Chapter 5

Euclidean structures and physics

Although much was known about the physical world prior to Newton that knowledge was highly unorganized and formulated in such a way that is was difficult to use and understand[1]. The advent of Newton changed all that. In 1665-1666 Newton transformed the way people thought about the physical world, years later he published his many ideas in "Principia mathematica philosphiae naturalia" (1686). His contribution was to formulate three basic laws or principles which along with his universal law of gravitation would prove sufficient to derive and explain all mechanical systems both on earth and in the heavens known at the time. These basic laws may be stated as follows:

1. **Newton's First Law:** Every particle persists in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by impressed forces.

2. **Newton's Second Law:** The rate of change of motion is proportional to the motive force impressed; and is made in the direction of the straight line in which that force is impressed.

3. **Newton's Third Law:** To every action there is an equal reaction; or the mutual actions of two bodies upon each other are always equal but oppositely directed.

Until the early part of the last century Newton's laws proved adequate. We now know, however that they are only accurate within prescribed limits. They do not apply for things that are very small like an atom or for things that are very fast like cosmic rays or light itself. Nevertheless Newton's laws are valid for the majority of our common macroscopic experiences in everyday life.

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[1] What follows is borrowed from Chapter 6 of my *Mathematical Models in Physics* notes which is turn borrowed from my advisor Dr. R.O. Fulp's notes for Math 430 at NCSU. I probably will not cover all of this in lecture but I thought it might be interesting to those of you who are more physically minded. I have repeated some mathematical definitions in this chapter in the interest of making this chapter more readable. This chapter gives you an example of the practice of Mathematical Physics. One common idea in Mathematical Physics is to take known physics and reformulate it in a proper mathematical context. Physicists don't tend to care about domains or existence so if we are to understand their calculations then we need to do some work in most cases.

It is implicitly presumed in the formulation of Newton's laws that we have a concept of a straight line, of uniform motion, of force and the like. Newton realized that Euclidean geometry was a necessity in his model of the physical world. In a more critical formulation of Newtonian mechanics one must address the issues implicit in the above formulation of Newton's laws. This is what we attempt in this chapter, we seek to craft a mathematically rigorous systematic statement of Newtonian mechanics.

5.1 Euclidean geometry

Note: we abandon the more careful notation of the previous chapters in what follows. In a nutshell we are setting $\mathbb{R}^3 = V^3$, this is usually done in physics. We can identify a given point with a vector that eminates from the origin to the point in question. It will be clear from the context if a point or a vector is intended.

Nowadays Euclidean geometry is imposed on a vector space via an inner product structure. Let $x_1, x_2, x_3, y_1, y_2, y_3, c \in \mathbb{R}$. As we discussed $\mathbb{R}^3$ is the set of 3-tuples and it is a vector space with respect to the operations,

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$c(x_1, x_2, x_3) = (cx_1, cx_2, cx_3)$$

where $x_1, x_2, x_3, y_1, y_2, y_3, c \in \mathbb{R}$. Also we have the dot-product,

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 + x_3 y_3$$

from which the *length* of a vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ can be calculated,

$$|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

meaning $|x|^2 = x \cdot x$. Also if $x, y \in \mathbb{R}^3$ are nonzero vectors then the angle between them is defined by the formula,

$$\theta = \cos^{-1}\left(\frac{x \cdot y}{|x||y|}\right)$$

In particular nonzero vectors x and y are *perpendicular or orthogonal* iff $\theta = 90^o$ which is so iff $\cos(\theta) = 0$ which is turn true iff $x \cdot y = 0$.

Definition 5.1.1.

A function $L : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be a **linear transformation** if and only if there is a 3×3 matrix A such that $L(x) = Ax$ for all $x \in \mathbb{R}^3$. Here Ax indicates multiplication by the matrix A on the column vector x

Definition 5.1.2.

An **orthogonal transformation** is a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies

$$L(x) \cdot L(y) = x \cdot y$$

for all $x, y \in \mathbb{R}^3$. Such a transformation is also called an **linear isometry of the Euclidean metric.**

The term *isometry* means the same measure, you can see why that's appropriate from the following,

$$|L(x)|^2 = L(x) \cdot L(x) = x \cdot x = |x|^2$$

for all $x \in \mathbb{R}^3$. Taking the square root of both sides yields $|L(x)| = |x|$; an orthogonal transformation preserves the lengths of vectors in $\mathbb{R}^3$. Using what we just learned its easy to show orthogonal transformations preserve angles as well,

$$\cos(\theta_L) = \frac{L(x) \cdot L(y)}{|L(x)||L(y)|} = \frac{x \cdot y}{|x||y|} = \cos(\theta)$$

Hence taking the inverse cosine of each side reveals that the angle θ_L between $L(x)$ and $L(y)$ is equal to the angle θ between x and y; $\theta_L = \theta$. Orthogonal transformations preserve angles.

Definition 5.1.3.

We say $l \subseteq \mathbb{R}^3$ is a **line** if there exist $a, v \in \mathbb{R}^3$ such that

$$l = \{x \in \mathbb{R}^{\,n \times n} \mid x = a + tv, \quad t \in \mathbb{R}\}.$$

Proposition 5.1.4.

If L is an orthonormal transformation then $L(l)$ is also a line in $\mathbb{R}^3$.

To prove this we simply need to find new a' and v' in $\mathbb{R}^3$ to demonstrate that $L(l)$ is a line. Take a point on the line, $x \in l$

$$\begin{aligned} L(x) &= L(a + tv) \\ &= L(a) + tL(v) \end{aligned} \tag{5.1}$$

thus $L(x)$ is on a line described by $x = L(a) + tL(v)$, so we can choose $a' = L(a)$ and $v' = L(v)$ it turns out; $L(l) = \{x \in \mathbb{R}^3 \mid x = a' + tv'\}$.

If one has a coordinate system with unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ along three mutually orthogonal axes then an orthogonal transformation will create three new mutually orthogonal unit vectors $L(\hat{\imath}) = \hat{\imath}', L(\hat{\jmath}) = \hat{\jmath}', L(\hat{k}) = \hat{k}'$ upon which one could lay out new coordinate axes. In this way orthogonal transformations give us a way of constructing new "rotated" coordinate systems from a given coordinate system. Moreover, it turns out that Newton's laws are preserved (have the same form) under orthogonal transformations. Transformations which are not orthogonal can greatly distort the form of Newton's laws.

Remark 5.1.5.

If we view vectors in $\mathbb{R}^3$ as column vectors then the dot-product of x with y can be written as $x \cdot y = x^T y$ for all $x, y \in \mathbb{R}^3$. Recall that x^T is the *transpose* of x, it changes the column vector x to the corresponding row vector x^T.

Let us consider an orthogonal transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ where $L(x) = Ax$. What condition on the matrix A follows from the the L being an orthogonal transformation ?

$$\begin{aligned} L(x) \cdot L(y) = x \cdot y \quad &\iff (Ax)^T(Ay) = x^T y \\ &\iff x^T(A^T A)y = x^T y \\ &\iff x^T(A^T A)y = x^T I y \\ &\iff x^T(A^T A - I)y = 0. \end{aligned} \tag{5.2}$$

But $x^T(A^T A - I)y = 0$ for *all* $x, y \in \mathbb{R}^3$ iff $A^T A - I = 0$ or $A^T A = I$. Thus L is orthogonal iff its matrix A satisfies $A^T A = I$. This is in turn equivalent to A having an inverse and $A^{-1} = A^T$.

Proposition 5.1.6.

The set of **orthogonal transformations** on $\mathbb{R}^3$ is denoted O(3). The operation of function composition on O(3) makes it a group. Likewise we also denote the set of all **orthogonal matrices** by O(3),

$$O(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^T A = I\}$$

it is also a group under matrix multiplication.

Usually we will mean the matrix version, it should be clear from the context, it's really just a question of notation since we know that L and A contain the same information thanks to linear algebra. Recall that every linear transformation L on a finite dimensional vector space can be represented by matrix multiplication of some matrix A.

Proposition 5.1.7.

The set of **special orthogonal matrices** on $\mathbb{R}^3$ is denoted SO(3),

$$SO(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^T A = I \text{ and } det(A) = 1\}$$

it is also a group under matrix multiplication and thus it is a subgroup of O(3). It is shown in standard linear algebra course that every special orthogonal matrix rotates $\mathbb{R}^3$ about some line. Thus, we will often refer to SO(3) as the **group of rotations**.

There are other transformations that do not change the geometry of $\mathbb{R}^3$.

Definition 5.1.8.

A **translation** is a function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x) = x + v$ where v is some fixed vector in $\mathbb{R}^3$ and x is allowed to vary over $\mathbb{R}^3$.

Clearly translations do not change the distance between two points $x, y \in \mathbb{R}^3$,

$$|T(x) - T(y)| = |x + v - (y - v)| = |x - y| = \text{ distance between } x \text{ and } y.$$

Also if x, y, z are points in $\mathbb{R}^3$ and θ is the angle between $y - x$ and $z - x$ then θ is also the angle between $T(y) - T(x)$ and $T(z) - T(x)$. Geometrically this is trivial, if we shift all points by the same vector then the difference vectors between points are unchanged thus the lengths and angles between vectors connecting points in $\mathbb{R}^3$ are unchanged.

Definition 5.1.9.

A function $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is called a **rigid motion** if there exists a vector $r \in \mathbb{R}^3$ and a rotation matrix $A \in SO(3)$ such that $\phi(x) = Ax + r$.

A rigid motion is the composite of a translation and a rotation therefore it will clearly preserve lengths and angles in $\mathbb{R}^3$. So rigid motions are precisely those transformations which preserve Euclidean geometry and consequently they are the transformations which will preserve Newton's laws. If Newton's laws hold in one coordinate system then we will find Newton's laws are also valid in a new coordinate system iff it is related to the original coordinate system by a rigid motion. We now proceed to provide a careful exposition of the ingredients needed to give a rigorous formulation of Newton's laws.

Definition 5.1.10.

We say that $\mathcal{E}$ is an **Euclidean structure** on a set S iff $\mathcal{E}$ is a family of bijections from S onto $\mathbb{R}^3$ such that,
(1.) $\mathcal{X}, \mathcal{Y} \in \mathcal{E}$ then $\mathcal{X} \circ \mathcal{Y}^{-1}$ is a rigid motion.
(2.) if $\mathcal{X} \in \mathcal{E}$ and ϕ is a rigid motion then $\phi \circ \mathcal{X} \in \mathcal{E}$.
Also a **Newtonian space** is an ordered pair $(S, \mathcal{E})$ where S is a set and $\mathcal{E}$ is an Euclidean structure on S.

Notice that if $\mathcal{X}, \mathcal{Y} \in \mathcal{E}$ then there exists an $A \in SO(3)$ and a vector $r \in \mathbb{R}^3$ so that we have $\mathcal{X}(p) = A\mathcal{Y}(p) + r$ for every $p \in S$. Explicitly in cartesian coordinates on $\mathbb{R}^3$ this means,

$$[\mathcal{X}_1(p), \mathcal{X}_2(p), \mathcal{X}_3(p)]^T = A[\mathcal{Y}_1(p), \mathcal{Y}_2(p), \mathcal{Y}_3(p)]^T + [r_1, r_2, r_3]^T.$$

Newtonian space is the mathematical model of space which is needed in order to properly formulate Newtonian mechanics. The first of Newton's laws states that an object which is subject to no forces must move along a straight line. This means that some observer should be able to show that the object moves along a line in space. We take this to mean that the observer chooses an inertial frame and makes measurements to decide wether or not the object executes straight line motion in the coordinates defined by that frame. If the observations are to be frame independent then the notion of a straight line in space should be independent of which inertial coordinate system is used to make the measurements. We intend to identify inertial coordinate systems as precisely those elements of $\mathcal{E}$. Thus we need to show that if l is a line as measured by $\mathcal{X} \in \mathcal{E}$ then l is also a line as measured by $\mathcal{Y} \in \mathcal{E}$.

Definition 5.1.11.

Let $(S, \mathcal{E})$ be a Newtonian space. A subset l of S is said to be **a line in S** iff $\mathcal{X}(l)$ is a line in $\mathbb{R}^3$ for some choice of $\mathcal{X} \in \mathcal{E}$.

The theorem below shows us that the choice made in the definition above is not special. In fact our definition of a line in S is coordinate independent. Mathematicians almost always work towards formulating geometry in a way which is independent of the coordinates employed, this is known as the coordinate free approach. Physicists in contrast almost always work in coordinates.

Theorem 5.1.12.

If l is a line in a Newtonian space $(S, \mathcal{E})$ then $\mathcal{Y}(l)$ is a line in $\mathbb{R}^3$ for **every** $\mathcal{Y} \in \mathcal{E}$.

Proof: Because l is a line in the S we know there exists $\mathcal{X} \in \mathcal{E}$ and $\mathcal{X}(l)$ is a line in $\mathbb{R}^3$. Let $\mathcal{Y} \in \mathcal{E}$ observe that,

$$\mathcal{Y}(l) = (\mathcal{Y} \circ \mathcal{X}^{-1} \circ \mathcal{X})(l) = (\mathcal{Y} \circ \mathcal{X}^{-1})(\mathcal{X}(l)).$$

Now since $\mathcal{X}, \mathcal{Y} \in \mathcal{E}$ we have that $\mathcal{Y} \circ \mathcal{X}^{-1}$ is a rigid motion on $\mathbb{R}^3$. Thus if we can show that rigid motions take lines to lines in $\mathbb{R}^3$ the proof will be complete. We know that there exist $A \in SO(3)$ and $r \in \mathbb{R}^3$ such that $(\mathcal{Y} \circ \mathcal{X}^{-1})(x) = Ax + r$. Let $x \in \mathcal{X}(l) = \{x \in \mathbb{R}^3 \mid x = p + tq \quad t \in \mathbb{R}$ and p,q are fixed vectors in $\mathbb{R}^3\}$, consider

$$\begin{aligned} (\mathcal{Y} \circ \mathcal{X}^{-1})(x) &= Ax + r \\ &= A(p + tq) + r \\ &= (Ap + r) + tAq \\ &= p' + tq' \qquad \text{letting } p' = Ap + r \text{ and } q' = Aq. \end{aligned} \tag{5.3}$$

The above hold for all $x \in \mathcal{X}(l)$, clearly we can see the line has mapped to a new line $\mathcal{Y}(l) = \{x \in \mathbb{R}^3 \mid x = p' + tq' \quad, t \in \mathbb{R}\}$. Thus we find what we had hoped for, lines are independent of the frame chosen from $\mathcal{E}$ in the sense that a line is always a line no matter which element of $\mathcal{E}$ describes it.

Definition 5.1.13.

An **observer** is a function from an interval $I \subset \mathbb{R}$ into $\mathcal{E}$. We think of such a function $\mathcal{X} : I \to \mathcal{E}$ as being a time-varying coordinate system on S. For each $t \in I$ we denote $\mathcal{X}(t)$ by $\mathcal{X}_t$; thus $\mathcal{X}_t : S \to \mathbb{R}^3$ for each $t \in I$ and $\mathcal{X}_t(p) = [\mathcal{X}_{t1}(p), \mathcal{X}_{t2}(p), \mathcal{X}_{t3}(p)]$ for all $p \in S$.

Assume that a material particle or more generally a "point particle" moves in space S in such a way that at time t the particle is centered at the point $\gamma(t)$. Then the mapping $\gamma : I \to S$ will be called the **trajectory** of the particle.

Definition 5.1.14.

Let us consider a particle with trajectory $\gamma : I \to S$. Further assume we have an observer $\mathcal{X} : I \to \mathcal{E}$ with $t \mapsto \mathcal{X}_t$ then:

(1.) $\mathcal{X}_t(\gamma(t))$ is the **position vector** of the particle at time $t \in I$ relative to the observer $\mathcal{X}$.

(2.) $\frac{d}{dt}\big[\mathcal{X}_t(\gamma(t))\big]|_{t=t_o}$ is called the **velocity** of the particle at time $t_o \in I$ relative to the observer $\mathcal{X}$, it is denoted $v_{\mathcal{X}}(t_o)$.

(3.) $\frac{d^2}{dt^2}\big[\mathcal{X}_t(\gamma(t))\big]|_{t=t_o}$ is called the **acceleration** of the particle at time $t_o \in I$ relative to the observer $\mathcal{X}$, it is denoted $a_{\mathcal{X}}(t_o)$.

Notice that position, velocity and acceleration are only defined with respect to an observer. We now will calculate how position, velocity and acceleration of a particle with trajectory $\gamma : I \to S$ relative to observer $\mathcal{Y} : I \to \mathcal{E}$ compare to those of another observer $\mathcal{X} : I \to \mathcal{E}$. To begin we note that each particular $t \in I$ we have $\mathcal{X}_t, \mathcal{Y}_t \in \mathcal{E}$ thus there exists a rotation matrix $A(t) \in SO(3)$ and a vector $v(t) \in \mathbb{R}^3$ such that,

$$\mathcal{Y}_t(p) = A(t)\mathcal{X}_t(p) + r(t)$$

for all $p \in S$. As we let t vary we will in general find that $A(t)$ and $r(t)$ vary, in other words we have A a matrix-valued function of time given by $t \mapsto A(t)$ and r a vector-valued function of time given by $t \mapsto r(t)$. Also note that the *origin* of the coordinate coordinate system $\mathcal{X}(p) = 0$ moves to $\mathcal{Y}(p) = r(t)$, this shows that the correct interpretation of $r(t)$ is that it is the position of the old coordinate's origin in the new coordinate system. Consider then $p = \gamma(t)$,

$$\boxed{\mathcal{Y}_t(\gamma(t)) = A(t)\mathcal{X}_t(\gamma(t)) + r(t)} \tag{5.4}$$

this equation shows how the position of the particle in $\mathcal{X}$ coordinates transforms to the new position in $\mathcal{Y}$ coordinates. We should not think that the particle has moved under this transformation, rather we have just changed our viewpoint of where the particle resides. Now move on to the transformation of velocity, (we assume the reader is familiar with differentiating matrix valued functions of a real variable, in short we just differentiate component-wise)

$$\begin{aligned} v_{\mathcal{Y}}(t) &= \tfrac{d}{dt}\big[\mathcal{Y}(\gamma(t))\big] \\ &= \tfrac{d}{dt}\big[A(t)\mathcal{X}_t(\gamma(t)) + r(t)\big] \\ &= \tfrac{d}{dt}[A(t)]\mathcal{X}_t(\gamma(t)) + A(t)\tfrac{d}{dt}[\mathcal{X}_t(\gamma(t))] + \tfrac{d}{dt}[r(t)] \\ &= A'(t)\mathcal{X}_t(\gamma(t)) + A(t)v_{\mathcal{X}}(t) + r'(t). \end{aligned} \tag{5.5}$$

Recalling the dot notation for time derivatives and introducing $\gamma_{\mathcal{X}} = \mathcal{X} \circ \gamma$,

$$\boxed{v_{\mathcal{Y}} = \dot{A}\gamma_{\mathcal{X}} + Av_{\mathcal{X}} + \dot{r}.} \tag{5.6}$$

We observe that the velocity according to various observes depends not only on the trajectory itself, but also the time evolution of the observer itself. The case $A = I$ is more familiar, since $\dot{A} = 0$ we have,

$$v_{\mathcal{Y}} = Iv_{\mathcal{X}} + \dot{r} = v_{\mathcal{X}} + \dot{r}. \tag{5.7}$$

The velocity according to the observer $\mathcal{Y}$ moving with velocity $\dot{r}$ relative to $\mathcal{X}$ is the sum of the velocity according to $\mathcal{X}$ and the velocity of the observer $\mathcal{Y}$. Obviously when $A \neq I$ the story is more complicated, but the case $A = I$ should be familiar from freshman mechanics.
Now calculate how the accelerations are connected,

$$\begin{aligned} a_{\mathcal{Y}}(t) &= \tfrac{d^2}{dt^2}\big[\mathcal{Y}(\gamma(t))\big] \\ &= \tfrac{d}{dt}\big[A'(t)\mathcal{X}_t(\gamma(t)) + A(t)v_{\mathcal{X}}(t) + r'(t)\big] \\ &= A''(t)\mathcal{X}_t(\gamma(t)) + A'(t)\tfrac{d}{dt}[\mathcal{X}_t(\gamma(t))] + A'(t)v_{\mathcal{X}}(t) + A(t)\tfrac{d}{dt}[v_{\mathcal{X}}(t)] + r''(t) \\ &= A''(t)\mathcal{X}_t(\gamma(t)) + 2A'(t)v_{\mathcal{X}}(t) + +A(t)a_{\mathcal{X}}(t) + r''(t) \end{aligned} \tag{5.8}$$

Therefore we relate acceleration in $\mathcal{X}$ to the acceleration in $\mathcal{Y}$ as follows,

$$\boxed{a_{\mathcal{Y}} = Aa_{\mathcal{X}} + \ddot{r} + \ddot{A}\gamma_{\mathcal{X}} + 2\dot{A}v_{\mathcal{X}}.} \tag{5.9}$$

The equation above explains many things, if you take the junior level classical mechanics course you'll see what those things are. This equation does not look like the one used in mechanics for noninertial frames, it is nevertheless the same and if you're interested I'll show you.

Example 5.1.15. ..

(Examples added at
end of Chapter)

Example 5.1.16. ..

Definition 5.1.17.

If $\gamma : I \to S$ is the trajectory of a particle then we say the particle and $\mathcal{X} : I \to \mathcal{E}$ is an observer. We say the particle is in a **state of rest** relative to the observer $\mathcal{X}$ iff $v_{\mathcal{X}} = \frac{d}{dt}[\mathcal{X}_t(\gamma(t))] = 0$. We say the particle experiences **uniform rectilinear motion** relative to the observer $\mathcal{X}$ iff $t \mapsto \mathcal{X}_t(\gamma(t))$ is a straight line in $\mathbb{R}^3$ with velocity vector some nonzero constant vector.

We now give a rigorous definition for the existence of *force*, a little later we'll say how to calculate it.

Definition 5.1.18.

A particle **experiences a force** relative to an observer $\mathcal{X}$ iff the particle is neither in a state of rest nor is it in uniform rectilinear motion relative to $\mathcal{X}$. Otherwise we say the particle experiences no force relative to $\mathcal{X}$.

Definition 5.1.19.

An observer $\mathcal{X} : I \to \mathcal{E}$ is said to be an **inertial observer** iff there exists $\mathcal{X}_o \in \mathcal{E}$, $A \in SO(3)$, $v, w \in \mathbb{R}^3$ such that $\mathcal{X}_t = A\mathcal{X}_o + tv + w$ for all $t \in I$. A particle is called a **free particle** iff it experiences no acceleration relative to an inertial observer.

Observe that a constant mapping into $\mathcal{E}$ is an inertial observer and that general inertial observers are observers which are in motion relative to a "stationary observer" but the motion is "constant velocity" motion. We will refer to a constant mapping $\mathcal{X} : I \to \mathcal{E}$ as a **stationary observer**.

Theorem 5.1.20.

If $\mathcal{X} : I \to \mathcal{E}$ and $\mathcal{Y} : I \to \mathcal{E}$ are inertial observers then there exists $A \in SO(3)$, $v, w \in \mathbb{R}^3$ such that $\mathcal{Y}_t = A\mathcal{X}_t + tv + w$ for all $t \in I$. Moreover if a particle experiences no acceleration relative to $\mathcal{X}$ then it experiences no acceleration relative to $\mathcal{Y}$.

Proof: Since $\mathcal{X}$ and $\mathcal{Y}$ are inertial we have that there exist $\mathcal{X}_o$ and $\mathcal{Y}_o$ in $\mathcal{E}$ and fixed vectors $v_x, w_x, v_y, w_y \in \mathbb{R}^3$ and particular rotation matrices $A_x, A_y \in SO(3)$ such that

$$\mathcal{X}_t = A_x\mathcal{X}_o + tv_x + w_x \qquad\qquad \mathcal{Y}_t = A_y\mathcal{Y}_o + tv_y + w_y.$$

Further note that since $\mathcal{X}_o, \mathcal{Y}_o \in \mathcal{E}$ there exists fixed $Q \in SO(3)$ and $u \in \mathbb{R}^3$ such that $\mathcal{Y}_o = Q\mathcal{X}_o + u$. Thus, noting that $\mathcal{X}_o = A_x^{-1}(\mathcal{X}_t - tv_x - w_x)$ for the fourth line,

$$\begin{aligned} \mathcal{Y}_t &= A_y\mathcal{Y}_o + tv_y + w_y \\ &= A_y(Q\mathcal{X}_o + u) + tv_y + w_y \\ &= A_yQ\mathcal{X}_o + A_yu + tv_y + w_y \\ &= A_yQA_x^{-1}(\mathcal{X}_t - tv_x - w_x) + tv_y + A_yu + w_y \\ &= A_yQA_x^{-1}\mathcal{X}_t + t[v_y - A_yQA_x^{-1}v_x] - A_yQA_x^{-1}w_x + A_yu + w_y \end{aligned} \tag{5.10}$$

Thus define $A = A_yQA_x^{-1} \in SO(3)$, $v = v_y - A_yQA_x^{-1}v_x$, and $w = -A_yQA_x^{-1}w_x + A_yu + w_y$. Clearly $v, w \in \mathbb{R}^3$ and it is a short calculation to show that $A \in SO(3)$, we've left it as an exercise to the reader but it follows immediately if we already know that $SO(3)$ is a group under matrix multiplication (we have not proved this yet). Collecting our thoughts we have established the first half of the theorem, there exist $A \in SO(3)$ and $v, w \in \mathbb{R}^3$ such that,

$$\mathcal{Y}_t = A\mathcal{X}_t + tv + w$$

Now to complete the theorem consider a particle with trajectory $\gamma : I \to S$ such that $a_{\mathcal{X}} = 0$. Then by eqn.[5.9] we find, using our construction of A, v, w above,

$$\begin{aligned} a_{\mathcal{Y}} &= Aa_{\mathcal{X}} + \ddot{r} + \ddot{A}\gamma_{\mathcal{X}} + 2\dot{A}v_{\mathcal{X}} \\ &= A0 + 0 + 0\gamma_{\mathcal{X}} + 2(0)v_{\mathcal{X}} \\ &= 0. \end{aligned} \tag{5.11}$$

Therefore if the acceleration is zero relative to a particular inertial frame then it is zero for **all** inertial frames.

Consider that if a particle is either in a state of rest or uniform rectilinear motion then we can express it's trajectory γ relative to an observer $\mathcal{X} : I \to S$ by

$$\mathcal{X}_t(\gamma(t)) = tv + w$$

for all $t \in I$ and fixed $v, w \in \mathbb{R}^3$. In fact if $v = 0$ the particle is in a state of rest, whereas if $v \neq 0$ the particle is in a state of uniform rectilinear motion. Moreover,

$$\gamma_{\mathcal{X}}(t) = tv + w \iff v_{\mathcal{X}} = v \iff a_{\mathcal{X}} = 0.$$

Therefore we have shown that according to any inertial frame a particle that has zero acceleration necessarily travels in rectilinear motion or stays at rest.

Let us again ponder Newton's laws.

1. **Newton's First Law** Every particle persists in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by impressed forces.

2. **Newton's Second Law** The rate of change of motion is proportional to the motive force impressed; and is made in the direction of the straight line in which that force is impressed.

3. **Newton's Third Law** To every action there is an equal reaction; or the mutual actions of two bodies upon each other are always equal but oppositely directed.

It is easy to see that if the first law holds relative to one observer then it does not hold relative to another observer which is rotating relative to the first observer. So a more precise formulation of the first law would be that it holds relative to *some* observer, or some class of observers, but not relative to all observers. *We have just shown that if $\mathcal{X}$ is an inertial observer then a particle is* *either in a state of rest or uniform rectilinear motion relative to $\mathcal{X}$ iff its acceleration is zero.* If γ is the trajectory of the particle the second law says that the force F acting on the body is proportional to $m(dv_{\mathcal{X}}/dt) = ma_{\mathcal{X}}$. Thus the second law says that a body has zero acceleration iff the force acting on the body is zero (assuming $m \neq 0$). It seems to follow that the first law is a consequence of the second law. What then *does* the first law say that is *not* contained in the second law ?

The answer is that the *first law is not a mathematical axiom but a physical principle.* It says it should be *possible to physically construct, at least in principle, a set* of coordinate systems at each instant of time which may be modeled by the mathematical construct we have been calling an inertial observer. Thus the *first law can be reformulated* to read:

There exists an inertial observer

The second law is also subject to criticism. When one speaks of the force on a body what is it that one is describing? Intuitively we think of a force as something which pushes or pulls the particle off its natural course.

The truth is that a course which seems natural to one observer may not appear natural to another. One usually models forces as vectors. These vectors provide the push or pull. The components of a vector in this context are observer dependent. The second law could almost be relegated to a definition. The force on a particle at time t would be defined to be $ma_{\mathcal{X}}(t)$ relative to the observer $\mathcal{X}$. Generally physicists require that the second law hold **only** for inertial observers. One reason for this is that if $F_{\mathcal{X}}$ is the force on a particle according to an inertial observer $\mathcal{X}$ and $F_{\mathcal{Y}}$ is the force on the same particle measured relative to the inertial observer $\mathcal{Y}$ then we claim $F_{\mathcal{Y}} = AF_{\mathcal{X}}$ where $\mathcal{X}$ and $\mathcal{Y}$ are related by

$$\mathcal{Y}_t = A\mathcal{X}_t + tv + w$$

for $v, w \in \mathbb{R}^3$ and $A \in SO(3)$ and for all t. Consider a particle traveling the trajectory γ we find it's accelerations as measured by $\mathcal{X}$ and $\mathcal{Y}$ are related by,

$$a_{\mathcal{Y}} = Aa_{\mathcal{X}}$$

where we have used eqn.[5.9] for the special case that A is a fixed rotation matrix and $r = tv + w$. Multiply by the mass to obtain that $ma_{\mathcal{Y}} = A(ma_{\mathcal{X}})$ thus $F_{\mathcal{Y}} = AF_{\mathcal{X}}$. Thus the form of Newton's law is maintained under admissible transformations of observer.

Remark 5.1.21.

The invariance of the form of Newton's laws in any inertial frame is known as the Galilean relativity principle. It states that no inertial frame is preferred in the sense that the physical laws are the same no matter which inertial frame you take observations from. This claim is limited to mechanical or electrostatic forces. The force between to moving charges due to a magnetic field does not act along the straight line connecting those charges. This exception was important to Einstein conceptually. Notice that if no frame is preferred then we can never, taking observations solely within an inertial frame, deduce the velocity of that frame. Rather we only can deduce relative velocities by comparing observations from different frames.

In contrast, if one defines the force relative to one observer $\mathcal{Z}$ which is rotating relative to $\mathcal{X}$ by $F_{\mathcal{Z}} = ma_{\mathcal{Z}}$ then one obtains a much more complex relation between $F_{\mathcal{X}}$ and $F_{\mathcal{Z}}$ which involves the force on the particle due to rotation. Such forces are called *fictitious forces* as they arise from the choice of noninertial coordinates, not a genuine force.

Example 5.1.22. ..

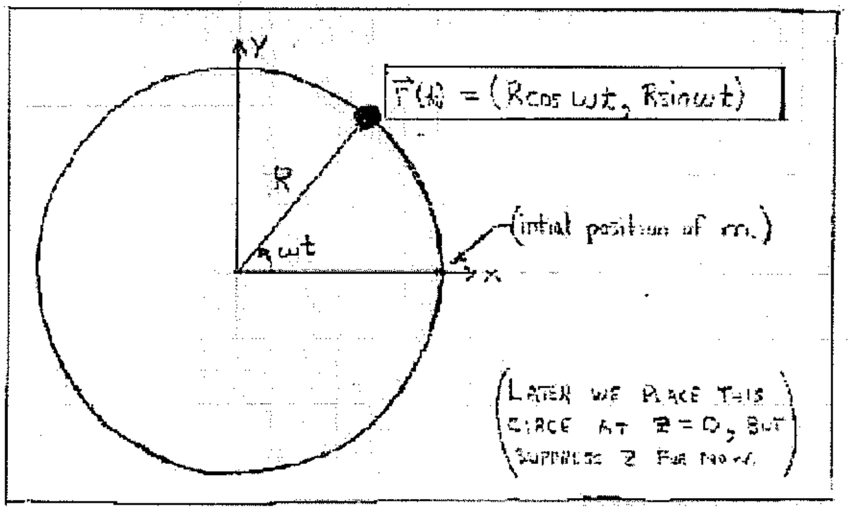
5.2 noninertial frames, a case study of circular motion

Some argue that any force proportional to mass may be viewed as a fictitious force, for example Hooke's law is F=kx, so you can see that the spring force is genuine. On the other hand gravity looks like $F = mg$ near the surface of the earth so some would argue that it is fictitious, however the conclusion of that thought takes us outside the realm of classical mechanics and the mathematics of this course. Anyway, if you are in a noninertial frame then for all intents and purposes fictitious forces are very real. The most familiar of these is probably the centrifugal force. Most introductory physics texts cast aspersion on the concept of centrifugal force (radially outward directed) because it is not a force observed from an inertial frame, rather it is a force due to noninertial motion. They say the centripetal (center seeking) force is really what maintains the motion and that there is no such thing as centrifugal force. I doubt most people are convinced by such arguments because it really feels like there is a force that wants to throw you out of a car when you take a hard turn. If there is no force then how can we feel it ? The desire of some to declare this force to be "fictional" stems from there belief that everything should be understood from the perspective of an inertial frame. Mathematically that is a convenient belief, but it certainly doesn't fit with everday experience. Ok, enough semantics. Lets examine circular motion in some depth.

For notational simplicity let us take $\mathbb{R}^3$ to be physical space and the identity mapping $\mathcal{X} = id$ to give us a stationary coordinate system on $\mathbb{R}^3$. Consider then the motion of a particle moving in a circle of radius R about the origin at a constant angular velocity of ω in the counterclockwise direction in the xy-plane. We will drop the third dimension for the most part throughout since it does not enter the calculations. If we assume that the particle begins at $(R, 0)$ at time zero then it follows that we can parametrize its path via the equations,

$$\begin{aligned} x(t) &= Rcos(\omega t) \\ y(t) &= Rsin(\omega t) \end{aligned} \tag{5.12}$$

this parametrization is geometric in nature and follows from the picture below, remember we took ω constant so that $\theta = \omega t$



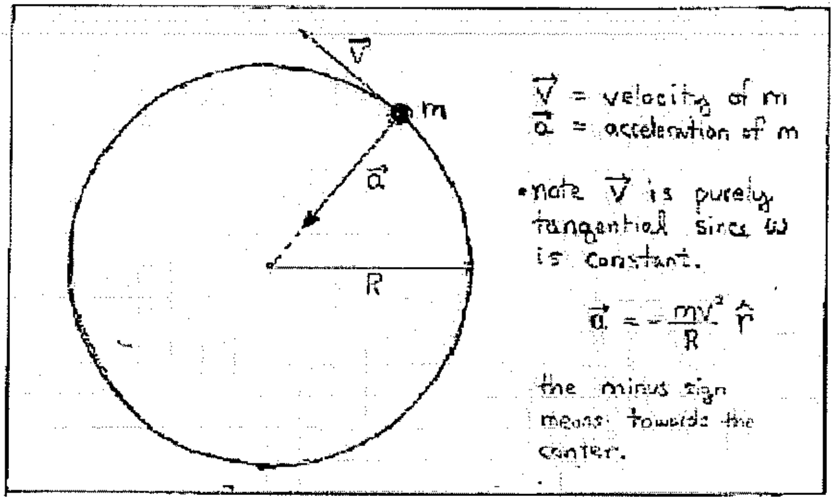
Now it is convenient to write $\vec{r}(t) = (x(t), y(t))$. Let us derive what the acceleration is for the particle, differentiate twice to obtain

$$\begin{aligned} \vec{r}''(t) &= (x''(t), y''(t)) \\ &= (-R\omega^2 cos(\omega t), -R\omega^2 sin(\omega t)) \\ &= -\omega^2 \vec{r}(t) \end{aligned}$$

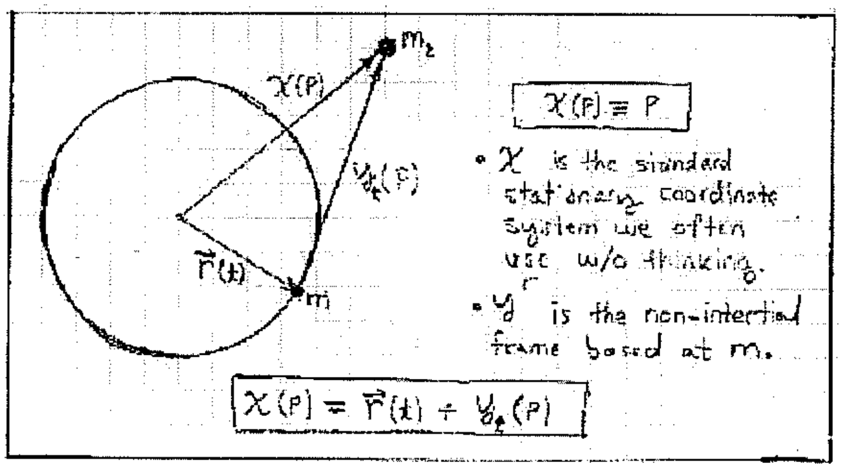
Now for pure circular motion the tangential velocity v is related to the angular velocity ω by $v = \omega R$. In other words $\omega = v/R$, radians per second is given by the length per second divided by the length of a radius. Substituting that into the last equation yields that,

$$\vec{a}(t) = \vec{r}''(t) = -\frac{v^2}{R^2} r(t) \tag{5.13}$$

The picture below summarizes our findings thus far.



Now define a second coordinate system that has its origin based at the rotating particle. We'll call this new frame $\mathcal{Y}$ whereas we have labeled the standard frame $\mathcal{X}$. Let $p \in \mathbb{R}^3$ be an arbitrary point then the following picture reveals how the descriptions of $\mathcal{X}$ and $\mathcal{Y}$ are related.



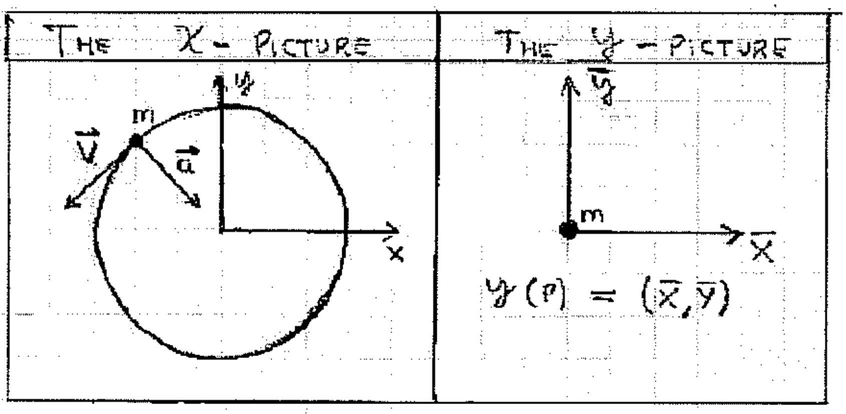
Clearly we find,

$$\mathcal{X}(p) = \mathcal{Y}(p) + \vec{r}(t) \tag{5.14}$$

note that the frames $\mathcal{X}$ and $\mathcal{Y}_t$ are not related by an rigid motion since $\vec{r}$ is not a constant function. Suppose that γ is the trajectory of a particle in $\mathbb{R}^3$, lets compare the acceleration of γ in frame $\mathcal{X}$ to that of it in $\mathcal{Y}_t$.

$$\begin{aligned} &\mathcal{X}(\gamma(t)) = \mathcal{Y}_t(\gamma(t)) + \vec{r}(t) \\ \implies\ &a_{\mathcal{X}}(t) = \gamma''(t) = a_{\mathcal{Y}_t}(t) + \vec{r}\,''(t) \end{aligned} \tag{5.15}$$

If we consider the special case of $\gamma(t) = r(t)$ we find the curious but trivial result that $\mathcal{Y}_t(r(t)) = 0$ and consequently $a_{\mathcal{Y}_t}(t) = 0$. Perhaps a picture is helpful,

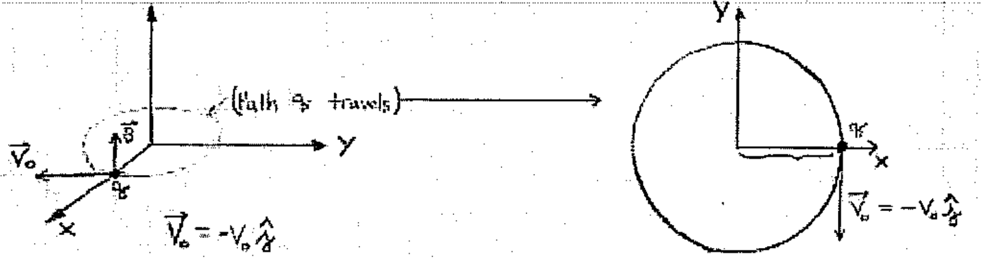


We have radically different pictures of the motion of the rotating particle, in the $\mathcal{X}$ picture the particle is accelerated and using our earlier calculation,

$$\boxed{a_{\mathcal{X}} = \vec{r}''(t) = \frac{-v^2}{R}\hat{r}}$$

on the other hand in the $\mathcal{Y}_t$ frame the mass just sits at the origin with $a_{\mathcal{Y}calt} = 0$. Since $F = ma$ we would conclude (ignoring our restriction to inertial frames for a moment) that the particle has an external force on it in the $\mathcal{X}$ frame but not in the $\mathcal{Y}$ frame. This clearly throws a wrench in the universality of the force concept, it is for this reason that we must restrict to inertial frames if we are to make nice broad sweeping statements as we have been able to in earlier sections. If we allowed noninertial frames in the basic set-up then it would be difficult to ever figure out what if any forces were in fact genuine. Dwelling on these matters actually led Einstein to his theory of general relativity where noninertial frames play a central role in the theory.

Anyway, lets think more about the circle. The relation we found in the $\mathcal{X}$ frame does not tell us *how* the particle is remaining in circular motion, rather only that if it is then it must have an acceleration which points towards the center of the circle with precisely the magnitude mv^2/R. I believe we have all worked problems based on this basic relation. An obvious question remains, which force makes the particle go in a circle? Well, we have not said enough about the particle yet to give a definitive answer to that question. In fact many forces could accomplish the task. You might imagine the particle is tethered by a string to the central point, or perhaps it is stuck in a circular contraption and the contact forces with the walls of the contraption are providing the force. A more interesting possibility for us is that the particle carries a charge and it is subject to a magnetic field in the z-direction. Further let us assume that the initial position of the charge q is $(mv/qB, 0, 0)$ and the initial velocity of the charged particle is v in the negative y-direction. I'll work this one out one paper because I can.



We assume that we have a charge q with mass m that has intial position $(mv_o/qB, 0, 0)$ with intial velocity $\vec{V}_o = (0, -V_o, 0)$ then the charge is subject to a constant background magnetic field of $\vec{B} = (0, 0, B)$ where $B > 0$. We show that the charge travels a circle of radius $R = mv_o/qB$. We begin with Newton's 2nd law paired with the Coulomb force law,

$$m\frac{d^2\vec{r}}{dt^2} = q\vec{V}\times\vec{B} = qB\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\times(0, 0, 1)$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{qB}{m}\left(\frac{dy}{dt}\hat{i} - \frac{dx}{dt}\hat{j}\right) = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

This gives us three 2nd order ODEs to solve, let $\alpha \equiv qB/m$.

$$\frac{d^2z}{dt^2} = 0 \text{ has sol}^{n}: \ z(t) = c_1 + c_2 t \ \Rightarrow \ z(t) = 0$$
(using intial conditions)

$$\left\{\begin{aligned} \frac{d^2y}{dt^2} &= -\alpha\frac{dx}{dt} \\ \frac{d^2x}{dt^2} &= \alpha\frac{dy}{dt} \end{aligned}\right\} \ \text{Coupled.}$$

Notice that $\frac{dy}{dt} = \frac{1}{\alpha}\frac{d^2x}{dt^2} \Rightarrow \frac{d^2y}{dt^2} = \frac{1}{\alpha}\frac{d^3x}{dt^3} = -\alpha\frac{dx}{dt}$.

Thus we need to solve $x''' = -\alpha^2 x'$, introduce $W \equiv x'$ then our DEq^{n} becomes $W'' = -\alpha^2 W$ this has well known sol^{n},

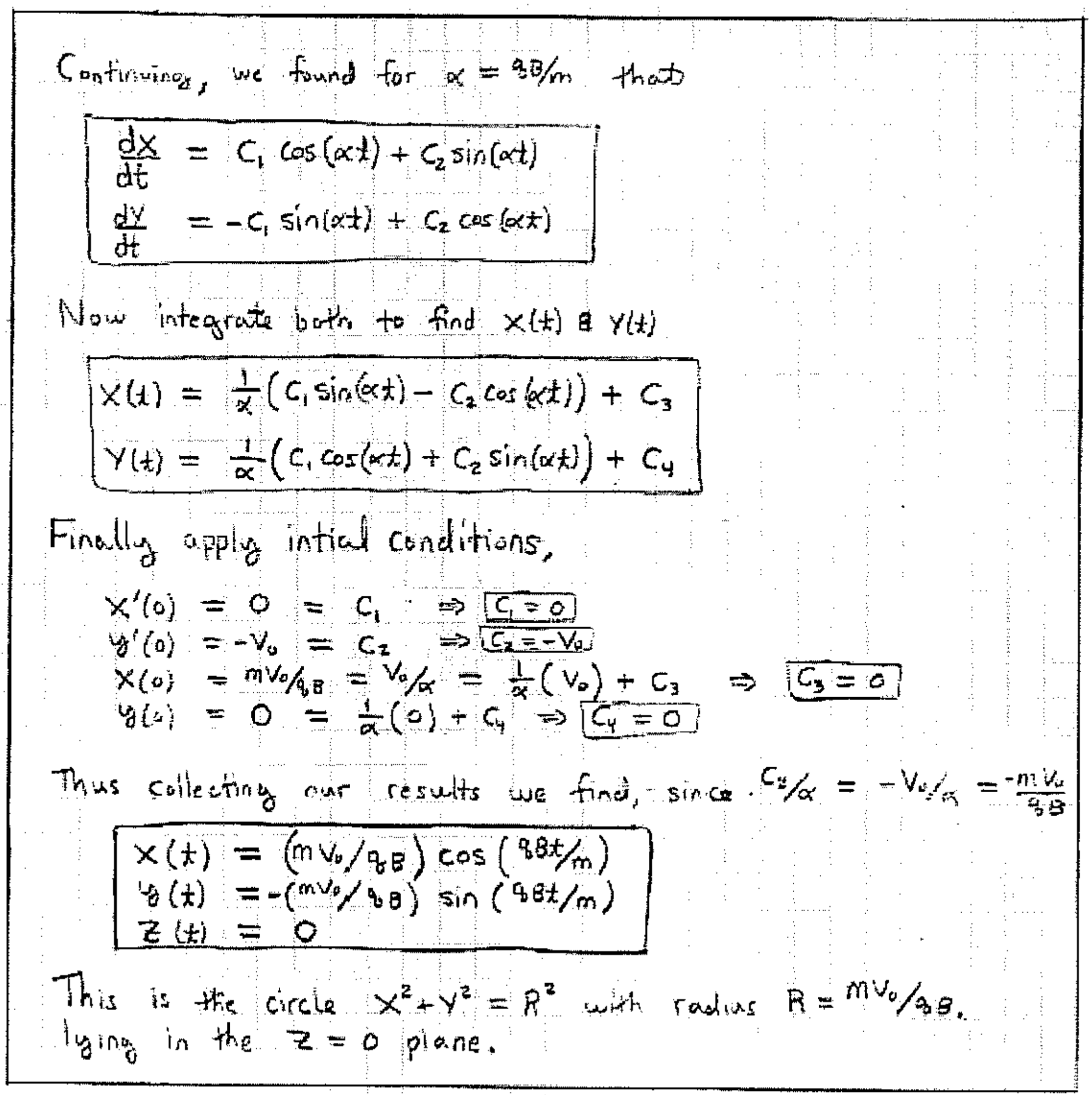
$$W = \boxed{\frac{dx}{dt} = c_1\cos(\alpha t) + c_2\sin(\alpha t)}$$

Then we can calculate everything else from this

$$\frac{dy}{dt} = \frac{1}{\alpha}\frac{d^2x}{dt^2} = \frac{1}{\alpha}\left(-c_1\alpha\sin(\alpha t) + c_2\alpha\cos(\alpha t)\right)$$

$$\Rightarrow \boxed{\frac{dy}{dt} = -c_1\sin(\alpha t) + c_2\cos(\alpha t)}$$

Continuing,



Continuing, we found for $\alpha = qB/m$ that

$$\boxed{\begin{aligned} \frac{dx}{dt} &= C_1 \cos(\alpha t) + C_2 \sin(\alpha t) \\ \frac{dy}{dt} &= -C_1 \sin(\alpha t) + C_2 \cos(\alpha t) \end{aligned}}$$

Now integrate both to find $x(t)$ & $y(t)$

$$\boxed{\begin{aligned} x(t) &= \frac{1}{\alpha}\left(C_1 \sin(\alpha t) - C_2 \cos(\alpha t)\right) + C_3 \\ y(t) &= \frac{1}{\alpha}\left(C_1 \cos(\alpha t) + C_2 \sin(\alpha t)\right) + C_4 \end{aligned}}$$

Finally apply intial conditions,

$$\begin{aligned} x'(0) &= 0 = C_1 \Rightarrow \boxed{C_1 = 0} \\ y'(0) &= -V_0 = C_2 \Rightarrow \boxed{C_2 = -V_0} \\ x(0) &= mV_0/qB = V_0/\alpha = \frac{1}{\alpha}(V_0) + C_3 \Rightarrow \boxed{C_3 = 0} \\ y(0) &= 0 = \frac{1}{\alpha}(0) + C_4 \Rightarrow \boxed{C_4 = 0} \end{aligned}$$

Thus collecting our results we find, since $C_2/\alpha = -V_0/\alpha = \frac{-mV_0}{qB}$

$$\boxed{\begin{aligned} x(t) &= (mV_0/qB)\cos(qBt/m) \\ y(t) &= -(mV_0/qB)\sin(qBt/m) \\ z(t) &= 0 \end{aligned}}$$

This is the circle $x^2 + y^2 = R^2$ with radius $R = mV_0/qB$. lying in the $z = 0$ plane.

It is curious that magnetic forces cannot be included in the Galilean relativity. For if the velocity of a charge is zero in one frame but not zero in another then does that mean that the particle has a non-zero force or no force? In the rest frame of the constant velocity charge apparently there is no magnetic force, yet in another inertially related frame where the charge is in motion there would be a magnetic force. How can this be? The problem with our thinking is we have not asked how the magnetic field transforms for one thing, but more fundamentally we will find that you cannot separate the magnetic force from the electric force. Later we'll come to a better understanding of this, there is no nice way of addressing it in Newtonian mechanics that I know of. It is an inherently relativistic problem, and Einstein attributes it as one of his motivating factors in dreaming up his special relativity.

> "What led me more or less directly to the special theory of relativity was the conviction that the electromotive force acting on a body in motion in a magnetic field was nothing else but an electric field"

Albert Einstein, 1952.

Rotating Coordinate Systems

The central idea is that different coordinate systems give descriptions of the same point,

$$\vec{r} = xe_1 + ye_2 + ze_3 = \bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3$$

The idea here is that $\bar{e}_1, \bar{e}_2, \bar{e}_3$ is a rotating frame and these coordinate systems share a common origin. Furthermore, we have in mind the coordinates x, y, z of some moving object so generally these are also functions of time. Time is time so differentiate,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[\bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3\right]$$

$$\frac{d\vec{r}}{dt} = \underbrace{\frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3}_{\text{(I)}} + \underbrace{\bar{x}\frac{d\bar{e}_1}{dt} + \bar{y}\frac{d\bar{e}_2}{dt} + \bar{z}\frac{d\bar{e}_3}{dt}}_{\text{(II)}}$$

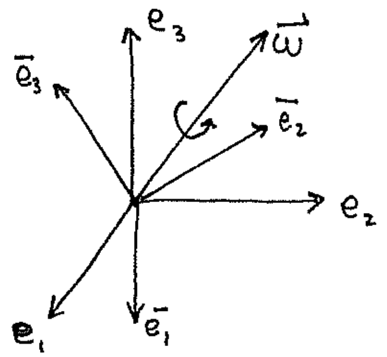
Note that in contrast the fixed, time independent frame e_1, e_2, e_3 has the simple familar form,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[xe_1 + ye_2 + ze_3\right]$$

$$= \underbrace{\frac{dx}{dt}e_1 + \frac{dy}{dt}e_2 + \frac{dz}{dt}e_3}_{\vec{V}_S}$$

$\vec{V}_S$ the velocity relative to the fixed coordinate system S

In contrast, $\vec{V}_{\bar{S}} = \frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3$ is velocity relative to the rotating coordinate system $\bar{S}$. The terms in (II) give the velocity of the $\bar{S}$-frame itself. It can be shown that $\frac{d\bar{e}_j}{dt} = \vec{\omega} \times \bar{e}_j$ for $j = 1, 2, 3$ where $\vec{\omega}$ is the angular velocity of the rotating frame



(imagine $\bar{e}_1, \bar{e}_2, \bar{e}_3$ rotate about the axis $\vec{\omega}$)

Here (II) $\longrightarrow \vec{\omega} \times (\underbrace{\bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3}_{\vec{r}})$

(using $\frac{d\bar{e}_j}{dt} = \vec{\omega} \times \bar{e}_j$ for $j = 1, 2, 3$)

Setting aside the question of why $\frac{d\bar{e}_j}{dt} = \vec{\omega}\times\bar{e}_j$, we find

$$\frac{d\vec{r}}{dt} = \frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3 + \bar{x}\frac{d\bar{e}_1}{dt} + \bar{y}\frac{d\bar{e}_2}{dt} + \bar{z}\frac{d\bar{e}_3}{dt}$$

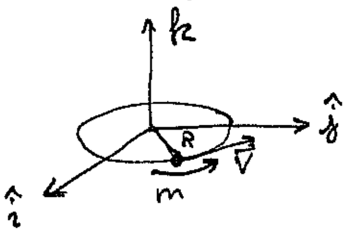
$$= \frac{d\bar{x}_1}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3 + \bar{x}(\vec{\omega}\times\bar{e}_1) + \bar{y}(\vec{\omega}\times\bar{e}_2) + \bar{z}(\vec{\omega}\times\bar{e}_3)$$

$$= \underbrace{\frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3}_{\vec{V}_{\bar{S}}} + \vec{\omega}\times(\underbrace{\bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3}_{\vec{r}})$$

We find $\boxed{\vec{V}_S = \vec{V}_{\bar{S}} + \vec{\omega}\times\vec{r}}$

This assumed a common origin.

Example:



Let $\bar{S}$ rotate at constant velocity $\vec{\omega}$ about the $\hat{k}$ vector Consider point m fixed at point in $\bar{S}$ frame; $\vec{r}(t) = \langle \underbrace{R\cos t}, R\sin t, 0\rangle$

oops. I just set $\omega = 1$ see ✱

$$\vec{V}_S = \omega\hat{k}\times(R\cos t\,\hat{i} + R\sin t\,\hat{j})$$
$$= \omega R\cos t(\hat{k}\times\hat{i}) + \omega R\sin t(\hat{k}\times\hat{j})$$
$$= (\omega R\cos t)\hat{j} - (\omega R\sin t)\hat{i}$$
$$= \underbrace{\omega R(\langle -\sin t, \cos t\rangle)}_{}$$

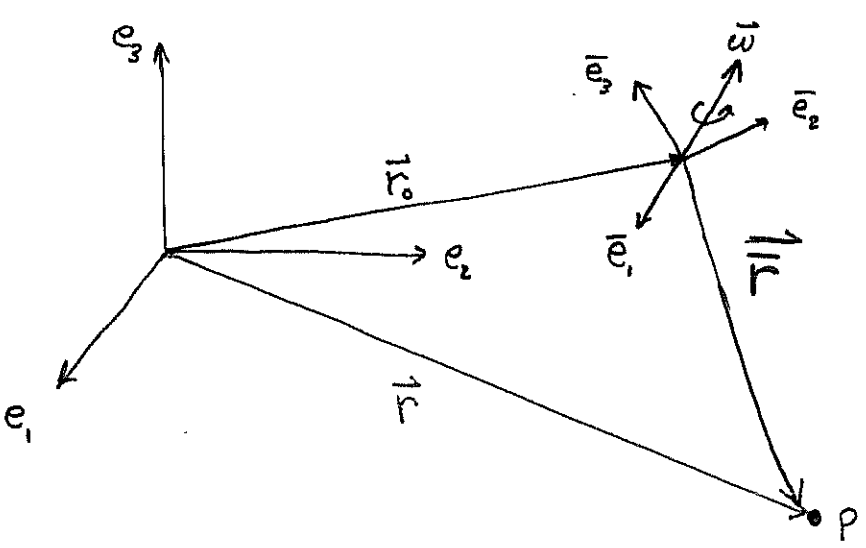
the tangential velocity, $\frac{d\vec{r}}{dt} = \langle -R\cos t, R\sin t\rangle$

I have shown that $\vec{V} = \vec{\omega}\times\vec{r}$ is true in this special case.

✱ $\vec{r}(t) = \langle R\cos\omega t, R\sin\omega t, 0\rangle$

$$\frac{d\vec{r}}{dt} = R\omega\langle -\sin\omega t, \cos\omega t, 0\rangle = \omega\hat{k}\times\vec{r}$$

(I forgot the ω above)



rotating coordinate system $\bar{S}$ with non-matching origin relative to fixed inertial frame $S:\{e_1, e_2, e_3\}$.

$$\vec{r} = x e_1 + y e_2 + z e_3$$

$$\vec{\vec{r}} = \bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3$$

From picture we see that the variable point P has

$$\vec{r} = \vec{r}_0 + \vec{\vec{r}}$$

Almost same calculation goes through,

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}_0}{dt} + \frac{d}{dt}\left(\bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3\right)$$

$$= \frac{d\vec{r}_0}{dt} + \underbrace{\frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3}_{} + \bar{x}\frac{d\bar{e}_1}{dt} + \bar{y}\frac{d\bar{e}_2}{dt} + \bar{z}\frac{d\bar{e}_3}{dt}$$

$$\boxed{\vec{V}_S = \frac{d\vec{r}_0}{dt} + \vec{V}_{\bar{S}} + \vec{\omega} \times \vec{\vec{r}}}$$

This formula relates the velocity measured relative to a fixed vs. rotating frame. The first two terms should be familar from freshman mechanics.

$\frac{d\vec{r}_0}{dt}$ = velocity of moving frame's origin

$\vec{V}_{\bar{S}}$ = velocity relative to moving frame

However, the $\vec{\omega} \times \vec{\vec{r}}$ is probably new to you in this context.

Comparing acceleration in fixed frame S verses rotating $\bar S$.

Continuing with the notation from the previous page,

$$\frac{d\vec r}{dt} = \vec V_s = \frac{d\vec r_o}{dt} + \frac{d\bar x}{dt}\bar e_1 + \frac{d\bar y}{dt}\bar e_2 + \frac{d\bar z}{dt}\bar e_3 + \vec\omega \times \vec{\bar r}$$

Now differentriate again, note this calculation is very much like the one we just completed,

$$\frac{d^2\vec r}{dt^2} = \frac{d^2\vec r_o}{dt^2} + \frac{d^2\bar x}{dt^2}\bar e_1 + \frac{d^2\bar y}{dt^2}\bar e_2 + \frac{d^2\bar z}{dt^2}\bar e_3 + \frac{d\bar x}{dt}\frac{d\bar e_1}{dt} + \frac{d\bar y}{dt}\frac{d\bar e_2}{dt} + \frac{d\bar z}{dt}\frac{d\bar e_3}{dt} + \frac{d}{dt}\left(\vec\omega \times \vec{\bar r}\right)$$

$$= \vec a_o + \vec a_{\bar s} + \frac{d\bar x}{dt}(\vec\omega \times \bar e_1) + \frac{d\bar y}{dt}(\vec\omega \times \bar e_2) + \frac{d\bar z}{dt}(\vec\omega \times \bar e_3) + \frac{d\vec\omega}{dt} \times \vec{\bar r} + \vec\omega \times \frac{d\vec{\bar r}}{dt}$$

$$= \vec a_o + \vec a_{\bar s} + \vec\omega \times \left(\frac{d\bar x}{dt}\bar e_1 + \frac{d\bar y}{dt}\bar e_2 + \frac{d\bar z}{dt}\bar e_3\right) + \frac{d\vec\omega}{dt} \times \vec{\bar r} + \vec\omega \times \frac{d\vec{\bar r}}{dt}$$

$$\boxed{\vec a_s = \vec a_o + \vec a_{\bar s} + 2\vec\omega \times \vec V_{\bar s} + \frac{d\vec\omega}{dt} \times \vec{\bar r} + \vec\omega \times (\vec\omega \times \vec{\bar r})}$$

The notation $\vec a_o = \frac{d^2\vec r_o}{dt^2}$ gives acceleration of origin of $\bar S$ relative to S.

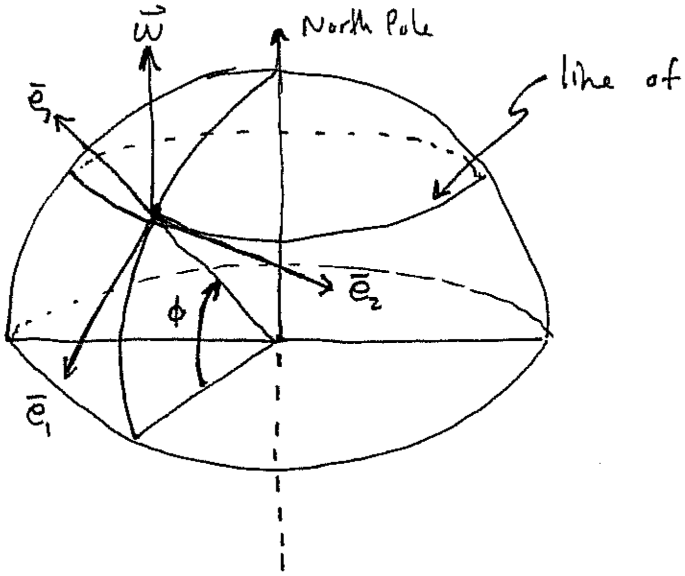
<u>Thought Experiment</u>: Suppose you took measurements relative to $\bar S$ and *assumed* it was an intertial frame. What "fake" forces would you encounter? Assuming Newton's 2nd Law,

$$m\vec a_{\bar s} = m\vec a_s - m\vec a_o - 2m\vec\omega \times \vec V_{\bar s} - m\frac{d\vec\omega}{dt} \times \vec{\bar r} - m\vec\omega \times (\vec\omega \times \vec{\bar r})$$

$$\boxed{m\vec a_{\bar s} = \underset{\substack{\uparrow \\ \text{real forces} \\ \text{like springs} \\ \text{or gravity}}}{\vec F_{net}} - \underset{\substack{\uparrow \\ \text{rectilinear} \\ \text{acceleration} \\ \text{of frame } \bar S}}{m\vec a_o} - \underset{\substack{\uparrow \\ \text{Coriolis} \\ \text{Force}}}{2m\vec\omega \times \vec V_{\bar s}} - \underset{\substack{\uparrow \\ \text{no-name} \\ \text{but you've} \\ \text{felt this} \\ \text{on a} \\ \text{tilt-a-whirl} \\ \text{as it stops} \\ \text{or starts}}}{m\frac{d\vec\omega}{dt} \times \vec{\bar r}} - \underset{\substack{\uparrow \\ \text{centrifugal} \\ \text{force.}}}{m\vec\omega \times (\vec\omega \times \vec{\bar r})}}$$

Rotating Frame of Reference we live in

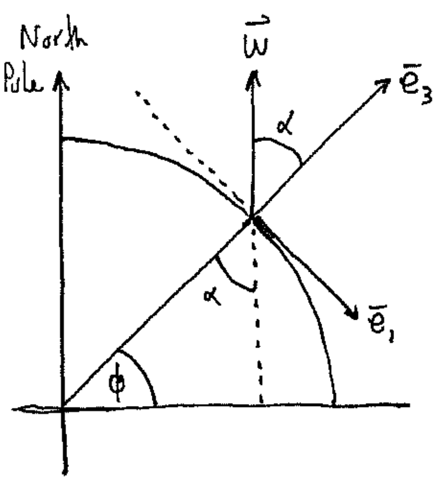
We've done almost all the math. Think about it, the earth is essentially moving at constant velocity relative to solar system over a time of <u>minutes</u>. So we can reasonable put a <u>fixed</u> coordinate system S at the center of the earth. Moreover, modulo earthquakes & tidal waves, the rotation of the earth is nearly constant in magnitude. This means the $\frac{d\vec{w}}{dt}$ term vanishes. Let's set up a rotating frame of reference (borrowed from McComb's "Dynamics and Relativity" Oxford Press.)



$\bar{e}_1$ points due South

$\bar{e}_2$ points due East

$\bar{e}_3$ points straight up



Continuing, we find the following Deqⁿ of motion near surface of earth at Latitude ϕ, either ignore or lump $\vec{\omega}\times(\vec{\omega}\times\vec{r})$ into the $-mg\bar{e}_3$ term,

$$m\left(\frac{d^2\bar{x}}{dt^2}\bar{e}_1+\frac{d^2\bar{y}}{dt^2}\bar{e}_2+\frac{d^2\bar{z}}{dt^2}\bar{e}_3\right)=\underset{\substack{\uparrow\\ \text{gravity}}}{-mg\bar{e}_3}-\underset{\substack{\uparrow\\ \text{Coriolis Force}}}{2m\vec{\omega}\times\vec{V}_{\bar{s}}}$$

Work out the Coriolis term,

$$\vec{\omega}=\omega e_3=\omega(e_3\cdot\bar{e}_1)\bar{e}_1+\omega\underset{0}{\cancel{(e_3\cdot\bar{e}_2)}}\bar{e}_2+\omega(e_3\cdot\bar{e}_3)\bar{e}_3$$

$$\Rightarrow \underbrace{\vec{\omega}=\omega\cos\left(\alpha+\tfrac{\pi}{2}\right)\bar{e}_1+\omega\cos\alpha\,\bar{e}_3}$$

See picture on previous page it's clear the angles are $\alpha+\frac{\pi}{2}$ and α between e_3 & $\bar{e}_1$ & e_3 & $\bar{e}_3$ respective.

However, $\alpha=\frac{\pi}{2}-\phi$ thus

$$\cos\left(\alpha+\tfrac{\pi}{2}\right)=\cos(\pi-\phi)=-\cos\phi$$

$$\cos\alpha=\cos\left(\tfrac{\pi}{2}-\phi\right)=+\sin\phi$$

We find that $\underline{\vec{\omega}=-\omega\cos\phi\,\bar{e}_1+\omega\sin\phi\,\bar{e}_3}$.

I did this so we can use $\bar{e}_1\times\bar{e}_2=\bar{e}_3$ etc... in the following,

$$\begin{aligned}\vec{\omega}\times\vec{V}_{\bar{s}}&=\left(-\omega\cos\phi\,\bar{e}_1+\omega\sin\phi\,\bar{e}_3\right)\times\left(\frac{d\bar{x}}{dt}\bar{e}_1+\frac{d\bar{y}}{dt}\bar{e}_2+\frac{d\bar{z}}{dt}\bar{e}_3\right)\\&=-\omega\cos\phi\frac{d\bar{y}}{dt}\bar{e}_3+\omega\cos\phi\frac{d\bar{z}}{dt}\bar{e}_2+\omega\sin\phi\frac{d\bar{x}}{dt}\bar{e}_2-\omega\sin\phi\frac{d\bar{y}}{dt}\bar{e}_1\\&=\left(-\omega\sin\phi\frac{d\bar{y}}{dt}\right)\bar{e}_1+\left(\omega\cos\phi\frac{d\bar{z}}{dt}+\omega\sin\phi\frac{d\bar{x}}{dt}\right)\bar{e}_2-\omega\cos\phi\frac{d\bar{y}}{dt}\bar{e}_3\end{aligned}$$

Putting this together with Newton's Law,

$$\begin{matrix}\bar{e}_1:\\[2em] \bar{e}_2:\\[2em] \bar{e}_3:\end{matrix}\;\boxed{\begin{aligned}m\frac{d^2\bar{x}}{dt^2}&=2m\omega\sin\phi\frac{d\bar{y}}{dt}\\ m\frac{d^2\bar{y}}{dt^2}&=-2m\omega\cos\phi\frac{d\bar{z}}{dt}-2m\omega\sin\phi\frac{d\bar{x}}{dt}\\ m\frac{d^2\bar{z}}{dt^2}&=2m\omega\cos\phi\frac{d\bar{y}}{dt}-mg\end{aligned}}$$

<u>Remark</u>: since $m,\omega,\cos\phi$ are all constants for a given problem <u>we</u> <u>can</u> solve this by reduction of order to a 6×6 nonhomog. <u>matrix</u> problem!

Approximate Sol^{n} for Coriolis Problem

Compared to mg the terms with $2m\omega$ are proportionally smaller. If we consider throwing an object vertically it stands to reason only $\frac{d\bar{z}}{dt}$ is nontrivial, the Coriolis force will create some nonzero $\frac{d\bar{x}}{dt}, \frac{d\bar{y}}{dt}$ as time progresses, but those terms are small. Hence we can solve:

$$m\frac{d^2\bar{x}}{dt^2} = 0$$

$$m\frac{d^2\bar{y}}{dt^2} = -2m\omega\cos\phi\frac{d\bar{z}}{dt}$$

$$m\frac{d^2\bar{z}}{dt^2} = -mg$$

We may simply integrate the $\bar{x}$ & $\bar{z}$ eq^{n}'s to find

$$\boxed{\begin{aligned}\bar{x}(t) &= x_0 \\ \bar{z}(t) &= z_0 - t - \tfrac{1}{2}gt^2\end{aligned}}$$

Same as usual. The interesting feature is the Eastward drift captured in the $\bar{y}$ - eq^{n}.

Hence $\frac{d\bar{z}}{dt} = \bar{V}_0 - gt$. This gives,

$$m\frac{d^2\bar{y}}{dt^2} = -2m\omega\cos\phi\left[\bar{V}_0 - gt\right]$$

$$\Rightarrow \frac{d^2\bar{y}}{dt^2} = -2\omega\cos\phi\,\bar{V}_0 + (2\omega\cos\phi\, g)t$$

$$\Rightarrow \frac{d\bar{y}}{dt} = -2\omega\cos\phi\,\bar{V}_0 t + \omega\cos\phi\, g t^2 \qquad \left(\text{assumed } \frac{d\bar{y}}{dt}(0) = 0.\right)$$

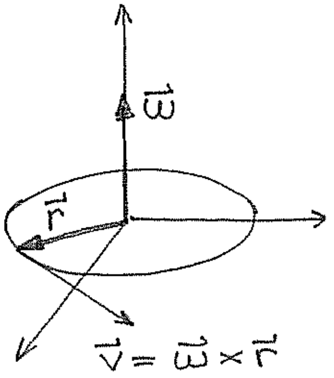
$$\Rightarrow \boxed{\bar{y}(t) = \left(\frac{1}{3}g\omega\cos\phi\right)t^3 - (\omega\cos\phi\,\bar{V}_0)t}$$

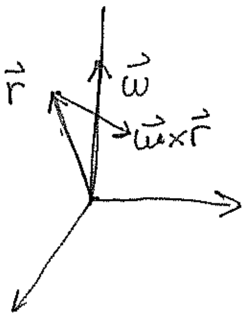
Or, if we drop a mass so $\bar{V}_0 = 0$ we have

$$\underbrace{\boxed{\bar{y}(t) = \frac{1}{3}(g\omega\cos\phi)t^3}}_{\substack{\text{Coriolis drift goes east in} \\ \text{Northern Hemisphere.}}}$$

(Notice the $-2m\vec{\omega}\times\vec{V}_{\bar{s}}$ points opposite direction below equator.)

Concerning <u>why</u> $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$ (I used $\frac{de_j}{dt} = \vec{\omega} \times e_j$ before)





$A_{\vec{\omega}}$ rotates at <u>constant</u> angular velocity ω about $\frac{\vec{\omega}}{\omega} = \hat{n}$

$$A_{\vec{\omega}}(t) = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{r}(t) = A_{\vec{\omega}}(t)\,\vec{r}_0 \rightsquigarrow \vec{r}(t) = [\cos\omega t\, x_0 - \sin\omega t\, y_0, \sin\omega t\, x_0 + \cos\omega t\, y_0, z$$

$$\frac{d\vec{r}}{dt} = \frac{\partial A}{dt}\vec{r}_0 = \begin{bmatrix} -\omega\sin\omega t & -\omega\cos\omega t & 0 \\ \omega\cos\omega t & -\omega\sin\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$= \begin{bmatrix} -\omega x_0 \sin\omega t - \omega y_0 \cos\omega t \\ \omega x_0 \cos\omega t - \omega y_0 \sin\omega t \\ 0 \end{bmatrix}$$

$$= \underline{\omega[-x_0\sin\omega t - y_0\cos\omega t, x_0\cos\omega t - y_0\sin\omega t, 0]}$$

$$\vec{\omega} = \omega[0,0,1]^T \rightsquigarrow \vec{\omega} \times \vec{r} = \omega\hat{k} \times (x\hat{i} + y\hat{j} + z_0\hat{k})$$

$$= \omega x \hat{j} - \omega y \hat{i}$$

$$= \underline{[-\omega x_0 \sin\omega t - \omega y_0 \cos\omega t, \omega x_0 \cos\omega t - \omega y_0 \sin\omega t, 0]}$$

$$\therefore \vec{v} = \vec{\omega} \times \vec{r}$$

<u>Remark</u>: this proof is almost general but it needs a little work....

(Just for fun, this is unfinished)

$$A^T A = I$$

$$A = e^{Bt}, \quad \frac{dA}{dt} = Be^{Bt}$$

$$\frac{dA^T}{dt} A + A^T \frac{dA}{dt} = 0$$

Assume $\gamma(t) = e^{Bt}$ then $\gamma(0) = I$ & $\gamma'(0) = B$.

If $\gamma^T \gamma = I$ then $\frac{d\gamma^T}{dt}(t)\, \gamma(t) + \gamma^T(t) \frac{d\gamma}{dt}(t) = 0$

$$\therefore \quad B^T + B = 0 \longrightarrow B^T = -B.$$

$$\Rightarrow \quad B = \begin{bmatrix} 0 & b_3 & b_2 \\ -b_3 & 0 & b_1 \\ -b_2 & -b_1 & 0 \end{bmatrix} = b_3 \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\epsilon_{ij3}} + b_2 \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{\epsilon_{ij2}} + b_1 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\epsilon_{ij1}}$$

Let $(J_k)_{ij} = \epsilon_{ijk}$.

<u>Claim</u>: $R_{\vec{\omega}} = \exp(\vec{J} \cdot \vec{\omega})$

$$R_{\omega \hat{k}} = \exp(\omega J_3)$$

Umm....

$$\begin{aligned} \frac{dR_{\vec{\omega}}}{dt} &= \frac{d}{dt} \exp(t(\vec{J} \cdot \vec{\omega})) \\ &= \exp(t\vec{J} \cdot \vec{\omega}) \frac{d}{dt}(t(\vec{J} \cdot \vec{\omega})) \\ &= \underline{\vec{J} \cdot \vec{\omega}} \exp(t\vec{J} \cdot \vec{\omega}) \\ &= \epsilon_{ijk} \omega_k R_{\vec{\omega}} \end{aligned}$$

$$\vec{r}(t) = R_{\vec{\omega}}(t)\, \vec{r}_0$$

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{dR_{\vec{\omega}}}{dt}(t)\, \vec{r}_0 \\ &= (\omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3) R_{\vec{\omega}}(t)\, \vec{r}_0 \\ &= (\omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3)\, \vec{r}(t). \\ &= (\omega_1, \omega_2, \omega_3) \times \vec{r}(t) \end{aligned}$$

$$J_1 \vec{r} = \epsilon_{ij1} r_j = \epsilon_{231} r_3 + \epsilon_{321} r_2 = r_3 - r_2 = z - y.$$