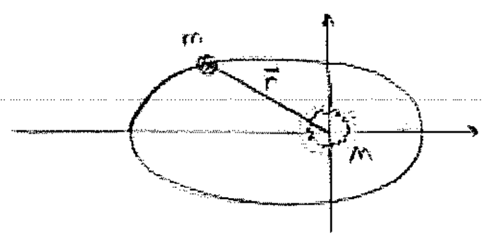
In antiquity there have been radically different views of the universe at large and the motion or lack of motion of the earth through it. At the time of Kepler the heliocentric view of Copernicus (1473–1543) had taken hold, but astronomers insisted that planets traveled in circles, then circles on top of circles on top of circles... This system of "perfect" circles were known as epicycles. Epicycles worked quite well but Kepler (1571–1630) found them unnatural. Kepler instead thought he could explain the motion of planets by a few simple rules. He found these rules empirically by studying the exquisite data taken by Tycho Brahe. These laws were chosen simply to fit the data. Only later were these laws derived from basic physical law. By the way, much of modern physics are still like Kepler's Laws, it is always the dream/goal/aspiration to derive known phenomenological law from basic principles. There is some controversy as to who first derived Kepler's Laws, many credit Newton himself others credit Johann Bernoulli in 1710. The incredible thing is that we can derive the laws in a few short pages. Our notation and understanding of vector calculus is several hundred years in advance, so ordinary folks like myself can grasp the proof.

Set-up

Keplers laws for the Sun and a single planet are:

1.) The orbit of the planet is elliptical with the sun at a focus.
2.) During equal times the planet sweeps out equal areas in the ellipse.
3.) $T^2 \propto a^3$ where T = period of planet's orbit, a = length of semimajor axis of ellipse.

We place the origin at the sun. We expect that



• My proof of Kepler's Laws follows Colley's of §3.1 fairly closely. (285)

Proposition: The motion of the planet lies in a plane which also contains the sun if we assume Newton's Universal Law of Gravitation governs the motion through Newton's Laws.

Proof: our goal is to show that $\vec{r}\times\vec{V}=\vec{C}$ for some constant vector $\vec{C}$. This will show that planet moves in a plane with normal $\vec{C}$. Note,

$$\frac{d}{dt}\left(\vec{r}\times\vec{V}\right)=\underbrace{\frac{d\vec{r}}{dt}\times\vec{V}}_{\vec{V}\times\vec{V}=0}+\vec{r}\times\frac{d\vec{V}}{dt}=\vec{r}\times\vec{a}.$$

Recall in our current notation that $\vec{r}=r\hat{r}$ and Newton tells us that,

$$\vec{F}=m\vec{a}=-\frac{GmM}{r^2}\hat{r}=-\frac{GmM}{r^3}\vec{r}$$

m = mass of planet
M = mass of sun
G = Gravitational Constant.

$$\therefore\quad \vec{a}=\frac{-GM}{r^3}\vec{r}\quad\text{thus }\vec{a}\parallel\vec{r}$$

$$\Rightarrow\ \vec{a}\times\vec{r}=0\ \Rightarrow\ \frac{d}{dt}\left(\vec{r}\times\vec{V}\right)=\vec{r}\times\vec{a}=0\quad\therefore\ \underline{\vec{r}\times\vec{V}=\vec{C}}$$

Th$^{\underline{m}}$/ Kepler's 1st Law: The planet's orbit is an ellipse with sun at one focus

Proof: this will take a little work so be patient, lets get a better hold on $\vec{C}$,

$$\vec{V}=\frac{d\vec{r}}{dt}=\frac{d}{dt}\left(r\hat{r}\right)=\frac{dr}{dt}\hat{r}+r\frac{d\hat{r}}{dt}=\dot{r}\hat{r}+r\frac{d\hat{r}}{dt}$$

Apply this to the following,

$$\vec{c}=\vec{r}\times\vec{v}=r\hat{r}\times\left[\dot{r}\hat{r}^{\nearrow 0}+r\frac{d\hat{r}}{dt}\right]=\underline{r^2\,\hat{r}\times\frac{d\hat{r}}{dt}=\vec{C}}\quad (I)$$

Calculate then, using (I)

$$\begin{aligned}\vec{a}\times\vec{C}&=\left(\frac{-GM}{r^2}\hat{r}\right)\times\left(r^2\,\hat{r}\times\frac{d\hat{r}}{dt}\right)\\ &=-GM\left[\hat{r}\times\left(\hat{r}\times\frac{d\hat{r}}{dt}\right)\right]\\ &=GM\left[\left(\hat{r}\times\frac{d\hat{r}}{dt}\right)\times\hat{r}\right]\quad : \text{recall } \overbrace{A\times(B\times C)=(A\cdot C)B-(A\cdot B)C}^{\text{see §9.4 \#30}}\\ &=GM\left[(\hat{r}\cdot\hat{r})\frac{d\hat{r}}{dt}-\underbrace{\left(\hat{r}\cdot\frac{d\hat{r}}{dt}\right)}_{0}\hat{r}\right]\quad : \hat{r}\cdot\hat{r}=1\ \Rightarrow\ 2\hat{r}\cdot\dot{\hat{r}}=0\Rightarrow\hat{r}\cdot\dot{\hat{r}}=0,\\ &=GM\frac{d\hat{r}}{dt}\\ &=\underline{\frac{d}{dt}\left(GM\hat{r}\right)=\vec{a}\times\vec{C}}\quad (II)\end{aligned}$$

(286)

Proof of Kepler's 1st Law continued

We may derive another identity for $\vec{a}\times\vec{c}$,

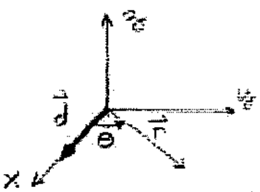
$$\vec{a}\times\vec{c} = \frac{d\vec{v}}{dt}\times\vec{c} + \vec{v}\times\frac{d\vec{c}}{dt} \quad \text{: added zero since } \frac{d\vec{c}}{dt} = 0.$$

$$= \frac{d}{dt}\left[\vec{v}\times\vec{c}\right] \quad \text{(III)} \quad \text{: using identity (k) on (265)}$$

Thus comparing (II) & (III) we find

$$\frac{d}{dt}\left(GM\hat{r}\right) = \frac{d}{dt}\left(\vec{v}\times\vec{c}\right) \quad \therefore \quad \vec{v}\times\vec{c} = GM\hat{r} + \vec{d} \quad \text{(IV)}$$

where $\vec{d}$ is a constant vector, it lies in the orbital plane since $\vec{v}\times\vec{c}$ and $\hat{r}$ do. Now chose coordinates in the orbital plane so that $\vec{d}$ lines up with the x-axis. Let θ be the usual θ in the xy-plane,



$$\hat{r}\cdot\vec{d} = |\hat{r}||\vec{d}|\cos\theta = d\cos\theta$$

where $|\vec{d}| = d$ in our notation here.

Now consider the length of $\vec{c}$ squared,

$$\begin{aligned} c^2 &= \vec{c}\cdot\vec{c} \\ &= (\vec{r}\times\vec{v})\cdot\vec{c} \\ &= \vec{r}\cdot(\vec{v}\times\vec{c}) \quad \text{: using identity (V) of (248)} \\ &= r\hat{r}\cdot\left[GM\hat{r}+\vec{d}\right] \quad \text{: using IV. we found just above.} \\ &= GMr + r\hat{r}\cdot\vec{d} \\ &= GMr + rd\cos\theta \\ &= r(GM + d\cos\theta) \end{aligned}$$

Therefore we solve for $r = \sqrt{x^2+y^2+z^2} = \sqrt{x^2+y^2}$ (we're in $z=0$) and obtain the eq$^{\underline{n}}$ of an ellipse (or parabola or hyperbola).

$$r = \frac{c^2}{GM + d\cos\theta} = \frac{c^2/GM}{1+(d/GM)\cos\theta} = \boxed{\frac{p}{1+e\cos\theta} = r}$$

where we define $p = c^2/GM$ and the eccentricity $e = d/GM$. This is an ellipse in polar coordinates. Since you've likely not seen that recently (or maybe never) we'll connect to ⮐

Proof of Kepler's 1st Law continued (287)

the usual Cartesian eq^{n}'s for the ellipse. The details will be of use to us in proving the 3rd Law of Kepler later on.

$$r = \frac{p}{1+e\cos\theta} \Rightarrow r = p - er\cos\theta$$

Trying to convert the polar coordinates (r,θ) to (x,y) where $x = r\cos\theta$ and $y = r\sin\theta$. We see, using $x = r\cos\theta$

$$r = p - ex$$

$$r^2 = x^2 + y^2 = p^2 - 2epx + e^2x^2$$

$$x^2 - e^2x^2 + y^2 + 2epx = p^2$$

$$x^2(1-e^2) + 2epx + y^2 = p^2$$

$$x^2 + \frac{2ep}{1-e^2}x + \frac{y^2}{1-e^2} = \frac{p^2}{1-e^2} \quad : \quad \underline{\text{assume } e \neq \pm 1}$$

$$\left(x - \frac{ep}{1-e^2}\right)^2 + \frac{y^2}{(1-e^2)} = \frac{p^2}{1-e^2} + \frac{e^2p^2}{(1-e^2)^2} = \frac{p^2 - e^2p^2 + e^2p^2}{(1-e^2)^2} = \frac{p^2}{(1-e^2)^2}$$

$$\therefore \boxed{\frac{\left(x - \frac{ep}{1-e^2}\right)^2}{p^2/(1-e^2)^2} + \frac{y^2}{p^2/(1-e^2)} = 1} \quad \underset{(0<e<1)}{\underline{\text{ellipse}}} \text{ or } \underset{(e>1)}{\underline{\text{hyperbola}}}$$

This is an ellipse with center $(ep/(1-e^2), 0)$ and it has semi-major axis length $a = p/(1-e^2)$ and semi-minor axis $b = p/\sqrt{1-e^2}$. Remark: recall that we defined $p = c^2/GM$ so $p > 0$ and we need not worry about ÷ by p. Now $e = d/GM > 0$ so we can rule out $e = -1$ as a problem. Notice we have divided by $\sqrt{1-e^2}$ as part of our sol^{n}, this only makes sense if $0 < e < 1$. The case $e = 1$ needs seperate treatment. Motion in the case $0 < e < 1$ is that of planets.

$\underline{e = 1)}$ $r = p - r\cos\theta \quad \therefore \quad r^2 = (p-x)^2 = p^2 - 2xp + x^2$

that is $x^2 + y^2 = p^2 - 2xp + x^2 \Rightarrow 2xp = p^2 - y^2$

$$\therefore \boxed{x = p/2 - y^2/2p} \text{ parabola}$$

Remark: One nice resource for background on conic-sections and polar coordinates is "Precalculus, Concepts through functions" Sullivan & Sullivan. There is just about all the cases you can imagine, rotated ellipses for example.

(288)

Th^m/ Kepler's 2nd Law: During equal times a planet sweeps through equal areas.

Proof: Pick a point P_o at angle θ_o. The later in the course we will learn that the area in polar coordinates swept by the region from θ_o to θ is simply

$$A(\theta) = \int_{\theta_o}^{\theta} \frac{1}{2} r^2 \, d\beta$$

We seek to show that $\frac{dA}{dt}$ = constant. Consider then

$$\frac{dA}{d\theta} = \frac{d}{d\theta}\int_{\theta_o}^{\theta} \frac{1}{2} r^2 \, d\beta = \frac{1}{2} r^2 \quad \text{by F.T.C.}$$

Then the chain rule tells us

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

Notice that $\hat{r} = \langle \cos\theta, \sin\theta \rangle$ thus diff. implicitly, remember $\theta = \theta(t)$.

$$\frac{d\hat{r}}{dt} = \langle -\sin\theta, \cos\theta \rangle \frac{d\theta}{dt} = \langle -\sin\theta, \cos\theta, 0 \rangle \frac{d\theta}{dt} \quad \text{(we've been suppressing the z-comp.)}$$

(285), (⊙) $\Rightarrow$ $\vec{C} = r^2\left(\hat{r} \times \frac{d\hat{r}}{dt}\right) = r^2 \frac{d\theta}{dt} \langle \cos\theta, \sin\theta, 0 \rangle \times \langle -\sin\theta, \cos\theta, 0 \rangle$

$$\vec{C} = r^2 \frac{d\theta}{dt} \langle 0, 0, 1 \rangle \quad \therefore \quad C = r^2 \frac{d\theta}{dt}$$

Hence $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{C}{2}$ = constant. //

Th^m/ Kepler's 3rd Law: $T^2 = Ka^3$ where T is the orbital period and a is the length of the semimajor axis, K = some constant

Proof: I proved back on pg. (138) in E? that the area of an ellipse is $A = \pi ab$. On the other hand we could say that $dA = \frac{dA}{dt} dt$ and integrate over a whole orbit to find

$$\pi ab = \int_0^T \frac{dA}{dt} dt = \int_0^T \frac{C}{2} dt = \frac{CT}{2} \quad \therefore \quad T = \frac{2\pi ab}{C} \quad \therefore \quad T^2 = \frac{4\pi^2 a^2 b^2}{C^2}$$

Notice that $a^2 = p^2/(1-e^2)^2$ and $b^2 = p^2/(1-e^2)$, also $C^2 = GMp$.

$$T^2 = \frac{4\pi^2}{GMp} \frac{p^2}{(1-e^2)^2} \cdot \frac{p^2}{(1-e^2)} = \frac{4\pi^2}{GM}\left(\frac{p}{1-e^2}\right)^3 = \boxed{\frac{4\pi^2 a^3}{GM} = T^2} \; //$$

It is interesting that $K = \frac{4\pi^2}{GM}$ is independent of the planets mass. all the planets orbit under the same K-value.

Remark: There is another method of proving Kepler's Laws that begins with the two-body Lagrangian for a central potential (well force really but $\vec{F} = f(r)\hat{r} \Rightarrow U = U(r) \dots$). In that derivation one need not assume the sun is at the origin. Instead you consider the center of mass to be at the origin and work out how the reduced mass μ orbits. Anyway its very beautiful, take Mechanics at the Junior/Senior level to see the more general derivation. Also they will actually find $r(t)$ explicitly as opposed to the indirect arguments we have offered (or rather stolen from Colley ☺.)