Name:

MATH 332-001, APRIL, 2010,	MIDTERM

Do not omit scratch work. I need to see all steps. Skipping details will result in a loss of credit. Closed book, open mind, one page front and back of notes allowed. Any calculator also allowed, no cell phones or laptops. Thanks and enjoy. You have 2 hours to attempt this exam. There are 3000pts required points to earn on this exam.

Problem 1 [300pts] Calculate or answer question (may be multiple answers, you just give a correct one)

10:37

(a.) Find f' if $f(x,y) = xy$, $f: \mathbb{R}^2 \to \mathbb{R}$

$$f'(x,y) = [y, x].$$

(b.) Find g' if $g(t) = (t, t^2)$, $g: \mathbb{R} \to \mathbb{R}^2$

$$g'(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix}.$$

(c.) Is $f(x,y,z) = (x+2y, y+z, 3x+z)$ a linear mapping?

$$f(x,y,z) = \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \therefore \quad f = L_A$$

$\therefore$ f linear.

(d.) How is the angle between two vectors in $\mathbb{R}^n$ defined?

For $\vec{v}, \vec{w}$ nonzero, $\theta = \cos^{-1}\left(\dfrac{\vec{v} \cdot \vec{w}}{||\vec{v}|| \, ||\vec{w}||} \right)$

(e.) Show $f(x,y) = (2y+1, 3x-2)$ is onto $\mathbb{R}^2$

Note $a = 2y+1$ & $b = 3x-2$ has $y = \frac{a-1}{2}$ & $x = \frac{b+2}{3}$.

Let $(a,b) \in \mathbb{R}^2$ note $\left(\frac{b+2}{3}, \frac{a-1}{2}\right) \in dom(f)$ and

$$f\left(\frac{b+2}{3}, \frac{a-1}{2}\right) = \left(2\left(\frac{a-1}{2}\right)+1, 3\left(\frac{b+2}{3}\right)-2\right)$$
$$= (a-1+1, b+2-2) = (a,b)$$

$\therefore$ f onto $\mathbb{R}^2$.

10:44

Problem 2 [300pts] Use the implicit function (or mapping) or inverse function (or mapping) theorem to answer the questions below. Make sure to verify the interesting hypothesis for the theorem that you use, I will provide grace for the fine print,

(a.) Can $x^2 + y^2 + z^2 = 6$ be solved for z as a function of (x, y) near $(1, 1, 2)$?

$$G(x,y,z) = x^2+y^2+z^2-6$$

Note $\frac{\partial G}{\partial z} = 2z \quad \therefore \quad G_z(1,1,2) = 4 \neq 0$

and since $G(1,1,2) = 1+1+4-6 = 0$ it follows $\exists$ an sol$^{\underline{n}}$ $z = h(x,y)$ of $G = 0$ near $(1,1,2)$.

(b.) Can $x^2 + y^2 + z^2 = 6$ be solved for z as a function of (x, y) ~~near~~ in an open nbhd of $(1, 1, 1)$?

Again $G_z = 2z$ and $G_z(1,1,1) = 2 \neq 0$

However, $G(1,1,1) = 1+1+1-6 = -3 \neq 0$

So $\nexists$ a sol$^{\underline{n}}$ through $(1,1,1)$ since $(1,1,1)$ is not a sol$^{\underline{n}}$.

(c.) Does the implicit function theorem yield that $x^2 + y^2 = 4$ be solved for y as a function of x ~~near~~ in an open nbhd. $(2, 0)$?

$$G(x,y) = x^2+y^2-4 = 0 \iff x^2+y^2 = 4$$

Note $\frac{\partial G}{\partial y} = 2y$ and $G_y(2,0) = 0 \quad \therefore$

$x^2+y^2 = 4$ cannot be solved in open nbhd. of $(2,0)$.

(d.) Where is the mapping $\cancel{F(x,y) = (x+y, e^{(x-1)^2} + e^{(y-1)^2})}$ locally invertible?

$$F(x,y) = \left(\exp(x-1)^2 , \; x + \exp(y-1)^2 \right)$$

$$F'(x,y) = \begin{bmatrix} 2(x-1)e^{(x-1)^2} & 0 \\ 1 & 2(y-1)e^{(y-1)^2} \end{bmatrix}$$

$$\det(F'(x,y)) = 4(x-1)(y-1)e^{(x-1)^2+(y-1)^2}$$

Thus F is locally invertible at each point in $\mathbb{R}^2 - \{(1,2)\}$.

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Problem 3 [300pts] Suppose $\gamma(t) = (\cos(t), \sin(t), t)$ for $t \geq 0$. Calculate T, N and B for the given curve. Also, find the arclength as a function of t.

$$\gamma'(t) = (-\sin t, \cos t, 1) \Rightarrow \underbrace{\|\gamma'(t)\|}_{} = \sqrt{2}.$$

$$s(t) = \int_0^t \sqrt{2}\, du = \boxed{\sqrt{2}\, t = s}$$

Moreover,

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \boxed{\frac{1}{\sqrt{2}}(-\sin t, \cos t, 1) = T(t)}$$

$$T'(t) = \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0)$$

$$\therefore \boxed{N(t) = (-\cos t, -\sin t, 0)}$$

$$B = T \times N$$

$$= \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle \times \langle -\cos t, -\sin t, 0 \rangle$$

$$= \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, \sin^2 t + \cos^2 t \rangle$$

$$= \boxed{\frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle = B(t)}$$

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Problem 4 [300pts] Given that $xyz + \cos(x^2 + z^2) = w$ calculate

$$\left(\frac{\partial w}{\partial y}\right)_{x,z} \quad \text{and} \quad \left(\frac{\partial y}{\partial w}\right)_{x,z}.$$

$$dw = yz\,dx + xz\,dy + xy\,dz - \sin(x^2+z^2)(2x\,dx + 2z\,dz)$$

$$= \underbrace{(yz - 2x\sin(x^2+z^2))}_{\left(\frac{\partial w}{\partial y}\right)_{x,z}}dx + \underbrace{xz}_{\left(\frac{\partial w}{\partial y}\right)_{x,z}}dy + (xy - 2z\sin(x^2+z^2))dz$$

$$\Rightarrow \underline{\left(\frac{\partial w}{\partial y}\right)_{x,z} = xz}, \quad \& \quad \underline{\left(\frac{\partial y}{\partial w}\right)_{x,z} = \frac{1}{yz - 2x\sin(x^2+z^2)}}.$$

Problem 5 [300pts] Suppose that

$$x^2 + y + z + e^w = 2 \quad \text{and} \quad x + y^2 + z + w^3 = 0.$$

Calculate $\left(\frac{\partial w}{\partial x}\right)_z$. State which variables are dependent and which are independent at the start of your calculation.

Use w, y dependent on independent x, z

$$2x\,dx + dy + dz + e^w dw = 0$$

$$dx + 2y\,dy + dz + 3w^2 dw = 0$$

$$dy + e^w dw = -2x\,dx - dz$$

$$2y\,dy + 3w^2 dw = -dx - dz$$

$$\begin{bmatrix} 1 & e^w \\ 2y & 3w^2 \end{bmatrix}\begin{bmatrix} dy \\ dw \end{bmatrix} = \begin{bmatrix} -2x\,dx - dz \\ -dx - dz \end{bmatrix}$$

$$dw = \frac{\det\begin{bmatrix} 1 & -2x\,dx - dz \\ 2y & -dx - dz \end{bmatrix}}{3w^2 - 2ye^w}$$

$$= -dx - dz + 2y(2x\,dx + dz)$$

$$= (4xy - 1)\,dx + (2y-1)\,dz \quad \therefore \boxed{\left(\frac{\partial w}{\partial x}\right)_z = 4xy - 1}$$

~~11:05~~
11:09

Problem 6 [300pts] Find point(s) on the circle $(x-1)^2+(y-1)^2=1$ which are nearest the line $x+y=8$. Use the method of Lagrange multipliers..

Let (x,y) be pt. on circle and (u,v) be point on line then $(x-1)^2+(y-1)^2-1=0$ & $u+v-8=0$ can be encoded by $G(x,y,u,v) = ((x-1)^2+(y-1)^2-1,\ u+v-8)=0$.

We wish to locate extrema of

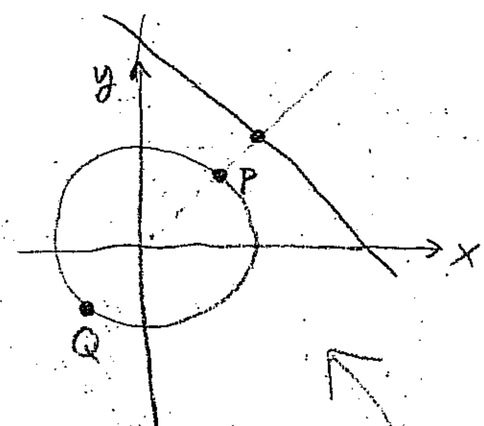
$$f(x,y,u,v) = (x-u)^2+(y-v)^2$$

Method of Lagrange,

$$\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2$$

$$\langle 2(x-u), 2(y-v), -2(x-u), -2(y-v)\rangle = \lambda_1\langle 2(x-1), 2(y-1), 0, 0\rangle + \lambda_2\langle 0,0,1,1\rangle$$

Hence we find 4 conditions,



$$2(x-u) = 2\lambda_1(x-1)$$
$$2(y-v) = 2\lambda_1(y-1)$$
$$-2(x-u) = \lambda_2$$
$$-2(y-v) = \lambda_2$$

~~11:13~~
11:27
11:33

$$x-u = y-v$$

$$\frac{x-u}{y-v} = \frac{x-1}{y-1} \Rightarrow 1 = \frac{x-1}{y-1} \Rightarrow \underline{x=y}.$$

Hence $(x-1)^2+(y-1)^2=1 \Rightarrow 2(x-1)^2=1$
$\Rightarrow 2(x^2-2x+1)=1$
$\Rightarrow 2x^2-4x+1=0$

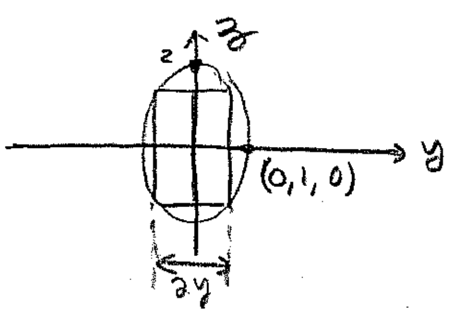
$$x = \frac{4\pm\sqrt{16-8}}{4} = 1 \pm \frac{\sqrt{8}}{4} = 1\pm\frac{1}{2}\sqrt{2}$$

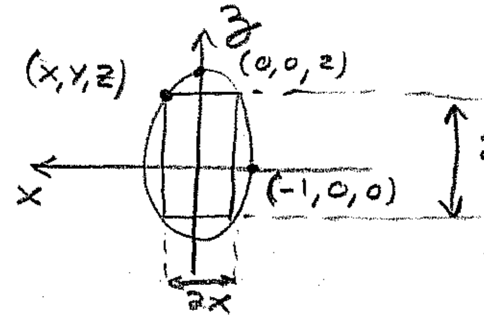
(+) with P
(−) with Q

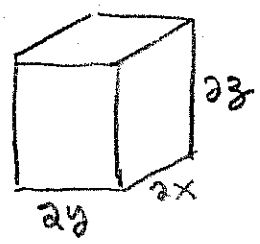
$\therefore\ (x,y) = (1+\sqrt{2}/2,\ 1+\sqrt{2}/2)$ is closest point.

Problem 7 [300pts] Find the dimensions of the largest box which can be inscribed in the ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 1$.

11:41 - 11:48







$$\underbrace{V(x,y,z) = 8xyz}_{\text{objective function.}}$$

$$\underbrace{G(x,y,z) = x^2 + y^2 - z^2/4 - 1 = 0}_{\text{constraint.}}$$

Lagrange's Method

$$\nabla V = \lambda \nabla G$$

$$\langle 8yz,\ 8xz,\ 8xy \rangle = \lambda \langle 2x,\ 2y,\ \tfrac{1}{2} z \rangle$$

$$\begin{aligned} 8yz &= 2\lambda x \\ 8xz &= 2\lambda y \\ 8xy &= \tfrac{1}{2}\lambda z \end{aligned}$$

$$\frac{y}{x} = \frac{x}{y} \quad \therefore \ \underline{x^2 = y^2} \ \text{(I)}$$

$$\frac{z}{y} = \frac{4y}{z} \quad \therefore \ \underline{z^2 = 4y^2} \ \text{(II)}$$

Hence subst. I & II into $G = 0$ yields

$$y^2 + y^2 - \frac{1}{4}(4y^2) = 1 \quad \therefore \ 3y^2 = 1 \Rightarrow y = \pm 1/\sqrt{3}$$

But, we defined x, y, z as lengths hence

$$x = \frac{1}{\sqrt{3}}, \quad y = \frac{1}{\sqrt{3}}, \quad z = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

Problem 8 [300pts] Suppose $f(x,y) = xy$ find the multivariate Taylor series for f centered at $(2,3)$.

11:34
11:40

$$f(x,y) = xy$$
$$= [(x-2)+2][(y-3)+3]$$
$$= (x-2)(y-3) + 3(x-2) + 2(y-3) + 6$$
$$= \underline{6 + 3(x-2) + 2(y-3) + (x-2)(y-3)}.$$

This is also derivable from

$$f(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-1) +$$
$$+ \frac{1}{2}\left(f_{xx}(1,2)(x-1)^2 + 2f_{xy}(1,2)(x-1)(y-2) + f_{yy}(1,2)(y-2)\right.$$

Either approach was acceptable.
(I'm sure $\exists$ other methods)

Problem 9 [300pts] Suppose $g(t) = (t^2, \cos(t))$ and $F(x,y) = (xy, x^2, y^2)$. Calculate $(F \circ g)'(t)$.

$$(F\circ G)'(t) = F'(g(t))\, g'(t) = \begin{bmatrix} y & x \\ 2x & 0 \\ 0 & 2y \end{bmatrix} \begin{bmatrix} 2t \\ -\sin t \end{bmatrix}$$

where $x = t^2$ & $y = \cos t$

$$= \begin{bmatrix} \cos t & t^2 \\ 2t^2 & 0 \\ 0 & 2\cos t \end{bmatrix} \begin{bmatrix} 2t \\ -\sin t \end{bmatrix}$$

$(3\times 2)(2\times 1)$

$$= \underline{\begin{bmatrix} 2t\cos t - t^2 \sin t \\ 4t^3 \\ -2\sin t \cos t \end{bmatrix}}.$$

Problem 10 [300pts] **Prove** that if $F_1, F_2 : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable at $b \in \mathbb{R}^n$ then $F_1 + F_2$ is differentiable at $b \in \mathbb{R}^n$ and $(F_1+F_2)'(b) = F_1'(b) + F_2'(b)$. You may assume the theorems concerning limits of mappings.

11:49–11:55

If F_1 & F_2 are diff. at $b \in \mathbb{R}^n$ then $\exists L_1, L_2$ linear transformations from $\mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{\| F_j(b+h) - F_j(b) - L_j(h) \|}{\| h \|} = 0$$

for $j = 1,2$. Consider then,

$$\lim_{h\to 0} \frac{\| (F_1+F_2)(b+h) - (F_1+F_2)(b) - (L_1+L_2)(h) \|}{\| h \|} \underset{\text{by linearity of } L_1 \& L_2}{=}$$

$$= \lim_{h\to 0} \frac{\| F_1(b+h) - F_1(b) - L_1(h) + F_2(b+h) - F_2(b) - L_2(h) \|}{\| h \|}$$

$$= \underbrace{\lim_{h\to 0} \frac{\| F_1(b,+h) - F_1(b) - L_1(h) \|}{\| h \|} + \lim_{h\to 0} \frac{\| L_2(b+h) - L_2(b) - L_2(h) \|}{\| h \|}}$$

$$= 0 + 0$$

$$= 0$$

both of these limits exist and are zero so this step is justified.

Therefore, we have shown that $L_1 + L_2$ is the best linear approx. to $\Delta(F_1+F_2)$ near b. This means

$$d(F_1+F_2)_b = L_1 + L_2 = (dF_1)_b + (dF_2)_b .$$

Or we can induce $(F_1+F_2)'(b) = F_1'(b) + F_2'(b)$ since the matrix of the sum of operators is the sum of their respective matrices.

Bonus [150pts] Compare and contrast the directional derivative and the differential. Give an example of a mapping whose directional derivatives exist yet the differential does not exist. What important property is your example necessarily missing?