

Lecture Notes for Advanced Calculus

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introduction and motivations for these notes

There are many excellent texts on portions of this subject. However, the particular path I choose this semester is not quite in line with any particular text. I required the text on Advanced Calculus by Edwards because it contains all the major theorems that traditionally are covered in an Advanced Calculus course.

My focus differs significantly. If I had students who had already completed a semester in real analysis then we could delve into the more analytic aspects of the subject. However, real analysis is not a prerequisite so we take a different path. Generically the story is as follows: a linear approximation replaces a complicated object very well so long as we are close to the base-point for the approximation. The first level of understanding is what I would characterize as *algebraic*, beyond that is the *analytic* understanding. I would argue that we must first have a firm grasp of the algebraic before we can properly attack the analytic aspects of the subject.

Edwards covers both the algebraic and the analytic. This makes his text hard to read in places because the full story is at some points technical. My goal is to focus on the algebraic. That said, I will try to at least point the reader to the section of Edward where the proof can be found.

Linear algebra is not a prerequisite for this course. However, I will use linear algebra. Matrices, linear transformations and vector spaces are necessary ingredients for a proper discussion of advanced calculus. I believe an interested student can easily assimilate the needed tools as we go so I am not terribly worried if you have not had linear algebra previously. I will make a point to include some baby¹ linear exercises to make sure everyone who is working at this course keeps up with the story that unfolds.

Doing the homework is doing the course. I cannot overemphasize the importance of thinking through the homework. I would be happy if you left this course with a working knowledge of:

- ✓ set-theoretic mapping language, fibers and images and how to picture relationships diagrammatically.
- ✓ continuity in view of the metric topology in n -space.
- ✓ the concept and application of the derivative and differential of a mapping.
- ✓ continuous differentiability
- ✓ inverse function theorem
- ✓ implicit function theorem
- ✓ tangent space and normal space via gradients

¹if you view this as an insult then you haven't met the right babies yet. Baby exercises are cute.

- ✓ extrema for multivariate functions, critical points and the Lagrange multiplier method
- ✓ multivariate Taylor series.
- ✓ quadratic forms
- ✓ critical point analysis for multivariate functions
- ✓ dual space and the dual basis.
- ✓ multilinear algebra.
- ✓ metric dualities and Hodge duality.
- ✓ the work and flux form mappings for \mathbb{R}^3 .
- ✓ basic manifold theory
- ✓ vector fields as derivations.
- ✓ Lie series and how vector fields generate symmetries
- ✓ differential forms and the exterior derivative
- ✓ integration of forms
- ✓ generalized Stokes's Theorem.
- ✓ surfaces
- ✓ fundamental forms and curvature for surfaces
- ✓ differential form formulation of classical differential geometry
- ✓ some algebra and calculus of supermathematics

Before we begin, I should warn you that I assume quite a few things from the reader. These notes are intended for someone who has already grappled with the problem of constructing proofs. I assume you know the difference between \Rightarrow and \Leftrightarrow . I assume the phrase "iff" is known to you. I assume you are ready and willing to do a proof by induction, strong or weak. I assume you know what \mathbb{R} , \mathbb{C} , \mathbb{Q} , \mathbb{N} and \mathbb{Z} denote. I assume you know what a subset of a set is. I assume you know how to prove two sets are equal. I assume you are familiar with basic set operations such as union and intersection (although we don't use those much). More importantly, I assume you have started to appreciate that mathematics is more than just calculations. Calculations without context, without theory, are doomed to failure. At a minimum theory and proper mathematics allows you to communicate analytical concepts to other like-educated individuals.

Some of the most seemingly basic objects in mathematics are insidiously complex. We've been taught they're simple since our childhood, but as adults, mathematical adults, we find the actual

definitions of such objects as \mathbb{R} or \mathbb{C} are rather involved. I will not attempt to provide foundational arguments to build numbers from basic set theory. I believe it is possible, I think it's well-thought-out mathematics, but we take the existence of the real numbers as an axiom for these notes. We assume that \mathbb{R} exists and that the real numbers possess all their usual properties. In fact, I assume \mathbb{R} , \mathbb{C} , \mathbb{Q} , \mathbb{N} and \mathbb{Z} all exist complete with their standard properties. In short, I assume we have numbers to work with. We leave the rigorization of numbers to a different course.

The format of these notes is similar to that of my calculus and linear algebra and advanced calculus notes from 2009-2011. However, I will make a number of definitions in the body of the text. Those sort of definitions are typically background-type definitions and I will make a point of putting them in bold so you can find them with ease.

I have avoided use of Einstein's implicit summation notation in the majority of these notes. This has introduced some clutter in calculations, but I hope the student finds the added detail helpful. Naturally if one goes on to study tensor calculations in physics then no such luxury is granted, you will have to grapple with the meaning of Einstein's convention. I suspect that is a minority in this audience so I took that task off the to-do list for this course.

The content of this course differs somewhat from my previous offering. The presentation of geometry and manifolds is almost entirely altered. Also, I have removed the chapter on Newtonian mechanics as well as the later chapter on variational calculus. Naturally, the interested student is invited to study those as independent studies past this course. If interested please ask.

I should mention that James Callahan's *Advanced Calculus: a geometric view* has influenced my thinking in this reformulation of my notes. His discussion of Morse's work was a useful addition to the critical point analysis.

I was inspired by Flander's text on differential form computation. It is my goal to implement some of his nicer calculations as an addition to my previous treatment of differential forms. In addition, I intend to incorporate material from Burns and Gidea's *Differential Geometry and Topology with a View to Dynamical Systems* as well as Munkres' *Analysis on Manifolds*. These additions should greatly improve the depth of the manifold discussion. I intend to go significantly deeper this year so the student can perhaps begin to appreciate manifold theory.

I plan to take the last few weeks of class to discuss supermathematics. This will serve as a sideways review for calculus on \mathbb{R}^n . In addition, I hope the exercise of abstracting calculus to supernumbers gives you some ideas about the process of abstraction in general. Abstraction is a cornerstone of modern mathematics and it is an essential skill for a mathematician. We may also discuss some of the historical motivations and modern applications of supermath to supersymmetric field theory.

NOTE To BE DELETED:

-add pictures from 2009 notes.

-change equations to numbered equations.

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Chapter 1

set-up

In this chapter we settle some basic terminology about sets and functions.

1.1 set theory

Let us denote sets by capital letters in as much as is possible. Often the lower-case letter of the same symbol will denote an element; $a \in A$ is to mean that the object a is in the set A . We can abbreviate $a_1 \in A$ and $a_2 \in A$ by simply writing $a_1, a_2 \in A$, this is a standard notation. The **union** of two sets A and B is denoted $A \cup B = \{x | x \in A \text{ or } x \in B\}$. The **intersection** of two sets is denoted $A \cap B = \{x | x \in A \text{ and } x \in B\}$. If a set S has no elements then we say S is the **empty set** and denote this by writing $S = \emptyset$. It is sometimes convenient to use unions or intersections of several sets:

$$\bigcup_{\alpha \in \Lambda} U_\alpha = \{x \mid \text{there exists } \alpha \in \Lambda \text{ with } x \in U_\alpha\}$$
$$\bigcap_{\alpha \in \Lambda} U_\alpha = \{x \mid \text{for all } \alpha \in \Lambda \text{ we have } x \in U_\alpha\}$$

we say Λ is the **index set** in the definitions above. If Λ is a finite set then the union/intersection is said to be a finite union/intersection. If Λ is a countable set then the union/intersection is said to be a countable union/intersection¹. Suppose A and B are both sets then we say A is a **subset** of B and write $A \subseteq B$ iff $a \in A$ implies $a \in B$ for all $a \in A$. If $A \subseteq B$ then we also say B is a **superset** of A . If $A \subseteq B$ then we say $A \subset B$ iff $A \neq B$ and $A \neq \emptyset$. Recall, for sets A, B we define $A = B$ iff $a \in A$ implies $a \in B$ for all $a \in A$ and conversely $b \in B$ implies $b \in A$ for all $b \in B$. This is equivalent to insisting $A = B$ iff $A \subseteq B$ and $B \subseteq A$. The **difference** of two sets A and B is denoted $A - B$ and is defined by $A - B = \{a \in A \mid \text{such that } a \notin B\}$ ².

¹recall the term **countable** simply means there exists a bijection to the natural numbers. The cardinality of such a set is said to be \aleph_0 .

²other texts sometimes use $A - B = A \setminus B$

We often make use of the following standard sets:

natural numbers (positive integers); $\mathbb{N} = \{1, 2, 3, \dots\}$.

natural numbers up to the number n ; $\mathbb{N}_n = \{1, 2, 3, \dots, n-1, n\}$.

integers; $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Note, $\mathbb{Z}_{>0} = \mathbb{N}$.

non-negative integers; $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$.

negative integers; $\mathbb{Z}_{<0} = \{-1, -2, -3, \dots\} = -\mathbb{N}$.

rational numbers; $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$.

irrational numbers; $\mathbb{J} = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$.

open interval from a to b ; $(a, b) = \{x \mid a < x < b\}$.

half-open interval; $(a, b] = \{x \mid a < x \leq b\}$ or $[a, b) = \{x \mid a \leq x < b\}$.

closed interval; $[a, b] = \{x \mid a \leq x \leq b\}$.

The final, and for us the most important, construction in set-theory is called the Cartesian product. Let A, B, C be sets, we define:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

By a slight abuse of notation³ we also define:

$$A \times B \times C = \{(a, b, c) \mid a \in A \text{ and } b \in B \text{ and } c \in C\}$$

In the case the sets comprising the cartesian product are the same we use an exponential notation for the construction:

$$A^2 = A \times A, \quad A^3 = A \times A \times A$$

We can extend to finitely many sets. Suppose A_i is a set for $i = 1, 2, \dots, n$ then we denote the Cartesian product by

$$A_1 \times A_2 \times \dots \times A_n = \times_{i=1}^n A_i$$

and define $\vec{x} \in \times_{i=1}^n A_i$ iff $\vec{x} = (a_1, a_2, \dots, a_n)$ where $a_i \in A_i$ for each $i = 1, 2, \dots, n$. An element \vec{x} as above is often called an **n-tuple**.

We define $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. I refer to \mathbb{R}^2 as "R-two" in conversational mathematics. Likewise, "R-three" is defined by $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$. We are ultimately interested in studying "R-n" where $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, 2, \dots, n\}$. In this course if we consider \mathbb{R}^n

³technically $A \times (B \times C) \neq (A \times B) \times C$ since objects of the form $(a, (b, c))$ are not the same as $((a, b), c)$, we ignore these distinctions and map both of these to the triple (a, b, c) without ambiguity in what follows

it is assumed from the context that $m \in \mathbb{N}$.

In terms of cartesian products you can imagine the x -axis as the number line then if we paste another numberline at each x value the union of all such lines constructs the plane; this is the picture behind $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Another interesting cartesian product is the **unit-square**; $[0, 1]^2 = [0, 1] \times [0, 1] = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Sometimes a rectangle in the plane with it's edges included can be written as $[x_1, x_2] \times [y_1, y_2]$. If we want to remove the edges use $(x_1, x_2) \times (y_1, y_2)$.

Moving to three dimensions we can construct the **unit-cube** as $[0, 1]^3$. A generic rectangular solid can sometimes be represented as $[x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$ or if we delete the edges: $(x_1, x_2) \times (y_1, y_2) \times (z_1, z_2)$.

1.2 vectors and geometry for n -dimensional space

Definition 1.2.1.

Let $n \in \mathbb{N}$, we define $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_j \in \mathbb{R} \text{ for } j = 1, 2, \dots, n\}$. If $v \in \mathbb{R}^n$ then we say v is an **n-vector**. The numbers in the vector are called the **components**; $v = (v_1, v_2, \dots, v_n)$ has j -th component v_j .

Notice, a consequence of the definition above and the construction of the Cartesian product⁴ is that two vectors v and w are equal iff $v_j = w_j$ for all $j \in \mathbb{N}_n$. Equality of two vectors is only true if all components are found to match. It is therefore logical to define addition and scalar multiplication in terms of the components of vectors as follows:

Definition 1.2.2.

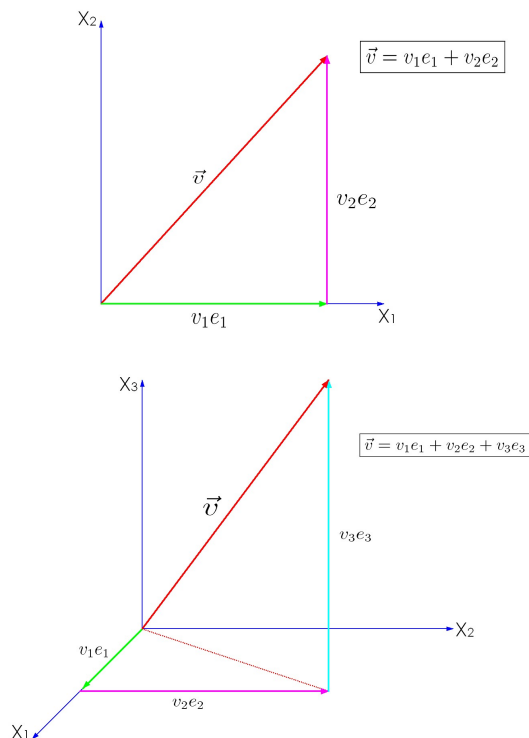
Define functions $+: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\cdot: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by the following rules: for each $v, w \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

$$(1.) (v + w)_j = v_j + w_j \qquad (2.) (cv)_j = cv_j$$

for all $j \in \{1, 2, \dots, n\}$. The operation $+$ is called **vector addition** and it takes two vectors $v, w \in \mathbb{R}^n$ and produces another vector $v + w \in \mathbb{R}^n$. The operation \cdot is called **scalar multiplication** and it takes a number $c \in \mathbb{R}$ and a vector $v \in \mathbb{R}^n$ and produces another vector $c \cdot v \in \mathbb{R}^n$. Often we simply denote $c \cdot v$ by juxtaposition cv .

If you are a gifted at visualization then perhaps you can add three-dimensional vectors in your mind. If you're mind is really unhinged maybe you can even add 4 or 5 dimensional vectors. The beauty of the definition above is that we have no need of pictures. Instead, algebra will do just fine. That said, let's draw a few pictures.

⁴see my Math 200 notes or ask me if interested, it's not entirely trivial



Notice these pictures go to show how you can break-down vectors into component vectors which point in the direction of the coordinate axis. Vectors of length one which point in the coordinate directions make up what is called the **standard basis**⁵ It is convenient to define special notation for the standard basis. First I define a useful shorthand,

Definition 1.2.3.

The symbol $\delta_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$ is called the **Kronecker delta**.

For example, $\delta_{22} = 1$ while $\delta_{12} = 0$.

Definition 1.2.4.

Let $e_i \in \mathbb{R}^{n \times 1}$ be defined by $(e_i)_j = \delta_{ij}$. The size of the vector e_i is determined by context. We call e_i the i -th standard basis vector.

Example 1.2.5. Let me expand on what I mean by "context" in the definition above:
In \mathbb{R} we have $e_1 = (1) = 1$ (by convention we drop the brackets in this case)

⁵the term "basis" is carefully developed in the linear algebra course. In a nutshell we need two things: (1.) the basis has to be big enough that we can add together the basis elements to make any thing in the set (2.) the basis is minimal so no single element in the basis can be formed by adding together other basis elements

In \mathbb{R}^2 we have $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

In \mathbb{R}^3 we have $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

In \mathbb{R}^4 we have $e_1 = (1, 0, 0, 0)$ and $e_2 = (0, 1, 0, 0)$ and $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$.

A **real linear combination** of $\{v_1, v_2, \dots, v_n\}$ is simply a finite weighted-sum of the objects from the set; $c_1v_1 + c_2v_2 + \dots + c_nv_n$ where $c_1, c_2, \dots, c_n \in \mathbb{R}$. If we take coefficients $c_1, c_2, \dots, c_n \in \mathbb{C}$ then it is said to be a **complex linear combination**. I invite the reader to verify that every vector in \mathbb{R}^n is a linear combination of e_1, e_2, \dots, e_n ⁶. It is not difficult to prove the following properties for vector addition and scalar multiplication: for all $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$,

$$\begin{array}{ll} (i.) & x + y = y + x, \\ (iii.) & x + 0 = x, \\ (v.) & 1x = x, \\ (vii.) & a(x + y) = ax + ay, \\ (ix.) & x + y \in \mathbb{R}^n \end{array} \quad \begin{array}{ll} (ii.) & (x + y) + z = x + (y + z) \\ (iv.) & x - x = 0 \\ (vi.) & (ab)x = a(bx), \\ (viii.) & (a + b)x = ax + bx \\ (x.) & cx \in \mathbb{R}^n \end{array}$$

These properties of \mathbb{R}^n are abstracted in linear algebra to form the definition of an abstract **vector space**. Naturally \mathbb{R}^n is a vector space, in fact it is the quintessential model for all other vector spaces. Fortunately \mathbb{R}^n also has a dot-product. The dot-product is a mapping from $\mathbb{R}^n \times \mathbb{R}^n$ to \mathbb{R} . We take in a pair of vectors and output a real number.

Definition 1.2.6. Let $x, y \in \mathbb{R}^n$ we define $x \cdot y \in \mathbb{R}$ by

$$x \cdot y = x_1y_1 + x_2y_2 + \dots + x_ny_n.$$

Example 1.2.7. Let $v = (1, 2, 3, 4, 5)$ and $w = (6, 7, 8, 9, 10)$

$$v \cdot w = 6 + 14 + 24 + 36 + 50 = 130$$

Example 1.2.8. Suppose we are given a vector $v \in \mathbb{R}^n$. We can select the j -th component by taking the dot-product of v with e_j . Observe that $e_i \cdot e_j = \delta_{ij}$ and consider,

$$v \cdot e_j = \left(\sum_{i=1}^n v_i e_i \right) \cdot e_j = \sum_{i=1}^n v_i e_i \cdot e_j = \sum_{i=1}^n v_i \delta_{ij} = v_1 \delta_{1j} + \dots + v_j \delta_{jj} + \dots + v_n \delta_{nj} = v_j.$$

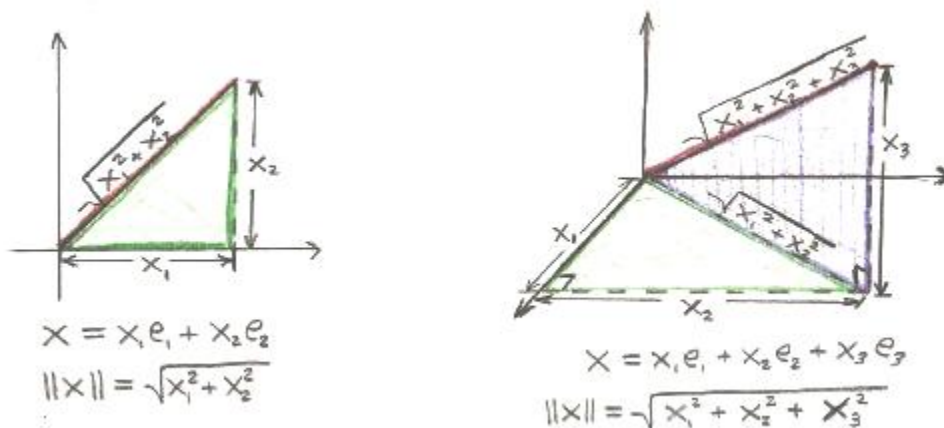
The dot-product with e_j has given us the length of the vector v in the j -th direction.

The length or norm of a vector and the angle between two vectors are induced from the dot-product:

Definition 1.2.9.

⁶the calculation is given explicitly in my linear notes

The **length** or **norm** of $x \in \mathbb{R}^n$ is a real number which is defined by $\|x\| = \sqrt{x \cdot x}$. Furthermore, let x, y be nonzero vectors in \mathbb{R}^n we define the **angle** θ between x and y by $\cos^{-1} \left[\frac{x \cdot y}{\|x\| \|y\|} \right]$. \mathbb{R}^n together with these definitions of length and angle forms a **Euclidean Geometry**.



Technically, before we make this definition we should make sure that the formulas given above even make sense. I have not shown that $x \cdot x$ is nonnegative and how do we know that argument of the inverse cosine is within its domain of $[-1, 1]$? I now state the propositions which justify the preceding definition. (proofs of the propositions below are found in my linear algebra notes)

Proposition 1.2.10.

Suppose $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then

1. $x \cdot y = y \cdot x$
2. $x \cdot (y + z) = x \cdot y + x \cdot z$
3. $c(x \cdot y) = (cx) \cdot y = x \cdot (cy)$
4. $x \cdot x \geq 0$ and $x \cdot x = 0$ iff $x = 0$

The formula $\cos^{-1} \left[\frac{x \cdot y}{\|x\| \|y\|} \right]$ is harder to justify. The inequality that we need for it to be reasonable is $\left| \frac{x \cdot y}{\|x\| \|y\|} \right| \leq 1$, otherwise we would not have a number in the $\text{dom}(\cos^{-1}) = \text{range}(\cos) = [-1, 1]$. An equivalent inequality is $|x \cdot y| \leq \|x\| \|y\|$ which is known as the **Cauchy-Schwarz** inequality.

Proposition 1.2.11.

If $x, y \in \mathbb{R}^n$ then $|x \cdot y| \leq \|x\| \|y\|$

Example 1.2.12. Let $v = [1, 2, 3, 4, 5]^T$ and $w = [6, 7, 8, 9, 10]^T$ find the angle between these vectors and calculate the unit vectors in the same directions as v and w . Recall that, $v \cdot w = 6 + 14 + 24 + 36 + 50 = 130$. Furthermore,

$$\|v\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2} = \sqrt{1 + 4 + 9 + 16 + 25} = \sqrt{55}$$

$$\|w\| = \sqrt{6^2 + 7^2 + 8^2 + 9^2 + 10^2} = \sqrt{36 + 49 + 64 + 81 + 100} = \sqrt{330}$$

We find unit vectors via the standard trick, you just take the given vector and multiply it by the reciprocal of its length. This is called **normalizing** the vector,

$$\hat{v} = \frac{1}{\sqrt{55}}[1, 2, 3, 4, 5]^T \quad \hat{w} = \frac{1}{\sqrt{330}}[6, 7, 8, 9, 10]^T$$

The angle is calculated from the definition of angle,

$$\theta = \cos^{-1}\left(\frac{130}{\sqrt{55}\sqrt{330}}\right) = 15.21^\circ$$

It's good we have this definition, 5-dimensional protractors are very expensive.

Proposition 1.2.13.

Let $x, y \in \mathbb{R}^n$ and suppose $c \in \mathbb{R}$ then

1. $\|cx\| = |c| \|x\|$
2. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)
3. $\|x\| \geq 0$
4. $\|x\| = 0$ iff $x = 0$

The four properties above make \mathbb{R}^n paired with $\|\cdot\| : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ a normed linear space. We'll see how differentiation can be defined given this structure. It turns out that we can define a reasonable concept of differentiation for other normed linear spaces. In this course we'll study how to differentiate functions to and from \mathbb{R}^n , matrix-valued functions and complex-valued functions of a real variable. Finally, if time permits, we'll study differentiation of functions of functions which is the central task of **variational calculus**. In each case the underlying linear structure along with the norm is used to define the limits which are necessary to set-up the derivatives. The focus of this course is the process and use of derivatives and integrals so I have not given proofs of the linear algebraic propositions in this chapter. The proofs and a deeper view of the meaning of these propositions is given at length in Math 321. If you haven't had linear then you'll just have to trust me on these propositions⁷

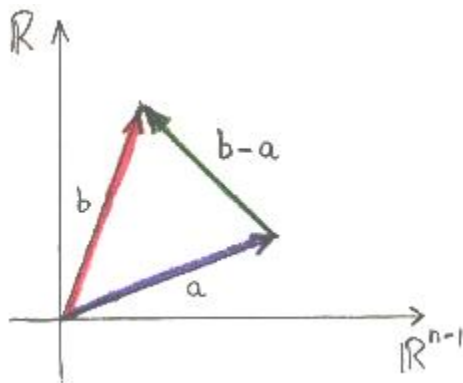
⁷or you could just read the linear notes if curious

Definition 1.2.14.

The **distance** between $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ is defined to be $d(a, b) \equiv \|b - a\|$.

If we draw a picture this definition is very natural. Here we are thinking of the points a, b as vectors from the origin then $b - a$ is the vector which points from a to b (this is algebraically clear since $a + (b - a) = b$). Then the distance between the points is the length of the vector that points from one point to the other. If you plug in two dimensional vectors you should recognize the distance formula from middle school math:

$$d((a_1, a_2), (b_1, b_2)) = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$$

**Proposition 1.2.15.**

Let $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be the distance function then

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$
3. $d(x, x) = 0$ iff $x = 0$
4. $d(x, y) + d(y, z) \geq d(x, z)$

In real analysis one studies a set paired with a distance function. Abstractly speaking such a pair is called a **metric space**. A vector space with a norm is called a normed linear space. Because we can always induce a distance function from the norm via the formula $d(a, b) = \|b - a\|$ every normed linear space is a metric space. The converse fails. Metric spaces need not be vector spaces, a metric space could just be formed from some subset of a vector space or something more exotic⁸. The absolute value function on \mathbb{R} defines distance function $d(a, b) = |b - a|$. In your real analysis

⁸there are many texts to read on metric spaces, one nice treatment is Rosenlicht's *Introduction to Analysis*, it's a good read

course you will study the structure of the metric space $(\mathbb{R}, |\cdot| : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R})$ in great depth. I include these comments here to draw your attention to the connection between this course and the real analysis course. I primarily use the norm in what follows, but it should be noted that many things could be written in terms of the distance function.

1.2.1 vector algebra for three dimensions

Every nonzero vector can be written as a unit vector scalar multiplied by its magnitude.

$$v \in V^n \text{ such that } v \neq 0 \Rightarrow v = \|v\| \hat{v} \text{ where } \hat{v} = \frac{1}{\|v\|} v.$$

You should recall that we can write any vector in V^3 as

$$v = \langle a, b, c \rangle = a \langle 1, 0, 0 \rangle + b \langle 0, 1, 0 \rangle + c \langle 0, 0, 1 \rangle = a\hat{i} + b\hat{j} + c\hat{k}$$

where we defined the $\hat{i} = \langle 1, 0, 0 \rangle$, $\hat{j} = \langle 0, 1, 0 \rangle$, $\hat{k} = \langle 0, 0, 1 \rangle$. You can easily verify that distinct Cartesian unit-vectors are orthogonal. Sometimes we need to produce a vector which is orthogonal to a given pair of vectors, it turns out the cross-product is one of two ways to do that in V^3 . We will see much later that this is special to three dimensions.

Definition 1.2.16.

If $A = \langle A_1, A_2, A_3 \rangle$ and $B = \langle B_1, B_2, B_3 \rangle$ are vectors in V^3 then the **cross-product** of A and B is a **vector** $A \times B$ which is defined by:

$$\vec{A} \times \vec{B} = \langle A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1 \rangle.$$

The magnitude of $\vec{A} \times \vec{B}$ can be shown to satisfy $\|\vec{A} \times \vec{B}\| = \|\vec{A}\| \|\vec{B}\| \sin(\theta)$ and the direction can be deduced by **right-hand-rule**. The right hand rule for the unit vectors yields:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{k} \times \hat{i} = \hat{j}, \quad \hat{j} \times \hat{k} = \hat{i}$$

If I wish to discuss both the point and the vector to which it corresponds we may use the notation

$$P = (a_1, a_2, \dots, a_n) \longleftrightarrow \vec{P} = \langle a_1, a_2, \dots, a_n \rangle$$

With this notation we can easily define directed line-segments as the vector which points from one point to another, also the distance between points is simply the length of the vector which points from one point to the other:

Definition 1.2.17.

Let $P, Q \in \mathbb{R}^n$. The directed line segment from P to Q is $\vec{PQ} = \vec{Q} - \vec{P}$. This vector is drawn from tail Q to the tip P where we denote the direction by drawing an arrowhead. The **distance between P and Q** is $d(P, Q) = \|\vec{PQ}\|$.

1.2.2 compact notations for vector arithmetic

I prefer the following notations over the hat-notation of the preceding section because this notation generalizes nicely to n -dimensions.

$$e_1 = \langle 1, 0, 0 \rangle \quad e_2 = \langle 0, 1, 0 \rangle \quad e_3 = \langle 0, 0, 1 \rangle .$$

Likewise the Kronecker delta and the Levi-Civita symbol are at times very convenient for abstract calculation:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \epsilon_{ijk} = \begin{cases} 1 & (i, j, k) \in \{(1, 2, 3), (3, 1, 2), (2, 3, 1)\} \\ -1 & (i, j, k) \in \{(3, 2, 1), (2, 1, 3), (1, 3, 2)\} \\ 0 & \text{if any index repeats} \end{cases}$$

An equivalent definition for the Levi-civita symbol is simply that $\epsilon_{123} = 1$ and it is antisymmetric with respect to the interchange of any pair of indices;

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{kji} = -\epsilon_{jik} = -\epsilon_{ikj}.$$

Now let us restate some earlier results in terms of the Einstein repeated index conventions⁹, let $\vec{A}, \vec{B} \in V^n$ and $c \in \mathbb{R}$ then

$\vec{A} = A_k e_k$	standard basis expansion
$e_i \cdot e_j = \delta_{ij}$	orthonormal basis
$(\vec{A} + \vec{B})_i = \vec{A}_i + \vec{B}_i$	vector addition
$(\vec{A} - \vec{B})_i = \vec{A}_i - \vec{B}_i$	vector subtraction
$(c\vec{A})_i = c\vec{A}_i$	scalar multiplication
$\vec{A} \cdot \vec{B} = A_k B_k$	dot product
$(\vec{A} \times \vec{B})_k = \epsilon_{ijk} A_i B_j$	cross product.

All but the last of the above are readily generalized to dimensions other than three by simply increasing the number of components. However, the cross product is special to three dimensions. I can't emphasize enough that the formulas given above for the dot and cross products can be utilized to yield great efficiency in abstract calculations.

Example 1.2.18. . .

⁹there are more details to be seen in the Appendix if you're curious

1.3 functions

Suppose A and B are sets, we say $f : A \rightarrow B$ is a **function** if for each $a \in A$ the function f assigns a single element $f(a) \in B$. Moreover, if $f : A \rightarrow B$ is a function we say it is a **B -valued function of an A -variable** and we say $A = \text{dom}(f)$ whereas $B = \text{codomain}(f)$. For example, if $f : \mathbb{R}^2 \rightarrow [0, 1]$ then f is real-valued function of \mathbb{R}^2 . On the other hand, if $f : \mathbb{C} \rightarrow \mathbb{R}^2$ then we'd say f is a vector-valued function of a complex variable. The term **mapping** will be used interchangeably with function in these notes¹⁰. Suppose $f : U \rightarrow V$ and $U \subseteq S$ and $V \subseteq T$ then we may consisely express the same data via the notation $f : U \subseteq S \rightarrow V \subseteq T$.

Sometimes we can take two given functions and construct a new function.

1. if $f : U \rightarrow V$ and $g : V \rightarrow W$ then $g \circ f : U \rightarrow W$ is the composite of g with f .
2. if $f, g : U \rightarrow V$ and V is a set with an operation of addition then we define $f \pm g : U \rightarrow V$ pointwise by the natural assignment $(f \pm g)(x) = f(x) \pm g(x)$ for each $x \in U$. We say that $f \pm g$ is the sum(+) or difference(-) of f and g .
3. if $f : U \rightarrow V$ and $c \in S$ where there is an operation of scalar multiplication by S on V then $cf : U \rightarrow V$ is defined pointwise by $(cf)(x) = cf(x)$ for each $x \in U$. We say that cf is scalar multiple of f by c .

Usually we have in mind $S = \mathbb{R}$ or $S = \mathbb{C}$ and often the addition is just that of vectors, however the definitions (2.) and (3.) apply equally well to matrix-valued functions or operators which is another term for function-valued functions. For example, in the first semester of calculus we study d/dx which is a function of functions; d/dx takes an input of f and gives the output df/dx . If we write $L = 3d/dx$ we have a new operator defined by $(3d/dx)[f] = 3df/dx$ for each function f in the domain of d/dx .

Definition 1.3.1.

¹⁰in my first set of advanced calculus notes (2010) I used the term function to mean the codomain was real numbers whereas mapping implied a codomain of vectors. I was following Edwards as he makes this convention in his text. I am not adopting that terminology any longer, I think it's better to use the term function as we did in Math 200 or 250. A function is an abstract construction which allows for a vast array of codomains.

Suppose $f : U \rightarrow V$. We define the **image of U_1 under f** as follows:

$$f(U_1) = \{ y \in V \mid \text{there exists } x \in U_1 \text{ with } f(x) = y \}.$$

The **range** of f is $f(U)$. The **inverse image of V_1 under f** is defined as follows:

$$f^{-1}(V_1) = \{ x \in U \mid f(x) \in V_1 \}.$$

The inverse image of a single point in the codomain is called a **fiber**. Suppose $f : U \rightarrow V$. We say f is **surjective** or **onto** V_1 iff there exists $U_1 \subseteq U$ such that $f(U_1) = V_1$. If a function is onto its codomain then the **function is surjective**. If $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in U_1 \subseteq U$ then we say **f is injective on U_1** or **1-1 on U_1** . If a function is injective on its domain then we say the **function is injective**. If a function is both injective and surjective then the function is called a **bijection** or a **1-1 correspondance**.

Example 1.3.2. Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $f(x, y) = x$ for each $(x, y) \in \mathbb{R}^2$. The function is not injective since $f(1, 2) = 1$ and $f(1, 3) = 1$ and yet $(1, 2) \neq (1, 3)$. Notice that the fibers of f are simply vertical lines:

$$f^{-1}(x_o) = \{(x, y) \in \text{dom}(f) \mid f(x, y) = x_o\} = \{(x_o, y) \mid y \in \mathbb{R}\} = \{x_o\} \times \mathbb{R}$$

Example 1.3.3. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = \sqrt{x^2} + 1$ for each $x \in \mathbb{R}$. This function is not surjective because $0 \notin f(\mathbb{R})$. In contrast, if we construct $g : \mathbb{R} \rightarrow [1, \infty)$ with $g(x) = f(x)$ for each $x \in \mathbb{R}$ then can argue that g is surjective. Neither f nor g is injective, the fiber of x_o is $\{-x_o, x_o\}$ for each $x_o \neq 0$. At all points except zero these maps are said to be **two-to-one**. This is an abbreviation of the observation that two points in the domain map to the same point in the range.

Definition 1.3.4.

Suppose $f : U \subseteq \mathbb{R}^p \rightarrow V \subseteq \mathbb{R}^n$ and suppose further that for each $x \in U$,

$$f(x) = (f_1(x), f_2(x), \dots, f_n(x)).$$

Then we say that $f = (f_1, f_2, \dots, f_n)$ and for each $j \in \mathbb{N}_p$ the functions $f_j : U \subseteq \mathbb{R}^p \rightarrow \mathbb{R}$ are called the **component functions** of f . Furthermore, we define the projection $\pi_j : \mathbb{R}^n \rightarrow \mathbb{R}$ to be the map $\pi_j(x) = x \cdot e_j$ for each $j = 1, 2, \dots, n$. This allows us to express each of the component functions as a composition $f_j = \pi_j \circ f$.

Example 1.3.5. Suppose $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $f(x, y, z) = (x^2 + y^2, z)$ for each $(x, y, z) \in \mathbb{R}^3$. Identify that $f_1(x, y, z) = x^2 + y^2$ whereas $f_2(x, y, z) = z$. You can easily see that $\text{range}(f) = [0, \infty) \times \mathbb{R}$. Suppose $R^2 \in [0, \infty)$ and $z_o \in \mathbb{R}$ then

$$f^{-1}(\{(R^2, z_o)\}) = S_1(R) \times \{z_o\}$$

where $S_1(R)$ denotes a circle of radius R . This result is a simple consequence of the observation that $f(x, y, z) = (R^2, z_o)$ implies $x^2 + y^2 = R^2$ and $z = z_o$.

Example 1.3.6. Let $a, b, c \in \mathbb{R}$ be particular constants. Suppose $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $f(x, y, z) = ax + by + cz$ for each $(x, y, z) \in \mathbb{R}^3$. Here there is just one component function so we could say that $f = f_1$ but we don't usually bother to make such an observation. If at least one of the constants a, b, c is nonzero then the fibers of this map are planes in three dimensional space with normal $\langle a, b, c \rangle$.

$$f^{-1}(\{d\}) = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\}$$

If $a = b = c = 0$ then the fiber of f is simply all of \mathbb{R}^3 and the range(f) = $\{0\}$.

The definition below explains how to put together functions with a common domain. The codomain of the new function is the cartesian product of the old codomains.

Definition 1.3.7.

Let $f : U_1 \subseteq \mathbb{R}^n \rightarrow V_1 \subseteq \mathbb{R}^p$ and $g : U_1 \subseteq \mathbb{R}^n \rightarrow V_2 \subseteq \mathbb{R}^q$ be a mappings then (f, g) is a mapping from U_1 to $V_1 \times V_2$ defined by $(f, g)(x) = (f(x), g(x))$ for all $x \in U_1$.

There's more than meets the eye in the definition above. Let me expand it a bit here:

$$(f, g)(x) = (f_1(x), f_2(x), \dots, f_p(x), g_1(x), g_2(x), \dots, g_q(x)) \quad \text{where } x = (x_1, x_2, \dots, x_n)$$

You might notice that Edwards uses π for the identity mapping whereas I use Id . His notation is quite reasonable given that the identity is the cartesian product of all the projection maps:

$$\pi = (\pi_1, \pi_2, \dots, \pi_n)$$

I've had courses where we simply used the coordinate notation itself for projections, in that notation have formulas such as $x(a, b, c) = a$, $x_j(a) = a_j$ and $x_j(e_i) = \delta_{ji}$.

Another way to modify a given function is to adjust the domain of a given mapping by restriction and extension.

Definition 1.3.8.

Let $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ be a mapping. If $R \subset U$ then we define the **restriction of f to R** to be the mapping $f|_R : R \rightarrow V$ where $f|_R(x) = f(x)$ for all $x \in R$. If $U \subseteq S$ and $V \subset T$ then we say a mapping $g : S \rightarrow T$ is an **extension of f** iff $g|_{dom(f)} = f$.

When I say $g|_{dom(f)} = f$ this means that these functions have matching domains and they agree at each point in that domain; $g|_{dom(f)}(x) = f(x)$ for all $x \in dom(f)$. Once a particular subset is chosen the restriction to that subset is a unique function. Of course there are usually many subsets of $dom(f)$ so you can imagine many different restrictions of a given function. The concept of extension is more vague, once you pick the enlarged domain and codomain it is not even necessarily the case that another extension to that same pair of sets will be the same mapping. To obtain uniqueness for extensions one needs to add more structure. This is one reason that complex variables are interesting, there are cases where the structure of the complex theory forces the extension of a complex-valued function of a complex variable to be unique. This is very surprising. Similarly a

linear transformation is uniquely defined by its values on a basis, it extends uniquely from that finite set of vectors to the infinite number of points in the vector space. This is very restrictive on the possible ways we can construct linear mappings. Maybe you can find some other examples of extensions as you collect your own mathematical storybook.

Definition 1.3.9.

Let $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ be a mapping, if there exists a mapping $g : f(U) \rightarrow U$ such that $f \circ g = Id_{f(U)}$ and $g \circ f = Id_U$ then g is the inverse mapping of f and we denote $g = f^{-1}$.

If a mapping is injective then it can be shown that the inverse mapping is well defined. We define $f^{-1}(y) = x$ iff $f(x) = y$ and the value x must be a single value if the function is one-one. When a function is not one-one then there may be more than one point which maps to a particular point in the range.

Notice that the inverse image of a set is well-defined even if there is no inverse mapping. Moreover, it can be shown that the fibers of a mapping are disjoint and their union covers the domain of the mapping:

$$f(y) \neq f(z) \Rightarrow f^{-1}\{y\} \cap f^{-1}\{z\} = \emptyset \qquad \bigcup_{y \in \text{range}(f)} f^{-1}\{y\} = \text{dom}(f).$$

This means that the fibers of a mapping *partition* the domain.

Example 1.3.10. . .

Definition 1.3.11.

Let $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ be a mapping. Furthermore, suppose that $s : U \rightarrow U$ is a mapping which is constant on each fiber of f . In other words, for each fiber $f^{-1}\{y\} \subseteq U$ we have some constant $u \in U$ such that $s(f^{-1}\{y\}) = u$. The subset $s^{-1}(U) \subseteq U$ is called a **cross section** of the fiber partition of f .

How do we construct a cross section for a particular mapping? For particular examples the details of the formula for the mapping usually suggests some obvious choice. However, in general if you accept the **axiom of choice** then you can be comforted in the existence of a cross section even in the case that there are infinitely many fibers for the mapping.

Example 1.3.12. . .

Proposition 1.3.13.

Let $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ be a mapping. The restriction of f to a cross section S of U is an injective function. The mapping $\tilde{f} : U \rightarrow f(U)$ is a surjection. The mapping $\tilde{f}|_S : S \rightarrow f(U)$ is a bijection.

The proposition above tells us that we can take any mapping and cut down the domain and/or codomain to reduce the function to an injection, surjection or bijection. If you look for it you'll see this result behind the scenes in other courses. For example, in linear algebra if we throw out the kernel of a linear mapping then we get an injection. The idea of a local inverse is also important to the study of calculus.

Example 1.3.14. . .

Definition 1.3.15.

Let $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ be a mapping then we say a mapping g is a **local inverse** of f iff there exists $S \subseteq U$ such that $g = (f|_S)^{-1}$.

Usually we can find local inverses for functions in calculus. For example, $f(x) = \sin(x)$ is not 1-1 therefore it is not invertible. However, it does have a local inverse $g(y) = \sin^{-1}(y)$. If we were more pedantic we wouldn't write $\sin^{-1}(y)$. Instead we would write $g(y) = (\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]})^{-1}(y)$ since the inverse sine is actually just a local inverse. To construct a local inverse for some mapping we must locate some subset of the domain upon which the mapping is injective. Then relative to that subset we can reverse the mapping. The inverse mapping theorem (which we'll study mid-course) will tell us more about the existence of local inverses for a given mapping.

1.4 elementary topology and limits

In this section we describe the *metric topology* for \mathbb{R}^n . In the study of functions of one real variable we often need to refer to open or closed intervals. The definition that follows generalizes those

concepts to n -dimensions. I have included a short discussion of general topology in the Appendix if you'd like to learn more about the term.

Definition 1.4.1.

An **open ball** of radius ϵ centered at $a \in \mathbb{R}^n$ is the subset all points in \mathbb{R}^n which are less than ϵ units from a , we denote this open ball by

$$B_\epsilon(a) = \{x \in \mathbb{R}^n \mid \|x - a\| < \epsilon\}$$

The **closed ball** of radius ϵ centered at $a \in \mathbb{R}^n$ is likewise defined

$$\overline{B}_\epsilon(a) = \{x \in \mathbb{R}^n \mid \|x - a\| \leq \epsilon\}$$

Notice that in the $n = 1$ case we observe an open ball is an open interval: let $a \in \mathbb{R}$,

$$B_\epsilon(a) = \{x \in \mathbb{R} \mid \|x - a\| < \epsilon\} = \{x \in \mathbb{R} \mid |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

In the $n = 2$ case we observe that an open ball is an open disk: let $(a, b) \in \mathbb{R}^2$,

$$B_\epsilon((a, b)) = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y) - (a, b)\| < \epsilon\} = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{(x - a)^2 + (y - b)^2} < \epsilon\}$$

For $n = 3$ an open-ball is a sphere without the outer shell. In contrast, a closed ball in $n = 3$ is a solid sphere which includes the outer shell of the sphere.

Definition 1.4.2.

Let $D \subseteq \mathbb{R}^n$. We say $y \in D$ is an **interior point** of D iff there exists some open ball centered at y which is completely contained in D . We say $y \in \mathbb{R}^n$ is a **limit point** of D iff every open ball centered at y contains points in $D - \{y\}$. We say $y \in \mathbb{R}^n$ is a **boundary point** of D iff every open ball centered at y contains points not in D and other points which are in $D - \{y\}$. We say $y \in D$ is an **isolated point** of D if there exist open balls about y which do not contain other points in D . The set of all interior points of D is called the **interior** of D . Likewise the set of all boundary points for D is denoted ∂D . The **closure** of D is defined to be $\overline{D} = D \cup \{y \in \mathbb{R}^n \mid y \text{ a limit point}\}$

If you're like me the paragraph above doesn't help much until I see the picture below. All the terms are aptly named. The term "limit point" is given because those points are the ones for which it is natural to define a limit.

Example 1.4.3. . .

Definition 1.4.4.

Let $A \subseteq \mathbb{R}^n$ is an **open set** iff for each $x \in A$ there exists $\epsilon > 0$ such that $x \in B_\epsilon(x)$ and $B_\epsilon(x) \subseteq A$. Let $B \subseteq \mathbb{R}^n$ is an **closed set** iff its complement $\mathbb{R}^n - B = \{x \in \mathbb{R}^n \mid x \notin B\}$ is an open set.

Notice that $\mathbb{R} - [a, b] = (\infty, a) \cup (b, \infty)$. It is not hard to prove that open intervals are open hence we find that a closed interval is a closed set. Likewise it is not hard to prove that open balls are open sets and closed balls are closed sets. I may ask you to prove the following proposition in the homework.

Proposition 1.4.5.

A closed set contains all its limit points, that is $A \subseteq \mathbb{R}^n$ is closed iff $A = \overline{A}$.

Example 1.4.6. . . .

In calculus I the limit of a function is defined in terms of deleted open intervals centered about the limit point. We can define the limit of a mapping in terms of deleted open balls centered at the limit point.

Definition 1.4.7.

Let $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ be a mapping. We say that f has limit $b \in \mathbb{R}^m$ at limit point a of U iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $x \in \mathbb{R}^n$ with $0 < \|x - a\| < \delta$ implies $\|f(x) - b\| < \epsilon$. In such a case we can denote the above by stating that

$$\lim_{x \rightarrow a} f(x) = b.$$

In calculus I the limit of a function is defined in terms of deleted open intervals centered about the limit point. We just defined the limit of a mapping in terms of deleted open balls centered at the limit point. The term "deleted" refers to the fact that we assume $0 < \|x - a\|$ which means we do not consider $x = a$ in the limiting process. In other words, the limit of a mapping considers values close to the limit point but not necessarily the limit point itself. The case that the function is defined at the limit point is special, when the limit and the mapping agree then we say the mapping is continuous at that point.

Example 1.4.8. . . .

Definition 1.4.9.

Let $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ be a mapping. If $a \in U$ is a limit point of f then we say that f is **continuous at a** iff

$$\lim_{x \rightarrow a} f(x) = f(a)$$

If $a \in U$ is an isolated point then we also say that f is continuous at a . The mapping f is **continuous on S** iff it is continuous at each point in S . The **mapping f is continuous** iff it is continuous on its domain.

Notice that in the $m = n = 1$ case we recover the definition of continuous functions from calc. I.

Proposition 1.4.10.

Let $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ be a mapping with component functions f_1, f_2, \dots, f_m hence $f = (f_1, f_2, \dots, f_m)$. If $a \in U$ is a limit point of f then

$$\lim_{x \rightarrow a} f(x) = b \quad \Leftrightarrow \quad \lim_{x \rightarrow a} f_j(x) = b_j \text{ for each } j = 1, 2, \dots, m.$$

Proof: (\Rightarrow) Suppose $\lim_{x \rightarrow a} f(x) = b$. Then for each $\epsilon > 0$ choose $\delta > 0$ such that $0 < \|x - a\| < \delta$ implies $\|f(x) - b\| < \epsilon$. This choice of δ suffices for our purposes as:

$$|f_j(x) - b_j| = \sqrt{(f_j(x) - b_j)^2} \leq \sqrt{\sum_{j=1}^m (f_j(x) - b_j)^2} = \|f(x) - b\| < \epsilon.$$

Hence we have shown that $\lim_{x \rightarrow a} f_j(x) = b_j$ for all $j = 1, 2, \dots, m$.

(\Leftarrow) Suppose $\lim_{x \rightarrow a} f_j(x) = b_j$ for all $j = 1, 2, \dots, m$. Let $\epsilon > 0$. Note that $\epsilon/m > 0$ and therefore by the given limits we can choose $\delta_j > 0$ such that $0 < \|x - a\| < \delta$ implies $\|f_j(x) - b_j\| < \sqrt{\epsilon/m}$. Choose $\delta = \min\{\delta_1, \delta_2, \dots, \delta_m\}$ clearly $\delta > 0$. Moreover, notice $0 < \|x - a\| < \delta \leq \delta_j$ hence requiring $0 < \|x - a\| < \delta$ automatically induces $0 < \|x - a\| < \delta_j$ for all j . Suppose that $x \in \mathbb{R}^n$ and $0 < \|x - a\| < \delta$ it follows that

$$\|f(x) - b\| = \left\| \sum_{j=1}^m (f_j(x) - b_j)e_j \right\| = \sqrt{\sum_{j=1}^m |f_j(x) - b_j|^2} < \sum_{j=1}^m (\sqrt{\epsilon/m})^2 < \sum_{j=1}^m \epsilon/m = \epsilon.$$

Therefore, $\lim_{x \rightarrow a} f(x) = b$ and the proposition follows. \square

We can analyze the limit of a mapping by analyzing the limits of the component functions:

Example 1.4.11. . .

The following follows immediately from the preceding proposition.

Proposition 1.4.12.

Suppose that $f : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}^m$ is a mapping with component functions f_1, f_2, \dots, f_m . Let $a \in U$ be a limit point of f then f is continuous at a iff f_j is continuous at a for $j = 1, 2, \dots, m$. Moreover, f is continuous on S iff all the component functions of f are continuous on S . Finally, a mapping f is continuous iff all of its component functions are continuous. .

The proof of the proposition is in Edwards, it's his Theorem 7.2. It's about time I proved something.

Proposition 1.4.13.

The projection functions are continuous. The identity mapping is continuous.

Proof: Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $x \in \mathbb{R}^n$ such that $0 < \|x - a\| < \delta$ then it follows that $\|x - a\| < \epsilon$. Therefore, $\lim_{x \rightarrow a} x = a$ which means that $\lim_{x \rightarrow a} Id(x) = Id(a)$ for all $a \in \mathbb{R}^n$. Hence Id is continuous on \mathbb{R}^n which means Id is continuous. Since the projection functions are component functions of the identity mapping it follows that the projection functions are also continuous (using the previous proposition). \square

Definition 1.4.14.

The **sum** and **product** are functions from \mathbb{R}^2 to \mathbb{R} defined by

$$s(x, y) = x + y \quad p(x, y) = xy$$

Proposition 1.4.15.

The sum and product functions are continuous.

Preparing for the proof: Let the limit point be (a, b) . Consider what we wish to show: given a point (x, y) such that $0 < \|(x, y) - (a, b)\| < \delta$ we wish to show that

$$|s(x, y) - (a + b)| < \epsilon \quad \text{or for the product} \quad |p(x, y) - (ab)| < \epsilon$$

follow for appropriate choices of δ . Think about the sum for a moment,

$$|s(x, y) - (a + b)| = |x + y - a - b| \leq |x - a| + |y - b|$$

I just used the triangle inequality for the absolute value of real numbers. We see that if we could somehow get control of $|x - a|$ and $|y - b|$ then we'd be getting closer to the prize. We have control of $0 < \|(x, y) - (a, b)\| < \delta$ notice this reduces to

$$\|(x - a, y - b)\| < \delta \Rightarrow \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

it is clear that $(x - a)^2 < \delta^2$ since if it was otherwise the inequality above would be violated as adding a nonnegative quantity $(y - b)^2$ only increases the radicand resulting in the squareroot to be larger than δ . Hence we may assume $(x - a)^2 < \delta^2$ and since $\delta > 0$ it follows $|x - a| < \delta$. Likewise,

$|y - b| < \delta$. Thus

$$|s(x, y) - (a + b)| = |x + y - a - b| < |x - a| + |y - b| < 2\delta$$

We see for the sum proof we can choose $\delta = \epsilon/2$ and it will work out nicely.

Proof: Let $\epsilon > 0$ and let $(a, b) \in \mathbb{R}^2$. Choose $\delta = \epsilon/2$ and suppose $(x, y) \in \mathbb{R}^2$ such that $\|(x, y) - (a, b)\| < \delta$. Observe that

$$\|(x, y) - (a, b)\| < \delta \Rightarrow \|(x - a, y - b)\|^2 < \delta^2 \Rightarrow |x - a|^2 + |y - b|^2 < \delta^2.$$

It follows $|x - a| < \delta$ and $|y - b| < \delta$. Thus

$$|s(x, y) - (a + b)| = |x + y - a - b| \leq |x - a| + |y - b| < \delta + \delta = 2\delta = \epsilon.$$

Therefore, $\lim_{(x,y) \rightarrow (a,b)} s(x, y) = a + b$. and it follows that the sum function is continuous at (a, b) . But, (a, b) is an arbitrary point thus s is continuous on \mathbb{R}^2 hence the sum function is continuous. \square .

Preparing for the proof of continuity of the product function: I'll continue to use the same notation as above. We need to study $|p(x, y) - (ab)| = |xy - ab| < \epsilon$. Consider that

$$|xy - ab| = |xy - ya + ya - ab| = |y(x - a) + a(y - b)| \leq |y||x - a| + |a||y - b|$$

We know that $|x - a| < \delta$ and $|y - b| < \delta$. There is one less obvious factor to bound in the expression. What should we do about $|y|$? I leave it to the reader to show that:

$$\boxed{|y - b| < \delta \Rightarrow |y| < |b| + \delta}$$

Now put it all together and hopefully we'll be able to "solve" for ϵ .

$$|xy - ab| \leq |y||x - a| + |a||y - b| < (|b| + \delta)\delta + |a|\delta = \delta^2 + \delta(|a| + |b|) \text{ " = " } \epsilon$$

I put solve in quotes because we have considerably more freedom in our quest for finding δ . We could just as well find δ which makes the " = " become an $<$. That said let's pursue equality,

$$\delta^2 + \delta(|a| + |b|) - \epsilon = 0 \quad \delta = \frac{-|a| - |b| \pm \sqrt{(|a| + |b|)^2 + 4\epsilon}}{2}$$

Since $\epsilon, |a|, |b| > 0$ it follows that $\sqrt{(|a| + |b|)^2 + 4\epsilon} < \sqrt{(|a| + |b|)^2} = |a| + |b|$ hence the (+) solution to the quadratic equation yields a positive δ namely:

$$\boxed{\delta = \frac{-|a| - |b| + \sqrt{(|a| + |b|)^2 + 4\epsilon}}{2}}$$

Yowsers, I almost made this a homework. There may be an easier route. You might notice we have run across a few little lemmas (I've boxed the punch lines for the lemmas) which are doubtless useful in other $\epsilon - \delta$ proofs. We should collect those once we're finished with this proof.

Proof: Let $\epsilon > 0$ and let $(a, b) \in \mathbb{R}^2$. By the calculations that prepared for the proof we know that the following quantity is positive, hence choose

$$\delta = \frac{-|a| - |b| + \sqrt{(|a| + |b|)^2 + 4\epsilon}}{2} > 0.$$

Note that¹¹,

$$\begin{aligned}
 |xy - ab| &= |xy - ya + ya - ab| &&= |y(x - a) + a(y - b)| &&\text{algebra} \\
 &\leq |y||x - a| + |a||y - b| &&\text{triangle inequality} \\
 &< (|b| + \delta)\delta + |a|\delta &&\text{by the boxed lemmas} \\
 &= \delta^2 + \delta(|a| + |b|) &&\text{algebra} \\
 &= \epsilon
 \end{aligned}$$

where we know that last step follows due to the steps leading to the boxed equation in the proof preparation. Therefore, $\lim_{(x,y) \rightarrow (a,b)} p(x, y) = ab$. and it follows that the product function is continuous at (a, b) . But, (a, b) is an arbitrary point thus p is continuous on \mathbb{R}^2 hence the product function is continuous. \square .

Lemma 1.4.16.

Assume $\delta > 0$.

1. If $a, x \in \mathbb{R}$ then $|x - a| < \delta \Rightarrow |x| < |a| + \delta$.
2. If $x, a \in \mathbb{R}^n$ then $\|x - a\| < \delta \Rightarrow |x_j - a_j| < \delta$ for $j = 1, 2, \dots, n$.

The proof of the proposition above is mostly contained in the remarks of the preceding two pages.

Example 1.4.17. . .

¹¹my notation is that when we stack inequalities the inequality in a particular line refers only to the immediate vertical successor.

Proposition 1.4.18.

Let $f : V \subseteq \mathbb{R}^p \rightarrow \mathbb{R}^m$ and $g : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^p$ be mappings. Suppose that $\lim_{x \rightarrow a} g(x) = b$ and suppose that f is continuous at b then

$$\lim_{x \rightarrow a} (f \circ g)(x) = f\left(\lim_{x \rightarrow a} g(x)\right).$$

The proof is in Edwards, see pages 46-47. Notice that the proposition above immediately gives us the important result below:

Proposition 1.4.19.

Let f and g be mappings such that $f \circ g$ is well-defined. The composite function $f \circ g$ is continuous for points $a \in \text{dom}(f \circ g)$ such that the following two conditions hold:

1. g is continuous at a
2. f is continuous at $g(a)$.

I make use of the earlier proposition that a mapping is continuous iff its component functions are continuous throughout the examples that follow. For example, I know (Id, Id) is continuous since Id was previously proved continuous.

Example 1.4.20. Note that if $f = p \circ (Id, Id)$ then $f(x) = (p \circ (Id, Id))(x) = p((Id, Id)(x)) = p(x, x) = x^2$. Therefore, the quadratic function $f(x) = x^2$ is continuous on \mathbb{R} as it is the composite of continuous functions.

Example 1.4.21. Note that if $f = p \circ (p \circ (Id, Id), Id)$ then $f(x) = p(x^2, x) = x^3$. Therefore, the cubic function $f(x) = x^3$ is continuous on \mathbb{R} as it is the composite of continuous functions.

Example 1.4.22. The power function is inductively defined by $x^1 = x$ and $x^n = xx^{n-1}$ for all $n \in \mathbb{N}$. We can prove $f(x) = x^n$ is continuous by induction on n . We proved the $n = 1$ case previously. Assume inductively that $f(x) = x^{n-1}$ is continuous. Notice that

$$x^n = xx^{n-1} = xf(x) = p(x, f(x)) = (p \circ (Id, f))(x).$$

Therefore, using the induction hypothesis, we see that $g(x) = x^n$ is the composite of continuous functions thus it is continuous. We conclude that $f(x) = x^n$ is continuous for all $n \in \mathbb{N}$.

We can play similar games with the sum function to prove that sums of power functions are continuous. In your homework you will prove constant functions are continuous. Putting all of these things together gives us the well-known result that polynomials are continuous on \mathbb{R} .

Proposition 1.4.23.

Let a be a limit point of mappings $f, g : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}$ and suppose $c \in \mathbb{R}$. If $\lim_{x \rightarrow a} f(x) = b_1 \in \mathbb{R}$ and $\lim_{x \rightarrow a} g(x) = b_2 \in \mathbb{R}$ then

1. $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$.
2. $\lim_{x \rightarrow a} (f(x)g(x)) = (\lim_{x \rightarrow a} f(x))(\lim_{x \rightarrow a} g(x))$.
3. $\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x)$.

Moreover, if f, g are continuous then $f + g, fg$ and cf are continuous.

Proof: Edwards proves (1.) carefully on pg. 48. I'll do (2.) here: we are given that $\lim_{x \rightarrow a} f(x) = b_1 \in \mathbb{R}$ and $\lim_{x \rightarrow a} g(x) = b_2 \in \mathbb{R}$ thus by Proposition 1.4.10 we find $\lim_{x \rightarrow a} (f, g)(x) = (b_1, b_2)$. Consider then,

$$\begin{aligned}
 \lim_{x \rightarrow a} (f(x)g(x)) &= \lim_{x \rightarrow a} (p(f, g)) && \text{defn. of product function} \\
 &= p(\lim_{x \rightarrow a} (f, g)) && \text{since } p \text{ is continuous} \\
 &= p(b_1, b_2) && \text{by Proposition 1.4.10.} \\
 &= b_1 b_2 && \text{definition of product function} \\
 &= (\lim_{x \rightarrow a} f(x))(\lim_{x \rightarrow a} g(x)).
 \end{aligned}$$

In your homework you proved that $\lim_{x \rightarrow a} c = c$ thus item (3.) follows from (2.). \square .

The proposition that follows does follow immediately from the proposition above, however I give a proof that again illustrates the idea we used in the examples. Reinterpreting a given function as a composite of more basic functions is a useful theoretical and calculational technique.

Proposition 1.4.24.

Assume $f, g : U \subseteq \mathbb{R}^n \rightarrow V \subseteq \mathbb{R}$ are continuous functions at $a \in U$ and suppose $c \in \mathbb{R}$.

1. $f + g$ is continuous at a .
2. fg is continuous at a .
3. cf is continuous at a .

Moreover, if f, g are continuous then $f + g, fg$ and cf are continuous.

Proof: Observe that $(f + g)(x) = (s \circ (f, g))(x)$ and $(fg)(x) = (p \circ (f, g))(x)$. We're given that f, g are continuous at a and we know s, p are continuous on all of \mathbb{R}^2 thus the composite functions $s \circ (f, g)$ and $p \circ (f, g)$ are continuous at a and the proof of items (1.) and (2.) is complete. To prove (3.) I refer the reader to their homework where it was shown that $h(x) = c$ for all $x \in U$ is a continuous function. We then find (3.) follows from (2.) by setting $g = h$ (function multiplication commutes for real-valued functions). \square .

We can use induction arguments to extend these results to arbitrarily many products and sums of power functions. To prove continuity of algebraic functions we'd need to do some more work with quotient and root functions. I'll stop here for the moment, perhaps I'll ask you to prove a few more fundamentals from calculus I. I haven't delved into the definition of exponential or log functions not to mention sine or cosine. We will assume that the basic functions of calculus are continuous on the interior of their respective domains. Basically if the formula for a function can be evaluated at the limit point then the function is continuous.

It's not hard to see that the comments above extend to functions of several variables and mappings. If the formula for a mapping is comprised of finite sums and products of power functions then we can prove such a mapping is continuous using the techniques developed in this section. If we have a mapping with a more complicated formula built from elementary functions then that mapping will be continuous provided its component functions have formulas which are sensibly calculated at the limit point. In other words, if you are willing to believe me that $\sin(x)$, $\cos(x)$, e^x , $\ln(x)$, $\cosh(x)$, $\sinh(x)$, \sqrt{x} , $\frac{1}{x^n}$, \dots are continuous on the interior of their domains then it's not hard to prove:

$$f(x, y, z) = \left(\sin(x) + e^x + \sqrt{\cosh(x^2) + \sqrt{y + e^x}}, \cosh(xyz), xe^{\sqrt{x + \frac{1}{yz}}} \right)$$

is a continuous mapping at points where the radicands of the square root functions are nonnegative. It wouldn't be very fun to write explicitly but it is clear that this mapping is the Cartesian product of functions which are the sum, product and composite of continuous functions.

Definition 1.4.25.

A polynomial in n -variables has the form:

$$f(x_1, x_2, \dots, x_n) = \sum_{i_1, i_2, \dots, i_n=0}^{\infty} c_{i_1, i_2, \dots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

where only finitely many coefficients $c_{i_1, i_2, \dots, i_n} \neq 0$. We denote the set of multinomials in n -variables as $\mathbb{R}(x_1, x_2, \dots, x_n)$.

Polynomials are $\mathbb{R}(x)$. Polynomials in two variables are $\mathbb{R}(x, y)$, for example,

$f(x, y) = ax + by$	$\deg(f) = 1$, linear function
$f(x, y) = ax + by + c$	$\deg(f) = 1$, affine function
$f(x, y) = ax^2 + bxy + cy^2$	$\deg(f) = 2$, quadratic form
$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + g$	$\deg(f) = 2$

If all the terms in the polynomial have the same number of variables then it is said to be **homogeneous**. In the list above only the linear function and the quadratic form were homogeneous.

Remark 1.4.26.

There are other topologies possible for \mathbb{R}^n . For example, one can prove that

$$\|v\|_1 = |v_1| + |v_2| + \cdots + |v_n|$$

gives a norm on \mathbb{R}^n and the theorems we proved transfer over almost without change by just trading $\|\cdot\|$ for $\|\cdot\|_1$. The unit "ball" becomes a diamond for the 1-norm. There are many other norms which can be constructed, infinitely many it turns out. However, it has been shown that the topology of all these different norms is equivalent. This means that open sets generated from different norms will be the same class of sets. For example, if you can fit an open disk around every point in a set then it's clear you can just as well fit an open diamond and vice-versa. One of the things that makes infinite dimensional linear algebra more fun is the fact that the topology generated by distinct norms need not be equivalent for infinite dimensions. There is a difference between the open sets generated by the Euclidean norm verses those generated by the 1-norm. Incidentally, my thesis work is mostly built over the 1-norm. It makes the supernumbers happy.

Chapter 2

linear algebra

Our goal in the first section of this chapter is to gain conceptual clarity on the meaning of the central terms from linear algebra. This is a birds-eye view of linear, my selection of topics here is centered around the goal of helping you to see the linear algebra in calculus. Once you see it then you can use the theory of linear algebra to help organize your thinking. Our ultimate goal is that organizational principle. Our goal here is not to learn all of linear algebra, rather we wish to use it as a tool for the right jobs as they arise this semester.

In the second section we summarize the tools of matrix computation. We will use matrix addition, multiplication and throughout this course. Inverse matrices and the noncommutative nature of matrix multiplication are illustrated. It is assumed that the reader has some previous experience in matrix computation, at least in highschool you should have spent some time.

In the third section the concept of a linear transformation is formalized. The formula for any linear transformation from \mathbb{R}^m to \mathbb{R}^m can be expressed as a matrix multiplication. We study this standard matrix in enough depth as to understand its application in for differentiation. A number of examples to visualize the role of a linear transformation are offered for breadth. Finally, isomorphisms and coordinate maps are discussed.

In the fourth section we define norms for vector spaces. We study how the norm allows us to define limits for an abstract vector space. This is important since it allows us to quantify continuity for abstract linear transformations as well as ultimately to define differentiation on a normed vector space in the chapter that follows.

2.1 vector spaces

Suppose we have a set V paired with an addition and scalar multiplication such that for all $x, y, z \in V$ and $a, b, c \in \mathbb{R}$:

$$\begin{array}{ll}
 (i.) & x + y = y + x, \\
 (iii.) & x + 0 = x, \\
 (v.) & 1x = x, \\
 (vii.) & a(x + y) = ax + ay, \\
 (ix.) & x + y \in \mathbb{R}^n \\
 (ii.) & (x + y) + z = x + (y + z) \\
 (iv.) & x - x = 0 \\
 (vi.) & (ab)x = a(bx), \\
 (viii.) & (a + b)x = ax + bx \\
 (x.) & cx \in \mathbb{R}^n
 \end{array}$$

then we say that V is a **vector space** over \mathbb{R} . To be a bit more precise, by (iii.) I mean to say that there exist some element $0 \in V$ such that $x + 0 = x$ for each $x \in V$. Also, (iv.) should be understood to say that for each $x \in V$ there exists another element $-x \in V$ such that $x + (-x) = 0$.

Example 2.1.1. \mathbb{R}^n is a vector space with respect to the standard vector addition and scalar multiplication.

Example 2.1.2. $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$ is a vector space where the usual complex number addition provides the vector addition and multiplication by a real number $c(a + ib) = ca + i(cb)$ clearly defines a scalar multiplication.

Example 2.1.3. The set of all $m \times n$ matrices is a vector space with respect to the usual matrix addition and scalar multiplication. We will elaborate on the details in an upcoming section.

Example 2.1.4. Suppose \mathcal{F} is the set of all functions from a set S to a vector space V then \mathcal{F} is naturally a vector space with respect to the function addition and multiplication by a scalar. Both of those operations are well-defined on the values of the function since we assumed the codomain of each function in \mathcal{F} is the vector space V .

There are many subspaces of function space which provide interesting examples of vector spaces. For example, the set of continuous functions:

Example 2.1.5. Let $C^0(\mathbb{R}^n, \mathbb{R}^m)$ be a set of continuous functions from \mathbb{R}^n to \mathbb{R}^m then $C^0(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space with respect to function addition and the usual multiplication. This fact relies on the sum and scalar multiple of continuous functions is once more continuous.

Definition 2.1.6.

We say a subset S of a vector space V is **linearly independent** (LI) iff for scalars c_1, c_2, \dots, c_k ,

$$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0 \quad \Rightarrow \quad c_1 = c_2 = \dots = 0$$

for each finite subset $\{v_1, v_2, \dots, v_k\}$ of S .

In the case that S is finite it suffices to show the implication for a linear combination of all the vectors in the set. Notice that if any vector in the set S can be written as a linear combination of the other vectors in S then that makes S fail the test for linear independence. Moreover, if a set S is not linearly independent then we say S is **linearly dependent**.

Example 2.1.7. The standard basis of \mathbb{R}^n is denoted $\{e_1, e_2, \dots, e_n\}$. We can show linear independence easily via the dot-product: suppose that $c_1e_1 + c_2e_2 + \dots + c_ne_n = 0$ and take the dot-product of both sides with e_j to obtain

$$(c_1e_1 + c_2e_2 + \dots + c_ne_n) \cdot e_j = 0 \cdot e_j \Rightarrow (c_1e_1 \cdot e_j + c_2e_2 \cdot e_j + \dots + c_ne_n \cdot e_j = 0 \Rightarrow c_j(1) = 0$$

but, j was arbitrary hence it follows that $c_1 = c_2 = \dots = c_n = 0$ which establishes the linear independence of the standard basis.

Example 2.1.8. Consider $S = \{1, i\} \subset \mathbb{C}$. We can argue S is LI as follows: suppose $c_1(1) + c_2(i) = 0$. Thus $c_1 + ic_2 = 0$ for some real numbers c_1, c_2 . Recall that a basic property of complex numbers is that if $z_1 = z_2$ then $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$ where $z_j = \text{Re}(z_j) + i\text{Im}(z_j)$. Therefore, the complex equation $c_1 + ic_2 = 0$ yields two real equations $c_1 = 0$ and $c_2 = 0$.

Example 2.1.9. Let $C^0(\mathbb{R})$ be the vector space of all continuous functions from \mathbb{R} to \mathbb{R} . Suppose S is the set of monic¹ monomials $S = \{1, t, t^2, t^3, \dots\}$. This is an infinite set. We can argue LI as follows: suppose $c_1t^{p_1} + c_2t^{p_2} + \dots + c_kt^{p_k} = 0$. For convenience relabel the powers p_1, p_2, \dots, p_k by $p_{i_1}, p_{i_2}, \dots, p_{i_k}$ such that $1 < p_{i_1} < p_{i_2} < \dots < p_{i_k}$. This notation just shuffles the terms in the finite sum around so that the first term has the lowest order: consider

$$\boxed{c_{i_1}t_{i_1}^{p_{i_1}} + c_{i_2}t_{i_2}^{p_{i_2}} + \dots + c_{i_k}t_{i_k}^{p_{i_k}} = 0} \quad \star$$

If $p_{i_1} = 0$ then evaluate \star at $t = 0$ to obtain $c_{i_1} = 0$. If $p_{i_1} > 0$ then differentiate \star p_{i_1} times and denote this new equation by $D^{p_{i_1}}\star$. Evaluate $D^{p_{i_1}}\star$ at $t = 0$ to find

$$p_{i_1}(p_{i_1} - 1) \dots 3 \cdot 2 \cdot 1 c_{i_1} = 0$$

hence $c_{i_1} = 0$. Since we set-up $p_{i_1} < p_{i_2}$ it follows that after p_{i_1} -differentiations the second summand is still nontrivial in $D^{p_{i_1}}\star$. However, we can continue differentiating \star until we reach $D^{p_{i_2}}\star$ and then constant term is $p_{i_2}!c_{i_2}$ so evaluation will show $c_{i_2} = 0$. We continue in this fashion until we have shown that $c_{i_j} = 0$ for $j = 1, 2, \dots, k$. It follows that S is a linearly independent set.

We spend considerable effort in linear algebra to understand LI from as many angles as possible. One equivalent formulation of LI is the ability to equate coefficients. In other words, a set of objects is LI iff whenever we have an equation with those objects we can equate coefficients. In calculus when we equate coefficients we implicitly assume that the functions in question are LI. Generally speaking two functions are LI if their graphs have distinct shapes which cannot be related by a simple vertical stretch.

Example 2.1.10. Consider $S = \{2^t, 3(1/2)^{-t}\}$ as a subset the vector space $C^0(\mathbb{R})$. To show linear dependence we observe that $c_12^t + c_23(1/2)^{-t} = 0$ yields $(c_1 + 3c_2)2^t = 0$. Hence $c_1 + 3c_2 = 0$ which means nontrivial solutions exist. Take $c_2 = 1$ then $c_1 = -3$. Of course the heart of the matter is that $3(1/2)^{-t} = 3(2^t)$ so the second function is just a scalar multiple of the first function.

¹monic means that the leading coefficient is 1.

If you've taken differential equations then you should recognize the concept of LI from your study of solution sets to differential equations. Given an n -th order linear differential equation we always have a goal of calculating n -LI solutions. In that context LI is important because it helps us make sure we do not count the same solution twice. The general solution is formed from a linear combination of the LI solution set. Of course this is not a course in differential equations, I include this comment to make connections to the other course. One last example on LI should suffice to make certain you at least have a good idea of the concept:

Example 2.1.11. Consider \mathbb{R}^3 as a vector space and consider the set $S = \{\vec{v}, \hat{i}, \hat{j}, \hat{k}\}$ where we could also denote $\hat{i} = e_1, \hat{j} = e_2, \hat{k} = e_3$ but I'm aiming to make your mind connect with your calculus III background. This set is clearly linearly dependent since we can write any vector \vec{v} as a linear combination of the standard unit-vectors: moreover, we can use dot-products to select the x, y and z components as follows:

$$\vec{v} = (\vec{v} \cdot \hat{i})\hat{i} + (\vec{v} \cdot \hat{j})\hat{j} + (\vec{v} \cdot \hat{k})\hat{k}$$

Linear independence helps us quantify a type of redundancy for vectors in a given set. The next definition is equally important and it is sort of the other side of the coin; spanning is a criteria which helps us insure a set of vectors will cover a vector space without missing anything.

Definition 2.1.12.

We say a subset S of a vector space V is a **spanning set** for V iff for each $v \in V$ there exist scalars c_1, c_2, \dots, c_k and vectors $v_1, v_2, \dots, v_k \in V$ such that $v = c_1v_1 + c_2v_2 + \dots + c_kv_k$. We denote $\text{Span}\{v_1, v_2, \dots, v_k\} = \{c_1v_1 + c_2v_2 + \dots + c_kv_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}$.

If $S \subset V$ and V is a vector space then it is immediately obvious that $\text{Span}(S) \subseteq V$. If S is a spanning set then it is obvious that $V \subseteq \text{Span}(S)$. It follows that when S is a spanning set for V we have $\text{Span}(S) = V$.

Example 2.1.13. It is easy to show that if $v \in \mathbb{R}^n$ then $v = v_1e_1 + v_2e_2 + \dots + v_ne_n$. It follows that $\mathbb{R}^n = \text{Span}\{e_i\}_{i=1}^n$.

Example 2.1.14. Let $1, i \in \mathbb{C}$ where $i^2 = -1$. Clearly $\mathbb{C} = \text{Span}\{1, i\}$.

Example 2.1.15. Let P be the set of polynomials. Since the sum of any two polynomials and the scalar multiple of any polynomial is once more a polynomial we find P is a vector space with respect to function addition and multiplication of a function by a scalar. We can argue that the set of monic monomials $\{1, t, t^2, \dots\}$ a spanning set for P . Why? Because if $f(t) \in P$ then that means there are scalars a_0, a_1, \dots, a_n such that $f(x) = a_0 + a_1t + a_2t^2 + \dots + a_nt^n$

Definition 2.1.16.

We say a subset β of a vector space V is a **basis** for V iff β is a linearly independent spanning set for V . If β is a finite set then V is said to be finite dimensional and the number of vectors in β is called the **dimension** of V . That is, if $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V then $\dim(V) = n$. If no finite basis exists for V then V is said to be **infinite dimensional**.

The careful reader will question why this concept of dimension is well-defined. Why can we not have bases of differing dimension for a given vector space? I leave this question for linear algebra, the theorem which asserts the uniqueness of dimension is one of the deeper theorems in the course. However, like most everything in linear, at some level it just boils down to solving some particular set of equations. You might tell Dr. Sprano it's just algebra. In any event, it is common practice to use the term **dimension** in courses where linear algebra is not understood. For example, \mathbb{R}^2 is a two-dimensional space. Or we'll say that \mathbb{R}^3 is a three-dimensional space. This terminology agrees with the general observation of the next example.

Example 2.1.17. *The standard basis $\{e_i\}_{i=1}^n$ for \mathbb{R}^n is a basis for \mathbb{R}^n and $\dim(\mathbb{R}^n) = n$. This result holds for all $n \in \mathbb{N}$. The line is one-dimensional, the plane is two-dimensional, three-space is three-dimensional etc...*

Example 2.1.18. *The set $\{1, i\}$ is a basis for \mathbb{C} . It follows that $\dim(\mathbb{C}) = 2$. We say that the complex numbers form a two-dimensional real vector space.*

Example 2.1.19. *The set of polynomials is clearly infinite dimensional. Contradiction shows this without much effort. Suppose P had a finite basis β . Choose the polynomial of largest degree (say k) in β . Notice that $f(t) = t^{k+1}$ is a polynomial and yet clearly $f(t) \notin \text{Span}(\beta)$ hence β is not a spanning set. But this contradicts the assumption β is a basis. Hence, by contradiction, no finite basis exists and we conclude the set of polynomials is infinite dimensional.*

There is a more general use of the term **dimension** which is beyond the context of linear algebra. For example, in calculus II or III you may have heard that a circle is one-dimensional or a surface is two-dimensional. Well, circles and surfaces are not usually vector spaces so the terminology is not taken from linear algebra. In fact, that use of the term dimension stems from manifold theory. I hope to discuss manifolds later in this course.

2.2 matrix calculation

An $m \times n$ matrix is an array of numbers with m -rows and n -columns. We define $\mathbb{R}^{m \times n}$ to be the set of all $m \times n$ matrices. The set of all n -dimensional column vectors is $\mathbb{R}^{n \times 1} = \mathbb{R}^n$. The set of all n -dimensional row vectors is $\mathbb{R}^{1 \times n}$. A given matrix $A \in \mathbb{R}^{m \times n}$ has mn -components A_{ij} . Notice that the components are numbers; $A_{ij} \in \mathbb{R}$ for all i, j such that $1 \leq i \leq m$ and $1 \leq j \leq n$. We should not write $A = A_{ij}$ because it is nonsense, however $A = [A_{ij}]$ is quite fine.

Suppose $A \in \mathbb{R}^{m \times n}$, note for $1 \leq j \leq n$ we have $\text{col}_j(A) \in \mathbb{R}^{m \times 1}$ whereas for $1 \leq i \leq m$ we find $\text{row}_i(A) \in \mathbb{R}^{1 \times n}$. In other words, an $m \times n$ matrix has n columns of length m and m rows of length n .

²We will use the convention that points in \mathbb{R}^n are column vectors. However, we will use the somewhat subtle notation $(x_1, x_2, \dots, x_n) = [x_1, x_2, \dots, x_n]^T$. This helps me write \mathbb{R}^n rather than $\mathbb{R}^{n \times 1}$ and I don't have to pepper transposes all over the place. If you've read my linear algebra notes you'll appreciate the wisdom of our convention.

Two matrices A and B are equal iff $A_{ij} = B_{ij}$ for all i, j . Given matrices A, B with components A_{ij}, B_{ij} and constant $c \in \mathbb{R}$ we define

$$(A + B)_{ij} = A_{ij} + B_{ij} \quad (cA)_{ij} = cA_{ij} \quad , \text{ for all } i, j.$$

The **zero matrix** in $\mathbb{R}^{m \times n}$ is denoted 0 and defined by $0_{ij} = 0$ for all i, j . The additive inverse of $A \in \mathbb{R}^{m \times n}$ is the matrix $-A$ such that $A + (-A) = 0$. The components of the additive inverse matrix are given by $(-A)_{ij} = -A_{ij}$ for all i, j . Likewise, if $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then the product $AB \in \mathbb{R}^{m \times p}$ is defined by³:

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$$

for each $1 \leq i \leq m$ and $1 \leq j \leq p$. In the case $m = p = 1$ the indices i, j are omitted in the equation since the matrix product is simply a number which needs no index. The identity matrix in $\mathbb{R}^{n \times n}$

is the $n \times n$ square matrix I whose components are the Kronecker delta; $I_{ij} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

The notation I_n is sometimes used. For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If the size of the identity matrix needs emphasis otherwise the size of the matrix I is to be understood from the context.

Let $A \in \mathbb{R}^{n \times n}$. If there exists $B \in \mathbb{R}^{n \times n}$ such that $AB = I$ and $BA = I$ then we say that A is **invertible** and $A^{-1} = B$. Invertible matrices are also called **nonsingular**. If a matrix has no inverse then it is called a **noninvertible** or **singular** matrix.

Let $A \in \mathbb{R}^{m \times n}$ then $A^T \in \mathbb{R}^{n \times m}$ is called the **transpose** of A and is defined by $(A^T)_{ji} = A_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. It is sometimes useful to know that $(AB)^T = B^T A^T$ and $(A^T)^{-1} = (A^{-1})^T$. It is also true that $(AB)^{-1} = B^{-1} A^{-1}$. Furthermore, note **dot-product** of $v, w \in V^n$ is given by $v \cdot w = v^T w$.

The ij -th **standard basis matrix** for $\mathbb{R}^{m \times n}$ is denoted E_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$. The matrix E_{ij} is zero in all entries except for the (i, j) -th slot where it has a 1. In other words, we define $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$. I invite the reader to show that the term basis is justified in this context⁴.

Given this basis we see that the vector space $\mathbb{R}^{m \times n}$ has $\dim(\mathbb{R}^{m \times n}) = mn$.

Theorem 2.2.1.

³this product is defined so the matrix of the composite of a linear transformation is the product of the matrices of the composed transformations. This is illustrated later in this section and is proved in my linear algebra notes.

⁴the theorem stated below contains the needed results and then some, you can find the proof is given in my linear algebra notes. It would be wise to just work it out in the 2×2 case as a warm-up if you are interested

If $A \in \mathbb{R}^{m \times n}$ and $v \in \mathbb{R}$ then

$$\begin{aligned}
 (i.) \quad v &= \sum_{i=1}^n v_n e_n & (ii.) \quad A &= \sum_{i=1}^m \sum_{j=1}^n A_{ij} E_{ij} \\
 (iii.) \quad [e_i^T A] &= \text{row}_i(A) & (iv.) \quad [A e_i] &= \text{col}_i(A) \\
 (v.) \quad E_{ij} E_{kl} &= \delta_{jk} E_{il} & (vi.) \quad E_{ij} &= e_i e_j^T & (vii.) \quad e_i^T e_j &= e_i \cdot e_j = \delta_{ij}.
 \end{aligned}$$

You can look in my linear algebra notes for the details of the theorem. I'll just expand one point here: Let $A \in \mathbb{R}^{m \times n}$ then

$$\begin{aligned}
 A &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \\
 &= A_{11} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} + A_{12} \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + A_{mn} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\
 &= A_{11} E_{11} + A_{12} E_{12} + \cdots + A_{mn} E_{mn}.
 \end{aligned}$$

The calculation above follows from repeated mn -applications of the definition of matrix addition and another mn -applications of the definition of scalar multiplication of a matrix.

Example 2.2.2. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. We see that A has 2 rows and 3 columns thus $A \in \mathbb{R}^{2 \times 3}$. Moreover, $A_{11} = 1$, $A_{12} = 2$, $A_{13} = 3$, $A_{21} = 4$, $A_{22} = 5$, and $A_{23} = 6$. It's not usually possible to find a formula for a generic element in the matrix, but this matrix satisfies $A_{ij} = 3(i-1) + j$ for all i, j ⁵. The columns of A are,

$$\text{col}_1(A) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \text{col}_2(A) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \text{col}_3(A) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

The rows of A are

$$\text{row}_1(A) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad \text{row}_2(A) = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

Example 2.2.3. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. Notice that

$$\text{row}_1(A) = \text{col}_1(A^T), \quad \text{row}_2(A) = \text{col}_2(A^T)$$

⁵In the statement "for all i, j " it is to be understood that those indices range over their allowed values. In the preceding example $1 \leq i \leq 2$ and $1 \leq j \leq 3$.

and

$$\text{col}_1(A) = \text{row}_1(A^T), \text{col}_2(A) = \text{row}_2(A^T), \text{col}_3(A) = \text{row}_3(A^T)$$

Notice $(A^T)_{ij} = A_{ji} = 3(j-1) + i$ for all i, j ; at the level of index calculations we just switch the indices to create the transpose.

Example 2.2.4. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. We calculate

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}.$$

Example 2.2.5. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. We calculate

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}.$$

Now multiply A by the scalar 5,

$$5A = 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

Example 2.2.6. Let $A, B \in \mathbb{R}^{m \times n}$ be defined by $A_{ij} = 3i + 5j$ and $B_{ij} = i^2$ for all i, j . Then we can calculate $(A + B)_{ij} = 3i + 5j + i^2$ for all i, j .

Example 2.2.7. Solve the following matrix equation,

$$0 = \begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x-1 & y-2 \\ z-3 & w-4 \end{bmatrix}$$

The definition of matrix equality means this single matrix equation reduces to 4 scalar equations: $0 = x - 1, 0 = y - 2, 0 = z - 3, 0 = w - 4$. The solution is $x = 1, y = 2, z = 3, w = 4$.

The definition of matrix multiplication $((AB)_{ij} = \sum_{k=1}^n A_{ik}B_{kj})$ is very nice for general proofs, but pragmatically I usually think of matrix multiplication in terms of *dot-products*. It turns out we can view the matrix product as a collection of dot-products: suppose $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then

$$AB = \begin{bmatrix} \text{row}_1(A) \cdot \text{col}_1(B) & \text{row}_1(A) \cdot \text{col}_2(B) & \cdots & \text{row}_1(A) \cdot \text{col}_p(B) \\ \text{row}_2(A) \cdot \text{col}_1(B) & \text{row}_2(A) \cdot \text{col}_2(B) & \cdots & \text{row}_2(A) \cdot \text{col}_p(B) \\ \vdots & \vdots & \cdots & \vdots \\ \text{row}_m(A) \cdot \text{col}_1(B) & \text{row}_m(A) \cdot \text{col}_2(B) & \cdots & \text{row}_m(A) \cdot \text{col}_p(B) \end{bmatrix}$$

Let me explain how this works. The formula above claims $(AB)_{ij} = \text{row}_i(A) \cdot \text{col}_j(B)$ for all i, j . Recall that $(\text{row}_i(A))_k = A_{ik}$ and $(\text{col}_j(B))_k = B_{kj}$ thus

$$(AB)_{ij} = \sum_{k=1}^n A_{ik}B_{kj} = \sum_{k=1}^n (\text{row}_i(A))_k (\text{col}_j(B))_k$$

Hence, using definition of the dot-product, $(AB)_{ij} = \text{row}_i(A) \cdot \text{col}_j(B)$. This argument holds for all i, j therefore the dot-product formula for matrix multiplication is valid.

Example 2.2.8. *The product of a 3×2 and 2×3 is a 3×3*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} [1,0][4,7]^T & [1,0][5,8]^T & [1,0][6,9]^T \\ [0,1][4,7]^T & [0,1][5,8]^T & [0,1][6,9]^T \\ [0,0][4,7]^T & [0,0][5,8]^T & [0,0][6,9]^T \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2.2.9. *The product of a 3×1 and 1×3 is a 3×3*

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 & 5 \cdot 1 & 6 \cdot 1 \\ 4 \cdot 2 & 5 \cdot 2 & 6 \cdot 2 \\ 4 \cdot 3 & 5 \cdot 3 & 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

Example 2.2.10. *Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. We calculate*

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} [1,2][5,7]^T & [1,2][6,8]^T \\ [3,4][5,7]^T & [3,4][6,8]^T \end{bmatrix} \\ &= \begin{bmatrix} 5 + 14 & 6 + 16 \\ 15 + 28 & 18 + 32 \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \end{aligned}$$

Notice the product of square matrices is square. For numbers $a, b \in \mathbb{R}$ it we know the product of a and b is commutative ($ab = ba$). Let's calculate the product of A and B in the opposite order,

$$\begin{aligned} BA &= \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} [5,6][1,3]^T & [5,6][2,4]^T \\ [7,8][1,3]^T & [7,8][2,4]^T \end{bmatrix} \\ &= \begin{bmatrix} 5 + 18 & 10 + 24 \\ 7 + 24 & 14 + 32 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix} \end{aligned}$$

*Clearly $AB \neq BA$ thus matrix multiplication is **noncommutative** or **nonabelian**.*

When we say that matrix multiplication is noncommutative that indicates that the product of two matrices does not *generally* commute. However, there are special matrices which commute with other matrices.

Example 2.2.11. Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We calculate

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Likewise calculate,

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since the matrix A was arbitrary we conclude that $IA = AI$ for all $A \in \mathbb{R}^{2 \times 2}$.

Example 2.2.12. Consider A, v, w from Example ??.

$$v + w = \begin{bmatrix} 5 \\ 7 \end{bmatrix} + \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \end{bmatrix}$$

Using the above we calculate,

$$A(v + w) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ 15 \end{bmatrix} = \begin{bmatrix} 11 + 30 \\ 33 + 60 \end{bmatrix} = \begin{bmatrix} 41 \\ 93 \end{bmatrix}.$$

In contrast, we can add Av and Aw ,

$$Av + Aw = \begin{bmatrix} 19 \\ 43 \end{bmatrix} + \begin{bmatrix} 22 \\ 50 \end{bmatrix} = \begin{bmatrix} 41 \\ 93 \end{bmatrix}.$$

Behold, $A(v + w) = Av + Aw$ for this example. It turns out this is true in general.

I collect all my favorite properties for matrix multiplication in the theorem below. To summarize, matrix math works as you would expect with the exception that matrix multiplication is not commutative. We must be careful about the order of letters in matrix expressions.

Theorem 2.2.13.

If $A, B, C \in \mathbb{R}^{m \times n}$, $X, Y \in \mathbb{R}^{n \times p}$, $Z \in \mathbb{R}^{p \times q}$ and $c_1, c_2 \in \mathbb{R}$ then

1. $(A + B) + C = A + (B + C)$,
2. $(AX)Z = A(XZ)$,
3. $A + B = B + A$,
4. $c_1(A + B) = c_1A + c_2B$,
5. $(c_1 + c_2)A = c_1A + c_2A$,
6. $(c_1c_2)A = c_1(c_2A)$,
7. $(c_1A)X = c_1(AX) = A(c_1X) = (AX)c_1$,
8. $1A = A$,
9. $I_m A = A = A I_n$,
10. $A(X + Y) = AX + AY$,
11. $A(c_1X + c_2Y) = c_1AX + c_2AY$,
12. $(A + B)X = AX + BX$,

Proof: I will prove a couple of these primarily to give you a chance to test your understanding of the notation. Nearly all of these properties are proved by breaking the statement down to components then appealing to a property of real numbers. Just a reminder, we assume that it is known that \mathbb{R} is an ordered field. Multiplication of real numbers is commutative, associative and distributes across addition of real numbers. Likewise, addition of real numbers is commutative, associative and obeys familiar distributive laws when combined with addition.

Proof of (1.): assume A, B, C are given as in the statement of the Theorem. Observe that

$$\begin{aligned}
 ((A + B) + C)_{ij} &= (A + B)_{ij} + C_{ij} && \text{defn. of matrix add.} \\
 &= (A_{ij} + B_{ij}) + C_{ij} && \text{defn. of matrix add.} \\
 &= A_{ij} + (B_{ij} + C_{ij}) && \text{assoc. of real numbers} \\
 &= A_{ij} + (B + C)_{ij} && \text{defn. of matrix add.} \\
 &= (A + (B + C))_{ij} && \text{defn. of matrix add.}
 \end{aligned}$$

for all i, j . Therefore $(A + B) + C = A + (B + C)$. \square

Proof of (6.): assume c_1, c_2, A are given as in the statement of the Theorem. Observe that

$$\begin{aligned}
 ((c_1c_2)A)_{ij} &= (c_1c_2)A_{ij} && \text{defn. scalar multiplication.} \\
 &= c_1(c_2A_{ij}) && \text{assoc. of real numbers} \\
 &= (c_1(c_2A))_{ij} && \text{defn. scalar multiplication.}
 \end{aligned}$$

for all i, j . Therefore $(c_1 c_2)A = c_1(c_2 A)$. \square

Proof of (10.): assume A, X, Y are given as in the statement of the Theorem. Observe that

$$\begin{aligned}
 ((A(X + Y))_{ij} &= \sum_k A_{ik}(X + Y)_{kj} && \text{defn. matrix multiplication,} \\
 &= \sum_k A_{ik}(X_{kj} + Y_{kj}) && \text{defn. matrix addition,} \\
 &= \sum_k (A_{ik}X_{kj} + A_{ik}Y_{kj}) && \text{dist. of real numbers,} \\
 &= \sum_k A_{ik}X_{kj} + \sum_k A_{ik}Y_{kj} && \text{prop. of finite sum,} \\
 &= (AX)_{ij} + (AY)_{ij} && \text{defn. matrix multiplication}(\times 2), \\
 &= (AX + AY)_{ij} && \text{defn. matrix addition,}
 \end{aligned}$$

for all i, j . Therefore $A(X + Y) = AX + AY$. \square

The proofs of the other items are similar, we consider the i, j -th component of the identity and then apply the definition of the appropriate matrix operation's definition. This reduces the problem to a statement about real numbers so we can use the properties of real numbers at the level of components. Then we reverse the steps. Since the calculation works for arbitrary i, j it follows the matrix equation holds true.

2.3 linear transformations

Definition 2.3.1.

Let V and W be vector spaces over \mathbb{R} . If a mapping $L : V \rightarrow W$ satisfies

1. $L(x + y) = L(x) + L(y)$ for all $x, y \in V$; this is called **additivity**.
2. $L(cx) = cL(x)$ for all $x \in V$ and $c \in \mathbb{R}$; this is called **homogeneity**.

then we say L is a linear transformation. If $V = W$ then we may say that L is a linear transformation on V .

Proposition 2.3.2.

If $A \in \mathbb{R}^{m \times n}$ and $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined by $L(x) = Ax$ for each $x \in \mathbb{R}^n$ then L is a linear transformation.

Proof: Let $A \in \mathbb{R}^{m \times n}$ and define $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $L(x) = Ax$ for each $x \in \mathbb{R}^n$. Let $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$L(x + y) = A(x + y) = Ax + Ay = L(x) + L(y)$$

and

$$L(cx) = A(cx) = cAx = cL(x)$$

thus L is a linear transformation. \square

Obviously this gives us a nice way to construct examples. The following proposition is really at the heart of all the geometry in this section.

Proposition 2.3.3.

Let $\mathcal{L} = \{p + tv \mid t \in [0, 1], p, v \in \mathbb{R}^n \text{ with } v \neq 0\}$ define a line segment from p to $p + v$ in \mathbb{R}^n . If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation then $T(\mathcal{L})$ is either a line-segment from $T(p)$ to $T(p + v)$ or a point.

Proof: suppose T and \mathcal{L} are as in the proposition. Let $y \in T(\mathcal{L})$ then by definition there exists $x \in \mathcal{L}$ such that $T(x) = y$. But this implies there exists $t \in [0, 1]$ such that $x = p + tv$ so $T(p + tv) = y$. Notice that

$$y = T(p + tv) = T(p) + T(tv) = T(p) + tT(v).$$

which implies $y \in \{T(p) + sT(v) \mid s \in [0, 1]\} = \mathcal{L}_2$. Therefore, $T(\mathcal{L}) \subseteq \mathcal{L}_2$. Conversely, suppose $z \in \mathcal{L}_2$ then $z = T(p) + sT(v)$ for some $s \in [0, 1]$ but this yields by linearity of T that $z = T(p + sv)$ hence $z \in T(\mathcal{L})$. Since we have that $T(\mathcal{L}) \subseteq \mathcal{L}_2$ and $\mathcal{L}_2 \subseteq T(\mathcal{L})$ it follows that $T(\mathcal{L}) = \mathcal{L}_2$. Note that \mathcal{L}_2 is a line-segment provided that $T(v) \neq 0$, however if $T(v) = 0$ then $\mathcal{L}_2 = \{T(p)\}$ and the proposition follows. \square

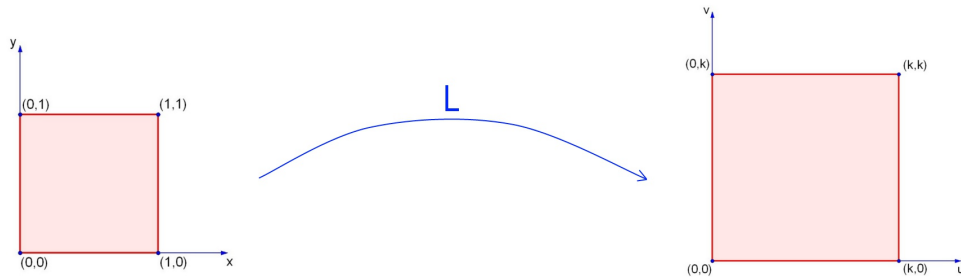
2.3.1 a gallery of linear transformations

My choice of mapping the unit square has no particular significance in the examples below. I merely wanted to keep it simple and draw your eye to the distinction between the examples. In each example we'll map the four corners of the square to see where the transformation takes the unit-square. Those corners are simply $(0, 0), (1, 0), (1, 1), (0, 1)$ as we traverse the square in a counter-clockwise direction.

Example 2.3.4. Let $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ for some $k > 0$. Define $L(v) = Av$ for all $v \in \mathbb{R}^2$. In particular this means,

$$L(x, y) = A(x, y) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}.$$

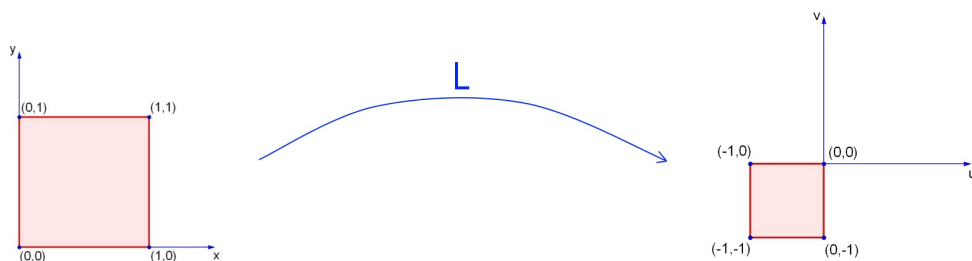
We find $L(0, 0) = (0, 0)$, $L(1, 0) = (k, 0)$, $L(1, 1) = (k, k)$, $L(0, 1) = (0, k)$. This mapping is called a **dilation**.



Example 2.3.5. Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^2$. In particular this means,

$$L(x, y) = A(x, y) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

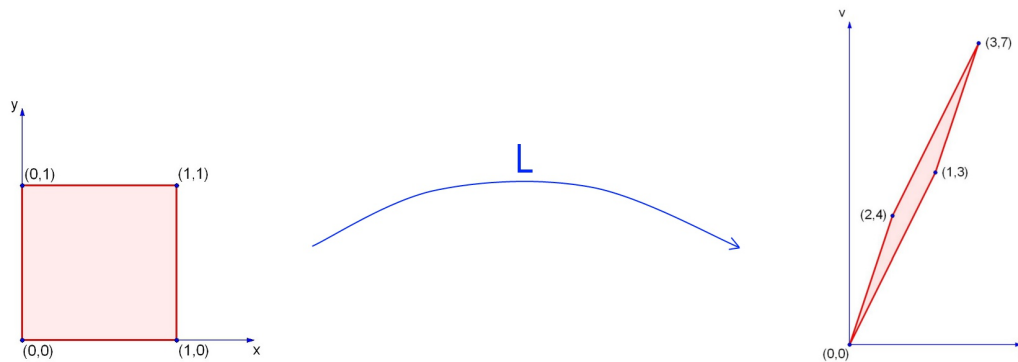
We find $L(0, 0) = (0, 0)$, $L(1, 0) = (-1, 0)$, $L(1, 1) = (-1, -1)$, $L(0, 1) = (0, -1)$. This mapping is called an **inversion**.



Example 2.3.6. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^2$. In particular this means,

$$L(x, y) = A(x, y) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix}.$$

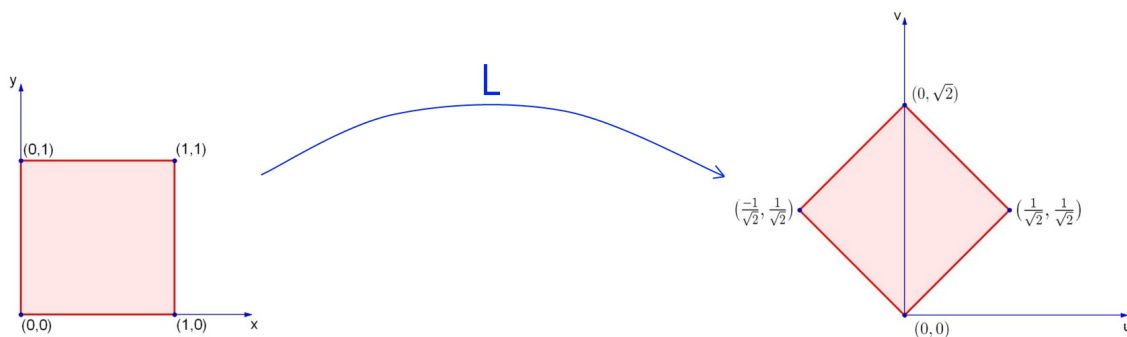
We find $L(0, 0) = (0, 0)$, $L(1, 0) = (1, 3)$, $L(1, 1) = (3, 7)$, $L(0, 1) = (2, 4)$. This mapping shall remain nameless, it is doubtless a combination of the other named mappings.



Example 2.3.7. Let $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^2$. In particular this means,

$$L(x, y) = A(x, y) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x - y \\ x + y \end{bmatrix}.$$

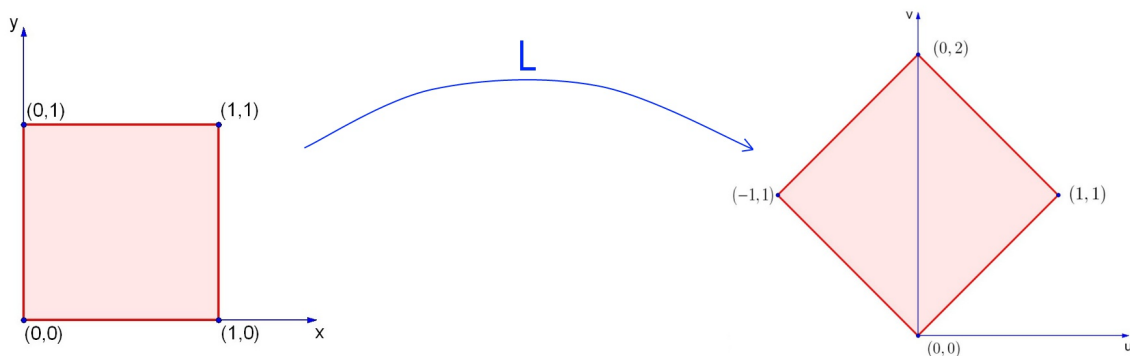
We find $L(0, 0) = (0, 0)$, $L(1, 0) = \frac{1}{\sqrt{2}}(1, 1)$, $L(1, 1) = \frac{1}{\sqrt{2}}(0, 2)$, $L(0, 1) = \frac{1}{\sqrt{2}}(-1, 1)$. This mapping is a rotation by $\pi/4$ radians.



Example 2.3.8. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^2$. In particular this means,

$$L(x, y) = A(x, y) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \end{bmatrix}.$$

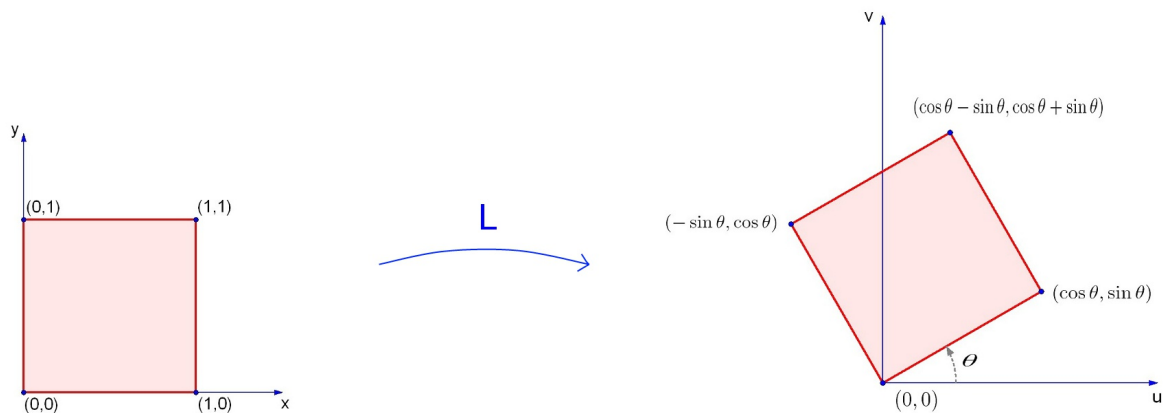
We find $L(0, 0) = (0, 0)$, $L(1, 0) = (1, 1)$, $L(1, 1) = (0, 2)$, $L(0, 1) = (-1, 1)$. This mapping is a rotation followed by a dilation by $k = \sqrt{2}$.



Example 2.3.9. Let $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^2$. In particular this means,

$$L(x, y) = A(x, y) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos(\theta) - y \sin(\theta) \\ x \sin(\theta) + y \cos(\theta) \end{bmatrix}.$$

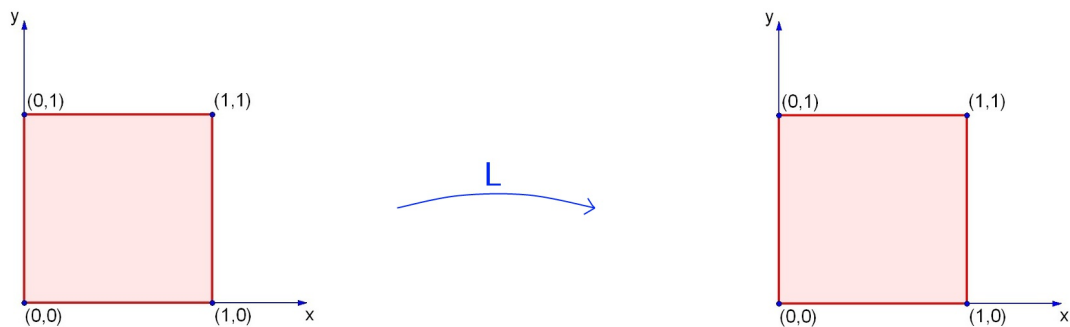
We find $L(0, 0) = (0, 0)$, $L(1, 0) = (\cos(\theta), \sin(\theta))$, $L(1, 1) = (\cos(\theta) - \sin(\theta), \cos(\theta) + \sin(\theta))$, $L(0, 1) = (\sin(\theta), \cos(\theta))$. This mapping is a rotation by θ in the counter-clockwise direction. Of course you could have derived the matrix A from the picture below.



Example 2.3.10. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^2$. In particular this means,

$$L(x, y) = A(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We find $L(0, 0) = (0, 0)$, $L(1, 0) = (1, 0)$, $L(1, 1) = (1, 1)$, $L(0, 1) = (0, 1)$. This mapping is a rotation by zero radians, or you could say it is a dilation by a factor of 1, ... usually we call this the identity mapping because the image is identical to the preimage.



Example 2.3.11. Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Define $P_1(v) = A_1 v$ for all $v \in \mathbb{R}^2$. In particular this means,

$$P_1(x, y) = A_1(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

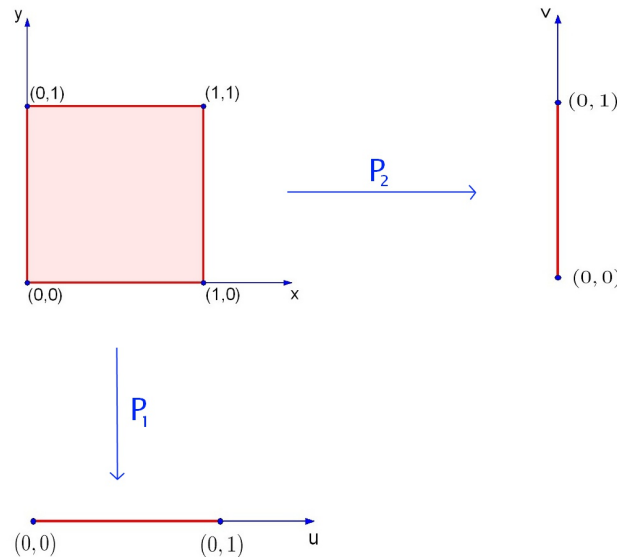
We find $P_1(0, 0) = (0, 0)$, $P_1(1, 0) = (1, 0)$, $P_1(1, 1) = (1, 0)$, $P_1(0, 1) = (0, 0)$. This mapping is a **projection** onto the first coordinate.

Let $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Define $P_2(v) = A_2 v$ for all $v \in \mathbb{R}^2$. In particular this means,

$$P_2(x, y) = A_2(x, y) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

We find $P_2(0, 0) = (0, 0)$, $P_2(1, 0) = (0, 0)$, $P_2(1, 1) = (0, 1)$, $P_2(0, 1) = (0, 1)$. This mapping is a **projection** onto the second coordinate.

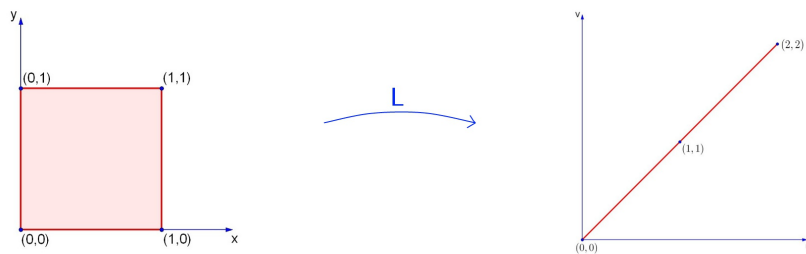
We can picture both of these mappings at once:



Example 2.3.12. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^2$. In particular this means,

$$L(x, y) = A(x, y) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x + y \end{bmatrix}.$$

We find $L(0, 0) = (0, 0)$, $L(1, 0) = (1, 1)$, $L(1, 1) = (2, 2)$, $L(0, 1) = (1, 1)$. This mapping is not a projection, but it does collapse the square to a line-segment.

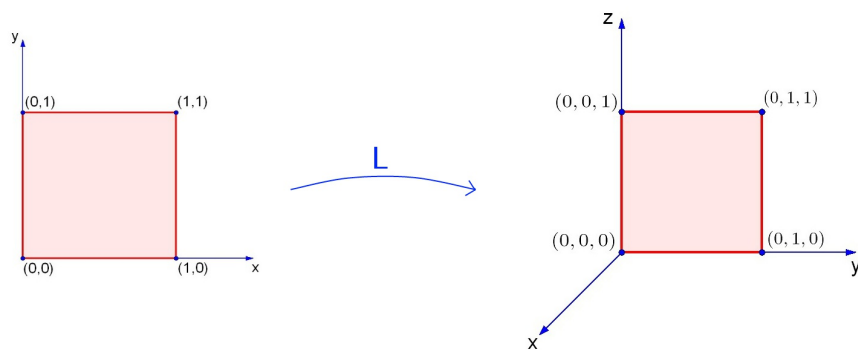
**Remark 2.3.13.**

The examples here have focused on linear transformations from \mathbb{R}^2 to \mathbb{R}^2 . It turns out that higher dimensional mappings can largely be understood in terms of the geometric operations we've seen in this section.

Example 2.3.14. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^2$. In particular this means,

$$L(x, y) = A(x, y) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ y \end{bmatrix}.$$

We find $L(0, 0) = (0, 0, 0)$, $L(1, 0) = (0, 1, 0)$, $L(1, 1) = (0, 1, 1)$, $L(0, 1) = (0, 0, 1)$. This mapping moves the xy -plane to the yz -plane. In particular, the horizontal unit square gets mapped to vertical unit square; $L([0, 1] \times [0, 1]) = \{0\} \times [0, 1] \times [0, 1]$. This mapping certainly is not surjective because no point with $x \neq 0$ is covered in the range.



Example 2.3.15. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^3$. In particular this means,

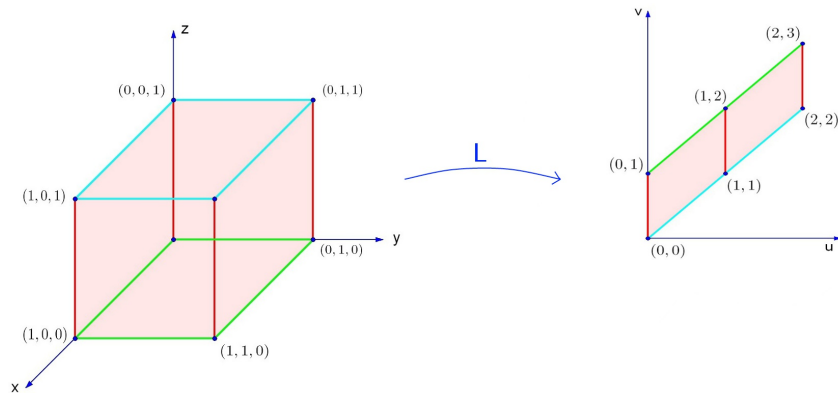
$$L(x, y, z) = A(x, y, z) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x + y + z \end{bmatrix}.$$

Let's study how L maps the unit cube. We have $2^3 = 8$ corners on the unit cube,

$$L(0,0,0) = (0,0), \quad L(1,0,0) = (1,1), \quad L(1,1,0) = (2,2), \quad L(0,1,0) = (1,1)$$

$$L(0,0,1) = (0,1), \quad L(1,0,1) = (1,2), \quad L(1,1,1) = (2,3), \quad L(0,1,1) = (1,2).$$

This mapping squished the unit cube to a shape in the plane which contains the points $(0,0)$, $(0,1)$, $(1,1)$, $(1,2)$, $(2,2)$, $(2,3)$. Face by face analysis of the mapping reveals the image is a parallelogram. This mapping is certainly not injective since two different points get mapped to the same point. In particular, I have color-coded the mapping of top and base faces as they map to line segments. The vertical faces map to one of the two parallelograms that comprise the image.

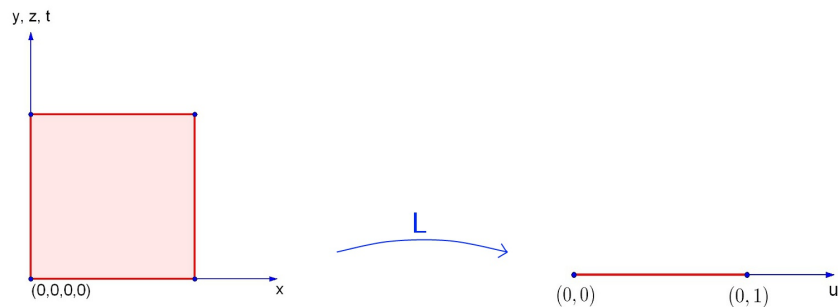


I have used terms like "vertical" or "horizontal" in the standard manner we associate such terms with three dimensional geometry. Visualization and terminology for higher-dimensional examples is not as obvious. However, with a little imagination we can still draw pictures to capture important aspects of mappings.

Example 2.3.16. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Define $L(v) = Av$ for all $v \in \mathbb{R}^4$. In particular this means,

$$L(x, y, z, t) = A(x, y, z, t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}.$$

Let's study how L maps the unit hypercube $[0,1]^4 \subset \mathbb{R}^4$. We have $2^4 = 16$ corners on the unit hypercube, note $L(1,a,b,c) = (1,1)$ whereas $L(0,a,b,c) = (0,0)$ for all $a,b,c \in [0,1]$. Therefore, the unit hypercube is squished to a line-segment from $(0,0)$ to $(1,1)$. This mapping is neither surjective nor injective. In the picture below the vertical axis represents the y, z, t -directions.



Example 2.3.17. Suppose $f(t, s) = (\sqrt{t}, s^2 + t)$ note that $f(1, 1) = (1, 2)$ and $f(4, 4) = (2, 20)$. Note that $(4, 4) = 4(1, 1)$ thus we should see $f(4, 4) = f(4(1, 1)) = 4f(1, 1)$ but that fails to be true so f is not a linear transformation.

Example 2.3.18. Let $L(x, y) = x^2 + y^2$ define a mapping from \mathbb{R}^2 to \mathbb{R} . This is not a linear transformation since

$$L(c(x, y)) = L(cx, cy) = (cx)^2 + (cy)^2 = c^2(x^2 + y^2) = c^2L(x, y).$$

We say L is a nonlinear transformation.

Example 2.3.19. Suppose $L : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $L(x) = mx + b$ for some constants $m, b \in \mathbb{R}$. Is this a linear transformation on \mathbb{R} ? Observe:

$$L(0) = m(0) + b = b$$

thus L is not a linear transformation if $b \neq 0$. On the other hand, if $b = 0$ then L is a linear transformation.

A mapping on \mathbb{R}^n which has the form $T(x) = x + b$ is called a **translation**. If we have a mapping of the form $F(x) = Ax + b$ for some $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}$ then we say F is an **affine transformation** on \mathbb{R}^n . Technically, in general, the line $y = mx + b$ is the graph of an affine function on \mathbb{R} . I invite the reader to prove that affine transformations also map line-segments to line-segments (or points).

2.3.2 standard matrices

Definition 2.3.20.

Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, the matrix $A \in \mathbb{R}^{m \times n}$ such that $L(x) = Ax$ for all $x \in \mathbb{R}^n$ is called the **standard matrix** of L . We denote this by $[L] = A$ or more compactly, $[L_A] = A$, we say that L_A is the linear transformation induced by A . Moreover, the components of the matrix A are found from $A_{ji} = (L(e_i))_j$.

Example 2.3.21. Given that $L([x, y, z]^T) = [x+2y, 3y+4z, 5x+6z]^T$ for $[x, y, z]^T \in \mathbb{R}^3$ find the standard matrix of L . We wish to find a 3×3 matrix such that $L(v) = Av$ for all $v = [x, y, z]^T \in \mathbb{R}^3$. Write $L(v)$ then collect terms with each coordinate in the domain,

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 3y+4z \\ 5x+6z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$$

It's not hard to see that,

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow A = [L] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{bmatrix}$$

Notice that the columns in A are just as you'd expect from the proof of theorem ???. $[L] = [L(e_1)|L(e_2)|L(e_3)]$. In future examples I will exploit this observation to save writing.

Example 2.3.22. Suppose that $L((t, x, y, z)) = (t + x + y + z, z - x, 0, 3t - z)$, find $[L]$.

$$\begin{aligned} L(e_1) &= L((1, 0, 0, 0)) = (1, 0, 0, 3) \\ L(e_2) &= L((0, 1, 0, 0)) = (1, -1, 0, 0) \\ L(e_3) &= L((0, 0, 1, 0)) = (1, 0, 0, 0) \\ L(e_4) &= L((0, 0, 0, 1)) = (1, 1, 0, -1) \end{aligned} \Rightarrow [L] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -1 \end{bmatrix}.$$

I invite the reader to check my answer here and see that $L(v) = [L]v$ for all $v \in \mathbb{R}^4$ as claimed.

Proposition 2.3.23.

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are linear transformations then $S + T$ and cT are linear transformations and

$$(1.) [T + S] = [T] + [S], \quad (2.) [T - S] = [T] - [S], \quad (3.) [cT] = c[T].$$

In words, the standard matrix of the sum, difference or scalar multiple of linear transformations is the sum, difference or scalar multiple of the standard matrices of the respective linear transformations.

Example 2.3.24. Suppose $T(x, y) = (x + y, x - y)$ and $S(x, y) = (2x, 3y)$. It's easy to see that

$$[T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } [S] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow [T + S] = [T] + [S] = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Therefore, $(T + S)(x, y) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + y \\ x + 2y \end{bmatrix} = (3x + y, x + 2y)$. Naturally this is the same formula that we would obtain through direct addition of the formulas of T and S .

Proposition 2.3.25.

$L_1 : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $L_2 : \mathbb{R}^n \rightarrow \mathbb{R}^p$ are linear transformations then $L_2 \circ L_1 : \mathbb{R}^m \rightarrow \mathbb{R}^p$ is a linear transformation with matrix $[L_2 \circ L_1]$ such that

$$[L_2 \circ L_1]_{ij} = \sum_{k=1}^n [L_2]_{ik} [L_1]_{kj}$$

for all $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, m$.

Example 2.3.26. Let $T : \mathbb{R}^{2 \times 1} \rightarrow \mathbb{R}^{2 \times 1}$ be defined by

$$T([x, y]^T) = [x + y, 2x - y]^T$$

for all $[x, y]^T \in \mathbb{R}^{2 \times 1}$. Also let $S : \mathbb{R}^{2 \times 1} \rightarrow \mathbb{R}^{3 \times 1}$ be defined by

$$S([x, y]^T) = [x, x, 3x + 4y]^T$$

for all $[x, y]^T \in \mathbb{R}^{2 \times 1}$. We calculate the composite as follows:

$$\begin{aligned} (S \circ T)([x, y]^T) &= S(T([x, y]^T)) \\ &= S([x + y, 2x - y]^T) \\ &= [x + y, x + y, 3(x + y) + 4(2x - y)]^T \\ &= [x + y, x + y, 11x - y]^T \end{aligned}$$

Notice we can write the formula above as a matrix multiplication,

$$(S \circ T)([x, y]^T) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 11 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow [S \circ T] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 11 & -1 \end{bmatrix}.$$

Notice that the standard matrices of S and T are:

$$[S] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 3 & 4 \end{bmatrix} \quad [T] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

It's easy to see that $[S \circ T] = [S][T]$ (as we should expect since these are linear operators)

Notice that $T \circ S$ is not even defined since the dimensions of the codomain of S do not match the domain of T . Likewise, the matrix product $[T][S]$ is not defined since there is a dimension mismatch; $(2 \times 2)(3 \times 2)$ is not a well-defined product of matrices.

2.3.3 coordinates and isomorphism

Let V be a finite dimensional vector space with basis $\beta = \{v_1, v_2, \dots, v_n\}$. The **coordinate map** $\Phi_\beta : V \rightarrow \mathbb{R}^n$ is defined by

$$\Phi_\beta(x_1v_1 + x_2v_2 + \dots + x_nv_n) = x_1e_1 + x_2e_2 + \dots + x_ne_n$$

for all $v = x_1v_1 + x_2v_2 + \dots + x_nv_n \in V$. Sometimes we have to adjust the numbering a bit for double-indices. For example:

Example 2.3.27. Let $\Phi : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{mn}$ be defined by

$$\Phi\left(\sum_{i,j} A_{ij}E_{ij}\right) = (A_{11}, \dots, A_{1n}, A_{21}, \dots, A_{2n}, \dots, A_{m1}, \dots, A_{mn})$$

This map simply takes the entries in the matrix and strings them out to a vector of length mn .

Example 2.3.28. Let $\Psi : \mathbb{C} \rightarrow \mathbb{R}^2$ be defined by $\Psi(x + iy) = (x, y)$. This is the coordinate map for the basis $\{1, i\}$.

Matrix multiplication is for vectors in \mathbb{R}^n . Direct matrix multiplication of an abstract vector makes no sense (how would you multiply a polynomial and a matrix?), however, since we can use the coordinate map to change the abstract vector to a vector in \mathbb{R}^n . The diagram below illustrates the idea for a linear transformation T from an abstract vector space V with basis β to another abstract vector space W with basis $\bar{\beta}$:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \Phi_\beta^{-1} \uparrow & & \downarrow \Phi_{\bar{\beta}} \\ \mathbb{R}^n & \xrightarrow{L_{[T]_{\beta, \bar{\beta}}}} & \mathbb{R}^n \end{array}$$

Let's walk through the formula $[T]_{\beta, \bar{\beta}} x = \Phi_{\bar{\beta}}(T(\Phi_\beta^{-1}(x)))$: we begin on the RHS with a column vector x , then Φ_β^{-1} lifts the column vector up to the abstract vector $\Phi_\beta^{-1}(x)$ in V . Next we operate by T which moves us over to the vector $T(\Phi_\beta^{-1}(x))$ which is in W . Finally the coordinate map $\Phi_{\bar{\beta}}$ pushes the abstract vector in W back to a column vector $\Phi_{\bar{\beta}}(T(\Phi_\beta^{-1}(x)))$ which is in $\mathbb{R}^{m \times 1}$. The same journey is accomplished by just multiplying x by the $m \times n$ matrix $[T]_{\beta, \bar{\beta}}$.

Example 2.3.29. Let $\beta = \{1, x, x^2\}$ be the basis for P_2 and consider the derivative mapping $D : P_2 \rightarrow P_2$. Find the matrix of D assuming that P_2 has coordinates with respect to β on both copies of P_2 . Define and observe

$$\Phi(x^n) = e_{n+1} \quad \text{whereas} \quad \Phi^{-1}(e_n) = x^{n-1}$$

for $n = 0, 1, 2$. Recall $D(ax^2 + bx + c) = 2ax + bx$.

$$\begin{aligned} \text{col}_1([D]_{\beta,\beta}) &= \Phi_\beta(D(\Phi_\beta^{-1}(e_1))) = \Phi_\beta(D(1)) = \Phi_\beta(0) = 0 \\ \text{col}_2([D]_{\beta,\beta}) &= \Phi_\beta(D(\Phi_\beta^{-1}(e_2))) = \Phi_\beta(D(x)) = \Phi_\beta(1) = e_1 \\ \text{col}_3([D]_{\beta,\beta}) &= \Phi_\beta(D(\Phi_\beta^{-1}(e_3))) = \Phi_\beta(D(x^2)) = \Phi_\beta(2x) = 2e_2 \end{aligned}$$

Therefore we find,

$$[D]_{\beta,\beta} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Calculate D^3 . Is this surprising?

A one-one correspondence is a map which is 1-1 and onto. If we can find such a mapping between two sets then it shows those sets have the same cardinality. Cardinality is a crude idea of size, it turns out that all finite dimensional vector spaces over \mathbb{R} have the same cardinality. On the other hand, not all vector spaces have the same dimension. Isomorphisms help us discern if two vector spaces have the same dimension.

Definition 2.3.30.

Let V, W be vector spaces then $\Phi : V \rightarrow W$ is an **isomorphism** if it is a 1-1 and onto mapping which is also a linear transformation. If there is an isomorphism between vector spaces V and W then we say those vector spaces are **isomorphic** and we denote this by $V \cong W$.

Other authors sometimes denote isomorphism by equality. But, I'll avoid that custom as I am reserving $=$ to denote set equality. Details of the first two examples below can be found in my linear algebra notes.

Example 2.3.31. Let $V = \mathbb{R}^3$ and $W = P_2$. Define a mapping $\Phi : P_2 \rightarrow \mathbb{R}^3$ by

$$\Phi(ax^2 + bx + c) = (a, b, c)$$

for all $ax^2 + bx + c \in P_2$. As vector spaces, \mathbb{R}^3 and polynomials of upto quadratic order are the same.

Example 2.3.32. Let S_2 be the set of 2×2 symmetric matrices. Let $\Psi : \mathbb{R}^3 \rightarrow S_2$ be defined by

$$\Psi(x, y, z) = \begin{bmatrix} x & y \\ y & z \end{bmatrix}.$$

Example 2.3.33. Let $L(\mathbb{R}^n, \mathbb{R}^m)$ denote the set of all linear transformations from \mathbb{R}^n to \mathbb{R}^m . $L(\mathbb{R}^n, \mathbb{R}^m)$ forms a vector space under function addition and scalar multiplication. There is a natural isomorphism to $m \times n$ matrices. Define $\Psi : L(\mathbb{R}^n, \mathbb{R}^m) \rightarrow \mathbb{R}^{m \times n}$ by $\Psi(T) = [T]$ for all linear transformations $T \in L(\mathbb{R}^n, \mathbb{R}^m)$. In other words, linear transformations and matrices are the same as vector spaces.

The quantification of "same" is a large theme in modern mathematics. In fact, the term *isomorphism* as we use it here is more accurately phrased *vector space isomorphism*. There are other kinds of isomorphisms which preserve other interesting structures like Group, Ring or Lie Algebra isomorphism. But, I think we've said more than enough for this course.

2.4 normed vector spaces

Definition 2.4.1.

Suppose V is a vector space. If $\|\cdot\| : V \times V \rightarrow \mathbb{R}$ is a function such that for all $x, y \in V$ and $c \in \mathbb{R}$:

1. $\|cx\| = |c| \|x\|$
2. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)
3. $\|x\| \geq 0$
4. $\|x\| = 0$ iff $x = 0$

then we say $(V, \|\cdot\|)$ is a normed vector space. When there is no danger of ambiguity we also say that V is a normed vector space.

The norms below are basically thieved from the usual Euclidean norm on \mathbb{R}^n .

Example 2.4.2. \mathbb{R}^n can be given the Euclidean norm which is defined by $\|x\| = \sqrt{x \cdot x}$ for each $x \in \mathbb{R}^n$.

Example 2.4.3. \mathbb{R}^n can also be given the 1-norm which is defined by $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$ for each $x \in \mathbb{R}^n$.

We use the Euclidean norm by default.

Example 2.4.4. Consider \mathbb{C} as a two dimensional real vector space. Let $a + ib \in \mathbb{C}$ and define $\|a + ib\| = \sqrt{a^2 + b^2}$. This is a norm for \mathbb{C} .

Example 2.4.5. Let $A \in \mathbb{R}^{m \times n}$. For each $A = [A_{ij}]$ we define

$$\|A\| = \sqrt{A_{11}^2 + A_{12}^2 + \cdots + A_{mn}^2} = \sqrt{\sum_{i=1}^m \sum_{j=1}^n A_{ij}^2}.$$

This is the Frobenius norm for matrices.

Each of the norms above allows us to define a distance function and hence open sets and limits for functions. An open ball in $(V, \|\cdot\|_V)$ is defined

$$B_\epsilon(x_o) = \{y \in V \mid \|y - x_o\|_V < \epsilon\}.$$

We define the deleted open ball by removing the center from the open ball $B_\epsilon(x_o)_o = B_\epsilon(x_o) - \{x_o\} = \{y \in V \mid 0 < \|y - x_o\|_V < \epsilon\}$. We say x_o is a limit point of a function f iff there exists a deleted open ball which is contained in the $\text{dom}(f)$. We say $U \subseteq V$ is an open set iff for each $u \in U$ there exists an open ball $B_\epsilon(u) \subseteq U$. Limits are also defined in the same way as in \mathbb{R}^n , if $f : V \rightarrow W$ is a function from normed space $(V, \|\cdot\|_V)$ to normed vector space $(W, \|\cdot\|_W)$ then we say $\lim_{x \rightarrow x_o} f(x) = L$ iff for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in V$ subject to $0 < \|x - x_o\|_V < \delta$ it follows $\|f(x) - f(x_o)\|_W < \epsilon$. If $\lim_{x \rightarrow x_o} f(x) = f(x_o)$ then we say that f is a continuous function at x_o .

Let $(V, \|\cdot\|_V)$ be a normed vector space, a function from \mathbb{N} to V is called a **sequence**. Suppose $\{a_n\}$ is a sequence then we say $\lim_{n \rightarrow \infty} a_n = L \in V$ iff for each $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $\|a_n - L\|_V < \epsilon$ for all $n \in \mathbb{N}$ with $n > M$. If $\lim_{n \rightarrow \infty} a_n = L \in V$ then we say $\{a_n\}$ is a **convergent sequence**. We say $\{a_n\}$ is a **Cauchy sequence** iff for each $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $\|a_m - a_n\|_V < \epsilon$ for all $m, n \in \mathbb{N}$ with $m, n > M$. In other words, a sequence is Cauchy if the terms in the sequence get arbitrarily close as we go sufficiently far out in the list. Many concepts we cover in calculus II are made clear with proofs built around the concept of a Cauchy sequence. The interesting thing about Cauchy is that for some spaces of numbers we can have a sequence which converges but is not Cauchy. For example, if you think about the rational numbers \mathbb{Q} we can construct a sequence of truncated decimal expansions of π :

$$\{a_n\} = \{3, 3.1, 3.14, 3.141, 3.1415 \dots\}$$

note that $a_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and yet the $a_n \rightarrow \pi \notin \mathbb{Q}$. When spaces are missing their limit points they are in some sense incomplete. For this reason we say a metric space which contains all its limit points is known as a **complete space**. Moreover, a normed vector space which is complete is known as a **Banach space**. Fortunately all the main examples of this course are built on the real numbers which are complete, this induces completeness for \mathbb{C}, \mathbb{R}^n and $\mathbb{R}^{m \times n}$. I may guide you through the proof that $\mathbb{R}, \mathbb{C}, \mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ are Banach spaces in a homework. When you take real analysis you'll spend some time thinking through the Cauchy concept.

Proposition 1.4.23 was given for the specific case of functions whose range is in \mathbb{R} . We might be able to mimick the proof of that proposition for the case of normed spaces. We do have a composition of limits theorem and I bet the sum function is continuous on a normed space. Moreover, if the range happens to be a Banach algebra⁶ then I would wager the product function is continuous. Put these together and we get the normed vector space version of Prop. 1.4.23. That said, a direct proof works nicely here so I'll just forego the more clever route here.

Proposition 2.4.6.

⁶if W is a Banach space that also has a product $m : W \times W \rightarrow W$ such that $\|w_1 w_2\| \leq \|w_1\| \|w_2\|$ then W is a Banach algebra.

Let V, W be normed vector spaces. Let a be a limit point of mappings $f, g : U \subseteq V \rightarrow W$ and suppose $c \in \mathbb{R}$. If $\lim_{x \rightarrow a} f(x) = b_1 \in W$ and $\lim_{x \rightarrow a} g(x) = b_2 \in W$ then

1. $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$.
2. $\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x)$.

Moreover, if f, g are continuous then $f + g$ and cf are continuous.

Proof: Let $\epsilon > 0$ and suppose $\lim_{x \rightarrow a} f(x) = b_1 \in W$ and $\lim_{x \rightarrow a} g(x) = b_2 \in W$. Choose $\delta_1, \delta_2 > 0$ such that $0 < \|x - a\| < \delta_1$ implies $\|f(x) - b_1\| < \epsilon/2$ and $0 < \|x - a\| < \delta_2$ implies $\|g(x) - b_2\| < \epsilon/2$. Choose $\delta = \min(\delta_1, \delta_2)$ and suppose $0 < \|x - a\| < \delta \leq \delta_1, \delta_2$ hence

$$\|(f + g)(x) - (b_1 + b_2)\| = \|f(x) - b_1 + g(x) - b_2\| \leq \|f(x) - b_1\| + \|g(x) - b_2\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Item (2.) follows. To prove (2.) note that if $c = 0$ the result is clearly true so suppose $c \neq 0$. Suppose $\epsilon > 0$ and choose $\delta > 0$ such that $\|f(x) - b_1\| < \epsilon/|c|$. Note that if $0 < \|x - a\| < \delta$ then

$$\|(cf)(x) - cb_1\| = \|c(f(x) - b_1)\| = |c|\|f(x) - b_1\| < |c|\epsilon/|c| = \epsilon.$$

The claims about continuity follow immediately from the limit properties and that completes the proof \square .

Perhaps you recognize these arguments from calculus I. The logic used to prove the basic limit theorems on \mathbb{R} is essentially identical.

Proposition 2.4.7.

Suppose V_1, V_2, V_3 are normed vector spaces with norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_3$ respectively. Let $f : \text{dom}(f) \subseteq V_2 \rightarrow V_3$ and $g : \text{dom}(g) \subseteq V_1 \rightarrow V_2$ be mappings. Suppose that $\lim_{x \rightarrow x_o} g(x) = y_o$ and suppose that f is continuous at y_o then

$$\lim_{x \rightarrow x_o} (f \circ g)(x) = f\left(\lim_{x \rightarrow x_o} g(x)\right).$$

Proof: Let $\epsilon > 0$ and choose $\beta > 0$ such that $0 < \|y - b\|_2 < \beta$ implies $\|f(y) - f(y_o)\|_3 < \epsilon$. We can choose such a β since f is continuous at y_o thus it follows that $\lim_{y \rightarrow y_o} f(y) = f(y_o)$. Next choose $\delta > 0$ such that $0 < \|x - x_o\|_1 < \delta$ implies $\|g(x) - y_o\|_2 < \beta$. We can choose such a δ because we are given that $\lim_{x \rightarrow x_o} g(x) = y_o$. Suppose $0 < \|x - x_o\|_1 < \delta$ and let $y = g(x)$ note $\|g(x) - y_o\|_2 < \beta$ yields $\|y - y_o\|_2 < \beta$ and consequently $\|f(y) - f(y_o)\|_3 < \epsilon$. Therefore, $0 < \|x - x_o\|_1 < \delta$ implies $\|f(g(x)) - f(y_o)\|_3 < \epsilon$. It follows that $\lim_{x \rightarrow x_o} (f(g(x))) = f(\lim_{x \rightarrow x_o} g(x))$. \square

The squeeze theorem relies heavily on the order properties of \mathbb{R} . Generally a normed vector space has no natural ordering. For example, is $1 > i$ or is $1 < i$ in \mathbb{C} ? That said, we can state a squeeze theorem for functions whose domain reside in a normed vector space. This is a generalization of

what we learned in calculus I. That said, the proof offered below is very similar to the typical proof which is not given in calculus I⁷

Proposition 2.4.8. *squeeze theorem.*

Suppose $f : \text{dom}(f) \subseteq V \rightarrow \mathbb{R}$, $g : \text{dom}(g) \subseteq V \rightarrow \mathbb{R}$, $h : \text{dom}(h) \subseteq V \rightarrow \mathbb{R}$ where V is a normed vector space with norm $\|\cdot\|$. Let $f(x) \leq g(x) \leq h(x)$ for all x on some $\delta > 0$ ball of⁸ $a \in V$ then we find that the limits at x_o follow the same ordering,

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \leq \lim_{x \rightarrow a} h(x).$$

Moreover, if $\lim_{x \rightarrow x_o} f(x) = \lim_{x \rightarrow x_o} h(x) = L \in \mathbb{R}$ then $\lim_{x \rightarrow x_o} g(x) = L$.

Proof: Suppose $f(x) \leq g(x)$ for all $x \in B_{\delta_1}(a)_o$ for some $\delta_1 > 0$ and also suppose $\lim_{x \rightarrow a} f(x) = L_f \in \mathbb{R}$ and $\lim_{x \rightarrow a} g(x) = L_g \in \mathbb{R}$. We wish to prove that $L_f \leq L_g$. Suppose otherwise towards a contradiction. That is, suppose $L_f > L_g$. Note that $\lim_{x \rightarrow a} [g(x) - f(x)] = L_g - L_f$ by the linearity of the limit. It follows that for $\epsilon = \frac{1}{2}(L_f - L_g) > 0$ there exists $\delta_2 > 0$ such that $x \in B_{\delta_2}(a)_o$ implies $|g(x) - f(x) - (L_g - L_f)| < \epsilon = \frac{1}{2}(L_f - L_g)$. Expanding this inequality we have

$$-\frac{1}{2}(L_f - L_g) < g(x) - f(x) - (L_g - L_f) < \frac{1}{2}(L_f - L_g)$$

adding $L_g - L_f$ yields,

$$-\frac{3}{2}(L_f - L_g) < g(x) - f(x) < -\frac{1}{2}(L_f - L_g) < 0.$$

Thus, $f(x) > g(x)$ for all $x \in B_{\delta_2}(a)_o$. But, $f(x) \leq g(x)$ for all $x \in B_{\delta_1}(a)_o$ so we find a contradiction for each $x \in B_{\delta}(a)$ where $\delta = \min(\delta_1, \delta_2)$. Hence $L_f \leq L_g$. The same proof can be applied to g and h thus the first part of the theorem follows.

Next, we suppose that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L \in \mathbb{R}$ and $f(x) \leq g(x) \leq h(x)$ for all $x \in B_{\delta_1}(a)$ for some $\delta_1 > 0$. We seek to show that $\lim_{x \rightarrow a} g(x) = L$. Let $\epsilon > 0$ and choose $\delta_2 > 0$ such that $|f(x) - L| < \epsilon$ and $|h(x) - L| < \epsilon$ for all $x \in B_{\delta}(a)_o$. We are free to choose such a $\delta_2 > 0$ because the limits of f and h are given at $x = a$. Choose $\delta = \min(\delta_1, \delta_2)$ and note that if $x \in B_{\delta}(a)_o$ then

$$f(x) \leq g(x) \leq h(x)$$

hence,

$$f(x) - L \leq g(x) - L \leq h(x) - L$$

but $|f(x) - L| < \epsilon$ and $|h(x) - L| < \epsilon$ imply $-\epsilon < f(x) - L$ and $h(x) - L < \epsilon$ thus

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon.$$

⁷this is lifted word for word from my calculus I notes, however here the meaning of open ball is considerably more general and the linearity of the limit which is referenced is the one proven earlier in this section

Therefore, for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in B_\delta(a)_o$ implies $|g(x) - L| < \epsilon$ so $\lim_{x \rightarrow a} g(x) = L$. \square

Our typical use of the theorem above applies to equations of norms from a normed vector space. The norm takes us from V to \mathbb{R} so the theorem above is essential to analyze interesting limits. We shall make use of it in the next chapter.

Proposition 2.4.9. *norm is continuous with respect to itself.*

Suppose V has norm $\|\cdot\|$ then $f : V \rightarrow \mathbb{R}$ defined by $f(x) = \|x\|$ defines a continuous function on V .

Proof: Suppose $x_o \in V$ and $\|x_o\| = l_o$. We wish to show that for each $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < \|x - x_o\| < \delta$ implies $|\|x\| - l_o| < \epsilon$. Let $\epsilon > 0$ and choose $\delta = \epsilon$ then $0 < \|x - x_o\| < \delta \dots$ stuck.XXX \square .

Finally, we should like to find a vector of the limits is the limit of the vector proposition for the context of normed spaces. It is generally true for normed vector spaces that if we know the limits of all the component functions converge to particular limits then the limit of a vector function is simply the vector of those limits. The converse is not so simple because the basis expansion for a normed vector space could fail to follow the pattern we expect from our study of \mathbb{R}^n .

Let \mathbb{R}^2 have basis $\beta = \{\mathcal{E}_1, \mathcal{E}_2\} = \{e_1, -3e_1 + e_2\}$ note the vector $v = 3\mathcal{E}_1 + \mathcal{E}_2 = 3e_1 + (-3e_1 + e_2) = e_2$. With respect to the β basis we find $v_1 = 3$ and $v_2 = 1$. The concept of length is muddled in these coordinates. If we tried (incorrectly) to use the pythagorean theorem we'd find $\|v\| = \sqrt{9+1} = \sqrt{10}$ and yet the length of the vector is clearly just 1 since $v = e_2 = (0, 1)$. The trouble with β is that it has different basis elements which overlap. To keep clear the euclidean idea of distance we must insist on the use of an **orthonormal** basis.

I'd rather not explain what that means at this point. Sufficient to say that if β is an orthonormal basis then the coordinates preserve essentially the euclidean idea of vector length. In particular, we can expect that if

$$v = \sum_{j=1}^m v_j f_j \quad \text{then} \quad \|v\|^2 = \sum_{j=1}^m v_j^2.$$

Proposition 2.4.10.

Let V, W be normed vector spaces and suppose W has basis $\beta = \{w_j\}_{j=1}^m$ such that when $v = \sum_{j=1}^m v_j w_j$ then $\|v\|^2 = \sum_{j=1}^m v_j^2$. Suppose that $F : \text{dom}(F) \subseteq V \rightarrow W$ is a mapping with component functions F_1, F_2, \dots, F_m with respect to the β basis ($F = \sum_{j=1}^m F_j w_j$). Let $a \in V$ be a limit point of F then

$$\lim_{x \rightarrow a} F(x) = B = \sum_{j=1}^m B_j w_j \quad \Leftrightarrow \quad \lim_{x \rightarrow a} F_j(x) = B_j \quad \text{for all } j = 1, 2, \dots, m.$$

Proof: Suppose $\lim_{x \rightarrow a} F(x) = B = \sum_{j=1}^m B_j w_j$. Since we assumed the basis of W was orthonormal we have that:

$$|F_k(x) - B_k|^2 \leq \sum_{j=1}^m |F_j(x) - B_j|^2 = \|F(x) - B\|^2$$

where in the first equality I simply added nonzero terms. With the inequality above in mind, let $\epsilon > 0$ and choose $\delta > 0$ such that $0 < \|x - a\| < \delta$ implies $\|F(x) - B\| < \epsilon$. It follows that $|F_k(x) - B_k|^2 < \epsilon^2$ and hence $|F_k(x) - B_k| < \epsilon$. The index k is arbitrary therefore, $\lim_{x \rightarrow a} F_k(x) = B_k$ for all $k \in \mathbb{N}_m$.

Conversely suppose $\lim_{x \rightarrow a} F_k(x) = B_k$ for all $k \in \mathbb{N}_m$. Let $M = \max\{\|w_1\|, \|w_2\|, \dots, \|w_m\|\}$. Let $\epsilon > 0$ and choose, by virtue of the given limits for the component functions, $\delta_k > 0$ such that $0 < \|x - a\| < \delta_k$ implies $|F_k(x) - B_k| < \frac{\epsilon}{mM}$. Choose $\delta = \min\{\delta_1, \delta_2, \dots, \delta_m\}$ and suppose $0 < \|x - a\| < \delta$. Consider

$$\|F(x) - B\| = \left\| \sum_{j=1}^m (F_j(x) - B_j) w_j \right\| \leq \sum_{j=1}^m \|(F_j(x) - B_j) w_j\| = \sum_{j=1}^m |F_j(x) - B_j| \|w_j\|$$

However, $\|w_j\| \leq M$ for all $j = 1, 2, \dots, m$ hence

$$\|F(x) - B\| \leq \sum_{j=1}^m M |F_j(x) - B_j| < \sum_{j=1}^m M \frac{\epsilon}{mM} = \sum_{j=1}^m \frac{\epsilon}{m} = \epsilon.$$

Therefore, $\lim_{x \rightarrow a} F(x) = B$. \square

I leave the case of non-orthonormal bases to the reader. In all the cases we consider it is possible and natural to choose orthogonal bases to describe the vector space. I'll avoid the temptation to do more here (there is more).⁹

⁹add a couple references for further reading here XXX

Chapter 3

differentiation

Our goal in this chapter is to describe differentiation for functions to and from normed linear spaces. It turns out this is actually quite simple given the background of the preceding chapter. The differential at a point is a linear transformation which best approximates the change in a function at a particular point. We can quantify "best" by a limiting process which is naturally defined in view of the fact there is a norm on the spaces we consider.

The most important example is of course the case $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. In this context it is natural to write the differential as a matrix multiplication. The matrix of the differential is what Edwards calls the *derivative*. Partial derivatives are also defined in terms of directional derivatives. The directional derivative is sometimes defined where the differential fails to exist. We will discuss how the criteria of continuous differentiability allows us to build the differential from the directional derivatives. We'll see how the Cauchy-Riemann equations of complex analysis are really just an algebraic result if we already have the theorem for continuous differentiability. We will see how this general concept of differentiation recovers all the derivatives you've seen previously in calculus and much more.

On the other hand, I postpone implicit differentiation for a future chapter where we have the existence theorems for implicit and inverse functions. I also postpone discussion of the geometry of the differential. In short, existence of the differential and the tangent space are essentially two sides of the same problem. In fact, the approach of this chapter is radically different than my first set of notes on advanced calculus. Last year I followed Edwards a bit more and built up to the definition of the differential on the basis of the directional derivative and geometry. I don't think students appreciate geometry or directional differentiation well enough to make that approach successful. Consequently, I begin with the unjustified definition of the derivative and then spend the rest of the chapter working out precise implications and examples that flow from the definition. I essentially ignore the question of motivating the definition we begin with. If you want motivation, think backward with this chapter or perhaps read Edwards or my old notes.

3.1 the differential

The definition¹ below says that $\triangle F = F(a + h) - F(a) \approx dF_a(h)$ when h is close to zero.

Definition 3.1.1.

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces. Suppose that U is open and $F : U \subseteq V \rightarrow W$ is a function then we say that F is **differentiable** at $a \in U$ iff there exists a linear mapping $L : V \rightarrow W$ such that

$$\lim_{h \rightarrow 0} \left[\frac{F(a + h) - F(a) - L(h)}{\|h\|_V} \right] = 0.$$

In such a case we call the linear mapping L the **differential at a** and we denote $L = dF_a$. In the case $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$ are given the standard euclidean norms, the matrix of the differential is called the **derivative of F at a** and we denote $[dF_a] = F'(a) \in \mathbb{R}^{m \times n}$ which means that $dF_a(v) = F'(a)v$ for all $v \in \mathbb{R}^n$.

Notice this definition gives an equation which implicitly defines dF_a . For the moment the only way we have to calculate dF_a is educated guessing.

Example 3.1.2. Suppose $T : V \rightarrow W$ is a linear transformation of normed vector spaces V and W . I propose $L = T$. In other words, I think we can show the best linear approximation to the change in a linear function is simply the function itself. Clearly L is linear since T is linear. Consider the difference quotient:

$$\frac{T(a + h) - T(a) - L(h)}{\|h\|_V} = \frac{T(a) + T(h) - T(a) - T(h)}{\|h\|_V} = \frac{0}{\|h\|_V}.$$

Note $h \neq 0$ implies $\|h\|_V \neq 0$ by the definition of the norm. Hence the limit of the difference quotient vanishes since it is identically zero for every nonzero value of h . We conclude that $dT_a = T$.

Example 3.1.3. Let $T : V \rightarrow W$ where V and W are normed vector spaces and define $T(v) = w_o$ for all $v \in V$. I claim the differential is the zero transformation. Linearity of $L(v) = 0$ is trivially verified. Consider the difference quotient:

$$\frac{T(a + h) - T(a) - L(h)}{\|h\|_V} = \frac{w_o - w_o - 0}{\|h\|_V} = \frac{0}{\|h\|_V}.$$

Using the arguments to the preceding example, we find $dT_a = 0$.

Typically the difference quotient is not identically zero. The pair of examples above are very special cases. I'll give a few more abstract examples later in this section. For now we turn to the question of how this general definition recovers the concept of differentiation we studied in calculus.

¹Some authors might put a norm in the numerator of the quotient. That is an equivalent condition since a function $g : V \rightarrow W$ has $\lim_{h \rightarrow 0} g(h) = 0$ iff $\lim_{h \rightarrow 0} \|g(h)\|_W = 0$

Example 3.1.4. Suppose $f : \text{dom}(f) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at x . It follows that there exists a linear function $df_x : \mathbb{R} \rightarrow \mathbb{R}$ such that²

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - df_x(h)}{|h|} = 0.$$

Since $df_x : \mathbb{R} \rightarrow \mathbb{R}$ is linear there exists a constant matrix m such that $df_x(h) = mh$. In this silly case the matrix m is a 1×1 matrix which otherwise known as a real number. Note that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - df_x(h)}{|h|} = 0 \quad \Leftrightarrow \quad \lim_{h \rightarrow 0^\pm} \frac{f(x+h) - f(x) - df_x(h)}{|h|} = 0.$$

In the left limit $h \rightarrow 0^-$ we have $h < 0$ hence $|h| = -h$. On the other hand, in the right limit $h \rightarrow 0^+$ we have $h > 0$ hence $|h| = h$. Thus, differentiability suggests that $\lim_{h \rightarrow 0^\pm} \frac{f(x+h) - f(x) - df_x(h)}{\pm h} = 0$. But we can pull the minus out of the left limit to obtain $\lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x) - df_x(h)}{h} = 0$. Therefore,

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x) - df_x(h)}{h} = 0.$$

We seek to show that $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = m$.

$$m = \lim_{h \rightarrow 0} \frac{mh}{h} = \lim_{h \rightarrow 0} \frac{df_x(h)}{h}$$

A theorem from calculus I states that if $\lim(f - g) = 0$ and $\lim(g)$ exists then so must $\lim(f)$ and $\lim(f) = \lim(g)$. Apply that theorem to the fact we know $\lim_{h \rightarrow 0} \frac{df_x(h)}{h}$ exists and

$$\lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} - \frac{df_x(h)}{h} \right] = 0.$$

It follows that

$$\lim_{h \rightarrow 0} \frac{df_x(h)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently,

$$df_x(h) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \text{defined } f'(x) \text{ in calc. I.}$$

Therefore, $\boxed{df_x(h) = f'(x)h}$. In other words, if a function is differentiable in the sense we defined at the beginning of this chapter then it is differentiable in the terminology we used in calculus I. Moreover, the derivative at x is precisely the matrix of the differential.

²unless we state otherwise, \mathbb{R}^n is assumed to have the euclidean norm, in this case $\|x\|_{\mathbb{R}} = \sqrt{x^2} = |x|$.

Example 3.1.5. Suppose $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is defined by $F(x, y) = (xy, x^2, x + 3y)$ for all $(x, y) \in \mathbb{R}^2$. Consider the difference function ΔF at (x, y) :

$$\Delta F = F((x, y) + (h, k)) - F(x, y) = F(x + h, y + k) - F(x, y)$$

Calculate,

$$\Delta F = ((x + h)(y + k), (x + h)^2, x + h + 3(y + k)) - (xy, x^2, x + 3y)$$

Simplify by cancelling terms which cancel with $F(x, y)$:

$$\Delta F = (xk + hy, 2xh + h^2, h + 3k)$$

Identify the linear part of ΔF as a good candidate for the differential. I claim that:

$$L(h, k) = (xk + hy, 2xh, h + 3k).$$

is the differential for f at (x, y) . Observe first that we can write

$$L(h, k) = \begin{bmatrix} y & x \\ 2x & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}.$$

therefore $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is manifestly linear. Use the algebra above to simplify the difference quotient below:

$$\lim_{(h,k) \rightarrow (0,0)} \left[\frac{\Delta F - L(h, k)}{\|(h, k)\|} \right] = \lim_{(h,k) \rightarrow (0,0)} \left[\frac{(0, h^2, 0)}{\|(h, k)\|} \right]$$

Note $\|(h, k)\| = \sqrt{h^2 + k^2}$ therefore we face the task of showing that $(0, h^2/\sqrt{h^2 + k^2}, 0) \rightarrow (0, 0, 0)$ as $(h, k) \rightarrow (0, 0)$. Recall from our study of limits that we can prove the vector tends to $(0, 0, 0)$ by showing the each component tends to zero. The first and third components are obviously zero however the second component requires study. Observe that

$$0 \leq \frac{h^2}{\sqrt{h^2 + k^2}} \leq \frac{h^2}{\sqrt{h^2}} = |h|$$

Clearly $\lim_{(h,k) \rightarrow (0,0)} (0) = 0$ and $\lim_{(h,k) \rightarrow (0,0)} |h| = 0$ hence the squeeze theorem for multivariate limits shows that $\lim_{(h,k) \rightarrow (0,0)} \frac{h^2}{\sqrt{h^2 + k^2}} = 0$. Therefore,

$$df_{(x,y)}(h, k) = \begin{bmatrix} y & x \\ 2x & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}.$$

Computation of less trivial multivariate limits is an art we'd like to avoid if possible. It turns out that we can actually avoid these calculations by computing partial derivatives. However, we still need a certain multivariate limit to exist for the partial derivative functions so in some sense it's unavoidable. The limits are there whether we like to calculate them or not. I want to give a few more abstract examples before I get into the partial differentiation. The purpose of this section is to showcase the generality of the definition for differential.

Example 3.1.6. Suppose $F(t) = U(t) + iV(t)$ for all $t \in \text{dom}(f)$ and both U and V are differentiable functions on $\text{dom}(F)$. By the arguments given in Example 3.1.4 it suffices to find $L : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$\lim_{h \rightarrow 0} \left[\frac{F(t+h) - F(t) - L(h)}{h} \right] = 0.$$

I propose that on the basis of analogy to Example 3.1.4 we ought to have $dF_t(h) = (U'(t) + iV'(t))h$. Let $L(h) = (U'(t) + iV'(t))h$. Observe that, using properties of \mathbb{C} , $L(h_1 + ch_2) =$

$$= (U'(t) + iV'(t))(h_1 + ch_2) = (U'(t) + iV'(t))h_1 + c(U'(t) + iV'(t))h_2 = L(h_1) + cL(h_2).$$

for all $h_1, h_2 \in \mathbb{R}$ and $c \in \mathbb{R}$. Hence $L : \mathbb{R} \rightarrow \mathbb{C}$ is linear. Moreover,

$$\begin{aligned} \frac{F(t+h) - F(t) - L(h)}{h} &= \frac{1}{h} \left(U(t+h) + iV(t+h) - U(t) + iV(t) - (U'(t) + iV'(t))h \right) \\ &= \frac{1}{h} \left(U(t+h) - U(t) - U'(t)h \right) + i \frac{1}{h} \left(V(t+h) - V(t) - V'(t)h \right) \end{aligned}$$

Consider the problem of calculating $\lim_{h \rightarrow 0} \frac{F(t+h) - F(t) - L(h)}{h}$. We use a lemma that a complex function converges to zero iff the real and imaginary parts of the function separately converge to zero (this might be a homework). By differentiability of U and V we find again using Example 3.1.4

$$\lim_{h \rightarrow 0} \frac{1}{h} \left(U(t+h) - U(t) - U'(t)h \right) = 0 \quad \lim_{h \rightarrow 0} \frac{1}{h} \left(V(t+h) - V(t) - V'(t)h \right) = 0.$$

Therefore, $dF_t(h) = (U'(t) + iV'(t))h$. Note that the quantity $U'(t) + iV'(t)$ is not a real matrix in this case. To write the derivative in terms of a real matrix multiplication we need to construct some further notation which makes use of the isomorphism between \mathbb{C} and \mathbb{R}^2 . Actually, it's pretty easy if you agree that $a + ib = (a, b)$ then $dF_t(h) = (U'(t), V'(t))h$ so the matrix of the differential is $(U'(t), V'(t)) \in \mathbb{R}^{1 \times 2}$ which makes sense since as $F : \mathbb{C} \approx \mathbb{R}^2 \rightarrow \mathbb{R}$.

Generally constructing the matrix for a function $f : V \rightarrow W$ where $V, W \neq \mathbb{R}$ involves a fair number of relatively ad-hoc conventions because the constructions necessarily involving choosing coordinates. The situation is similar in linear algebra. Writing abstract linear transformations in terms of matrix multiplication takes a little thinking. If you look back you'll notice that I did not bother to try to write a matrix Examples 3.1.2 or 3.1.3. The same is true for the final example of this section.

Example 3.1.7. Suppose $F : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ is defined by $F(X) = X^2$. Notice

$$\Delta F = F(X + H) - F(X) = (X + H)(X + H) - X^2 = XH + HX + H^2$$

I propose that F is differentiable at X and $L(H) = XH + HX$. Let's check linearity,

$$L(H_1 + cH_2) = X(H_1 + cH_2) + (H_1 + cH_2)X = XH_1 + H_1X + c(XH_2 + H_2X)$$

Hence $L : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ is a linear transformation. By construction of L the linear terms in the numerator cancel leaving just the quadratic term,

$$\lim_{H \rightarrow 0} \frac{F(X+H) - F(X) - L(H)}{\|H\|} = \lim_{H \rightarrow 0} \frac{H^2}{\|H\|}.$$

It suffices to show that $\lim_{H \rightarrow 0} \frac{\|H^2\|}{\|H\|} = 0$ since $\lim(\|g\|) = 0$ iff $\lim(g) = 0$ in a normed vector space. Fortunately the normed vector space $\mathbb{R}^{n \times n}$ is actually a **Banach algebra**. A vector space with a multiplication operation is called an algebra. In the current context the multiplication is simply matrix multiplication. A Banach algebra is a normed vector space with a multiplication that satisfies $\|XY\| \leq \|X\|\|Y\|$. Thanks to this inequality we can calculate our limit via the squeeze theorem. Observe $0 \leq \frac{\|H^2\|}{\|H\|} \leq \|H\|$. As $H \rightarrow 0$ it follows $\|H\| \rightarrow 0$ hence $\lim_{H \rightarrow 0} \frac{\|H^2\|}{\|H\|} = 0$. We find $dF_X(H) = XH + HX$.

XXX- need to adjust example below to reflect orthonormality assumption.

Example 3.1.8. Suppose V is a normed vector space with basis $\beta = \{f_1, f_2, \dots, f_n\}$. Furthermore, let $G : I \subseteq \mathbb{R} \rightarrow V$ be defined by

$$G(t) = \sum_{i=1}^n G_i(t) f_i$$

where $G_i : I \rightarrow \mathbb{R}$ is differentiable on I for $i = 1, 2, \dots, n$. I claim that if $T = \sum_{j=1}^n T_j f_j : \mathbb{R} \rightarrow V$ then $\lim_{t \rightarrow 0} T(t) = \sum_{j=1}^n l_j f_j$ iff $\lim_{t \rightarrow 0} T_j(t) = l_j$ for all $j = 1, 2, \dots, n$. In words, the limit of a vector-valued function can be parsed into a vector of limits. We've not proved this, I may make it a homework. With this in mind consider (again we can trade $|h|$ for h as we explained in-depth in Example 3.1.4) the difference quotient $\lim_{h \rightarrow 0} \left[\frac{G(t+h) - G(t) - h \sum_{i=1}^n \frac{dG_i}{dt} f_i}{h} \right]$, factoring out the basis yields:

$$\lim_{h \rightarrow 0} \left[\frac{\sum_{i=1}^n [G_i(t+h) - G_i(t) - h \frac{dG_i}{dt}] f_i}{h} \right] = \sum_{i=1}^n \left[\lim_{h \rightarrow 0} \frac{G_i(t+h) - G_i(t) - h \frac{dG_i}{dt}}{h} \right] f_i$$

The expression on the left is the limit of a vector whereas the expression on the right is a vector of limits. I make the equality by applying the claim. In any event, I hope you are not surprised that:

$$\boxed{dG_t(h) = h \sum_{i=1}^n \frac{dG_i}{dt} f_i}$$

The example above encompasses a number of cases at once:

1. $V = \mathbb{R}$, functions on \mathbb{R} , $f : \mathbb{R} \rightarrow \mathbb{R}$
2. $V = \mathbb{R}^n$, space curves in \mathbb{R} , $\vec{r} : \mathbb{R} \rightarrow \mathbb{R}^n$
3. $V = \mathbb{C}$, complex-valued functions of a real variable, $f = u + iv : \mathbb{R} \rightarrow \mathbb{C}$

4. $V = \mathbb{R}^{m \times n}$, matrix-valued functions of a real variable, $F : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$.

In short, when we differentiate a function which has a real domain then we can define the derivative of such a function by component-wise differentiation. It gets more interesting when the domain has several independent variables. We saw this in Examples 3.1.5 and 3.1.7.

Remark 3.1.9.

I have deliberately defined the derivative in slightly more generality than we need for this course. It's probably not much trouble to continue to develop the theory of differentiation for a normed vector space, however I will for the most part stop here. The theorems that follow are not terribly complicated in the notation of \mathbb{R}^n and traditionally this type of course only covers continuous differentiability, inverse and implicit function theorems in the context of mappings from \mathbb{R}^n to \mathbb{R}^m . For the reader interested in generalizing these results to the context of an abstract normed vector space feel free to discuss it with me sometime. This much we can conclude from our brief experience thus far, if we study functions whose domain is in \mathbb{R} then differentiation is accomplished component-wise in the range. This is good news since in all your previous courses I simply defined differentiation by the component-wise rule. This section at a minimum shows that idea is consistent with the larger theory we are working out in this chapter. It is likely I will have you work out the calculus of complex or matrix-valued functions of a real variable in the homework.

3.2 partial derivatives and the existence of the differential

In the preceding section we calculated the differential at a point via educated guessing. We should like to find better method to derive differentials. It turns out that we can systematically calculate the differential from partial derivatives of the component functions. However, certain topological conditions are required for us to properly paste together the partial derivatives of the component functions. We describe how the criteria of continuous differentiability achieves this goal. Much of this section was covered in calculus III but we do bring new generality and vision to the calculations described here.

3.2.1 directional derivatives

The directional derivative of a mapping F at a point $a \in \text{dom}(F)$ along v is defined to be the derivative of the curve $\gamma(t) = F(a + tv)$. In other words, the directional derivative gives you the instantaneous vector-rate of change in the mapping F at the point a along v . In the case that $m = 1$ then $F : \text{dom}(F) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ and the directional derivative gives the instantaneous rate of change of the function F at the point a along v . You probably insisted that $\|v\| = 1$ in calculus III but we make no such demand here. We define the directional derivative for mappings and vectors of non-unit length.

Definition 3.2.1.

Let $F : \text{dom}(F) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and suppose the limit below exists for $a \in \text{dom}(F)$ and $v \in \mathbb{R}^n$ then we define the **directional derivative of F at a along v** to be $D_v F(a) \in \mathbb{R}^m$ where

$$D_v F(a) = \lim_{h \rightarrow 0} \frac{F(a + hv) - F(a)}{h}$$

One great contrast we should pause to note is that the definition of the directional derivative is explicit whereas the definition of the differential was implicit. Many similarities do exist. For example: the directional derivative $D_v F(a)$ and the differential $df_a(v)$ are both is homogenous in v .

Proposition 3.2.2.

Let $F : \text{dom}(F) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ then if $D_v F(a)$ exists in \mathbb{R}^m then $D_{cv} F(a) = cD_v F(a)$

Proof: Let $F : \text{dom}(F) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and suppose $D_v F(a) \in \mathbb{R}^m$. This means we are given that $\lim_{h \rightarrow 0} \frac{F(a + hv) - F(a)}{h} = D_v F(a) \in \mathbb{R}^m$. If $c = 0$ then the proposition is clearly true. Consider, for nonzero $c \in \mathbb{R}$,

$$\lim_{h \rightarrow 0} \frac{F(a + h(cv)) - F(a)}{h} = c \lim_{ch \rightarrow 0} \frac{F(a + ch(v)) - F(a)}{ch}$$

Hence by the substitution $ch = k$ we find,

$$\lim_{h \rightarrow 0} \frac{F(a + h(cv)) - F(a)}{h} = c \lim_{k \rightarrow 0} \frac{F(a + k(v)) - F(a)}{k}$$

Therefore, the limit on the left of the equality exists as the limit on the right of the equality is given and we conclude $D_{cv} F(a) = cD_v F(a)$ for all $c \in \mathbb{R}$. \square

If we're given the derivative of a mapping then the directional derivative exists. The converse is not so simple as we shall discuss in the next subsection.

Proposition 3.2.3.

If $F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $a \in U$ then the directional derivative $D_v F(a)$ exists for each $v \in \mathbb{R}^n$ and $D_v F(a) = dF_a(v)$.

Proof: Suppose $a \in U$ such that dF_a is well-defined then we are given that

$$\lim_{h \rightarrow 0} \frac{F(a + h) - F(a) - dF_a(h)}{\|h\|} = 0.$$

This is a limit in \mathbb{R}^n , when it exists it follows that the limits that approach the origin along particular paths also exist and are zero. In particular we can consider the path $t \mapsto tv$ for $v \neq 0$ and $t > 0$, we find

$$\lim_{tv \rightarrow 0, t > 0} \frac{F(a + tv) - F(a) - dF_a(tv)}{\|tv\|} = \frac{1}{\|v\|} \lim_{t \rightarrow 0^+} \frac{F(a + tv) - F(a) - tdF_a(v)}{|t|} = 0.$$

Hence, as $|t| = t$ for $t > 0$ we find

$$\lim_{t \rightarrow 0^+} \frac{F(a + tv) - F(a)}{t} = \lim_{t \rightarrow 0} \frac{tdF_a(v)}{t} = dF_a(v).$$

Likewise we can consider the path $t \mapsto tv$ for $v \neq 0$ and $t < 0$

$$\lim_{tv \rightarrow 0, t < 0} \frac{F(a + tv) - F(a) - dF_a(tv)}{\|tv\|} = \frac{1}{\|v\|} \lim_{t \rightarrow 0^-} \frac{F(a + tv) - F(a) - tdF_a(v)}{|t|} = 0.$$

Note $|t| = -t$ thus the limit above yields

$$\lim_{t \rightarrow 0^-} \frac{F(a + tv) - F(a)}{-t} = \lim_{t \rightarrow 0^-} \frac{tdF_a(v)}{-t} \Rightarrow \lim_{t \rightarrow 0^-} \frac{F(a + tv) - F(a)}{t} = dF_a(v).$$

Therefore,

$$\lim_{t \rightarrow 0} \frac{F(a + tv) - F(a)}{t} = dF_a(v)$$

and we conclude that $D_v F(a) = dF_a(v)$ for all $v \in \mathbb{R}^n$ since the $v = 0$ case follows trivially. \square

Let's think about the problem we face. We want to find a nice formula for the differential. We now know that if it exists then the directional derivatives allow us to calculate the values of the differential in particular directions. The natural thing to do is to calculate the standard matrix for the differential using the preceding proposition. Recall that if $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ then the standard matrix was simply $[L] = [L(e_1)|L(e_2)|\cdots|L(e_n)]$ and thus the action of L is expressed nicely as a matrix multiplication; $L(v) = [L]v$. Similarly, $df_a : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear transformation and thus $df_a(v) = [df_a]v$ where $[df_a] = [df_a(e_1)|df_a(e_2)|\cdots|df_a(e_n)]$. Moreover, by the preceding proposition we can calculate $df_a(e_j) = D_{e_j}f(a)$ for $j = 1, 2, \dots, n$. Clearly the directional derivatives in the coordinate directions are of great importance. For this reason we make the following definition:

Definition 3.2.4.

Suppose that $F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a mapping then we say that F is **has partial derivative** $\frac{\partial F}{\partial x_i}(a)$ at $a \in U$ iff the directional derivative in the e_i direction exists at a . In this case we denote,

$$\frac{\partial F}{\partial x_i}(a) = D_{e_i}F(a).$$

Also we may use the notation $D_{e_i}F(a) = D_iF(a)$ or $\partial_i F = \frac{\partial F}{\partial x_i}$ when convenient. We also construct the partial derivative mapping $\partial_i F : V \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ as the mapping defined pointwise for each $v \in V$ where $\partial_i F(v)$ exists.

Let's expand this definition a bit. Note that if $F = (F_1, F_2, \dots, F_m)$ then

$$D_{e_i}F(a) = \lim_{h \rightarrow 0} \frac{F(a + he_i) - F(a)}{h} \Rightarrow [D_{e_i}F(a)] \cdot e_j = \lim_{h \rightarrow 0} \frac{F_j(a + he_i) - F_j(a)}{h}$$

for each $j = 1, 2, \dots, m$. But then the limit of the component function F_j is precisely the directional derivative at a along e_i hence we find the result

$$\frac{\partial F}{\partial x_i} \cdot e_j = \frac{\partial F_j}{\partial x_i} \quad \text{in other words,} \quad \boxed{\partial_i F = (\partial_i F_1, \partial_i F_2, \dots, \partial_i F_m)}.$$

Proposition 3.2.5.

If $F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $a \in U$ then the directional derivative $D_v F(a)$ can be expressed as a sum of partial derivative maps for each $v = \langle v_1, v_2, \dots, v_n \rangle \in \mathbb{R}^n$:

$$D_v F(a) = \sum_{j=1}^n v_j \partial_j F(a)$$

Proof: since F is differentiable at a the differential dF_a exists and $D_v F(a) = dF_a(v)$ for all $v \in \mathbb{R}^n$. Use linearity of the differential to calculate that

$$D_v F(a) = dF_a(v_1 e_1 + \dots + v_n e_n) = v_1 dF_a(e_1) + \dots + v_n dF_a(e_n).$$

Note $dF_a(e_j) = D_{e_j} F(a) = \partial_j F(a)$ and the prop. follows. \square

My primary interest in advanced calculus is the differential³. I discuss the directional derivative here merely to connect with your past calculations in calculus III where we explored the geometric and analytic significance of the directional derivative. I do not intend to revisit all of that here once more. Our focus is elsewhere. That said it's probably best to include the example below:

Example 3.2.6. Suppose $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ then $\nabla f = [\partial_x f, \partial_y f, \partial_z f]^T$ and we can write the directional derivative in terms of

$$D_v f = [\partial_x f, \partial_y f, \partial_z f]^T v = \nabla f \cdot v$$

if we insist that $\|v\| = 1$ then we recover the standard directional derivative we discuss in calculus III. Naturally the $\|\nabla f(a)\|$ yields the maximum value for the directional derivative at a if we limit the inputs to vectors of unit-length. If we did not limit the vectors to unit length then the directional derivative at a can become arbitrarily large as $D_v f(a)$ is proportional to the magnitude of v . Since our primary motivation in calculus III was describing rates of change along certain directions for some multivariate function it made sense to specialize the directional derivative to vectors of unit-length. The definition used in these notes better serves the theoretical discussion. If you read my calculus III notes you'll find a derivation of how the directional derivative in Stewart's calculus arises from the general definition of the derivative as a linear mapping. Look up page 305g. Incidentally, those notes may well be better than these in certain respects.

³this is why I have yet to give an example in this section, you should get out your calculus III notes if you need a refresher on directional derivatives

Proposition 3.2.7.

If $F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $a \in U$ then the differential dF_a has derivative matrix $F'(a)$ and it has components which are expressed in terms of partial derivatives of the component functions:

$$[dF_a]_{ij} = \partial_j F_i$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Perhaps it is helpful to expand the derivative matrix explicitly for future reference:

$$F'(a) = \begin{bmatrix} \partial_1 F_1(a) & \partial_2 F_1(a) & \cdots & \partial_n F_1(a) \\ \partial_1 F_2(a) & \partial_2 F_2(a) & \cdots & \partial_n F_2(a) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 F_m(a) & \partial_2 F_m(a) & \cdots & \partial_n F_m(a) \end{bmatrix}$$

Let's write the operation of the differential for a differentiable mapping at some point $a \in \mathbb{R}$ in terms of the explicit matrix multiplication by $F'(a)$. Let $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$,

$$dF_a(v) = F'(a)v = \begin{bmatrix} \partial_1 F_1(a) & \partial_2 F_1(a) & \cdots & \partial_n F_1(a) \\ \partial_1 F_2(a) & \partial_2 F_2(a) & \cdots & \partial_n F_2(a) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 F_m(a) & \partial_2 F_m(a) & \cdots & \partial_n F_m(a) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

You may recall the notation from calculus III at this point, omitting the a -dependence,

$$\nabla F_j = \text{grad}(F_j) = [\partial_1 F_j, \partial_2 F_j, \dots, \partial_n F_j]^T$$

So if the derivative exists we can write it in terms of a stack of gradient vectors of the component functions: (I used a transpose to write the stack side-ways),

$$F' = [\nabla F_1 | \nabla F_2 | \cdots | \nabla F_m]^T$$

Finally, just to collect everything together,

$$F' = \begin{bmatrix} \partial_1 F_1 & \partial_2 F_1 & \cdots & \partial_n F_1 \\ \partial_1 F_2 & \partial_2 F_2 & \cdots & \partial_n F_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 F_m & \partial_2 F_m & \cdots & \partial_n F_m \end{bmatrix} = [\partial_1 F | \partial_2 F | \cdots | \partial_n F] = \begin{bmatrix} (\nabla F_1)^T \\ (\nabla F_2)^T \\ \vdots \\ (\nabla F_m)^T \end{bmatrix}$$

Example 3.2.8. Recall that in Example 3.1.5 we showed that $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $F(x, y) = (xy, x^2, x + 3y)$ for all $(x, y) \in \mathbb{R}^2$ was differentiable. In fact we calculated that

$$dF_{(x,y)}(h, k) = \begin{bmatrix} y & x \\ 2x & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}.$$

If you recall from calculus III the mechanics of partial differentiation it's simple to see that

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(xy, x^2, x+3y) = (y, 2x, 1) = \begin{bmatrix} y \\ 2x \\ 1 \end{bmatrix}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(xy, x^2, x+3y) = (x, 0, 3) = \begin{bmatrix} x \\ 0 \\ 3 \end{bmatrix}$$

Thus $[dF] = [\partial_x F | \partial_y F]$ (as we expect given the derivations in this section!)

3.2.2 continuously differentiable, a cautionary tale

We have noted that differentiability on some set U implies all sorts of nice formulas in terms of the partial derivatives. Curiously the converse is not quite so simple. It is possible for the partial derivatives to exist on some set and yet the mapping may fail to be differentiable. We need an extra topological condition on the partial derivatives if we are to avoid certain pathological⁴ examples.

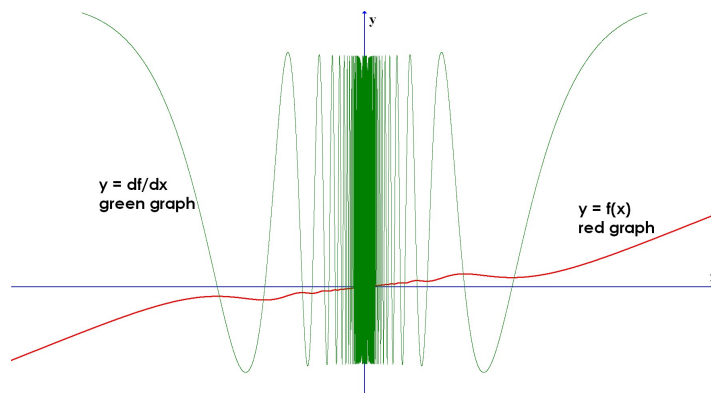
Example 3.2.9. I found this example in Hubbard's advanced calculus text(see Ex. 1.9.4, pg. 123). It is a source of endless odd examples, notation and bizarre quotes. Let $f(x) = 0$ and

$$f(x) = \frac{x}{2} + x^2 \sin \frac{1}{x}$$

for all $x \neq 0$. I can be shown that the derivative $f'(0) = 1/2$. Moreover, we can show that $f'(x)$ exists for all $x \neq 0$, we can calculate:

$$f'(x) = \frac{1}{2} + 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

Notice that $\text{dom}(f') = \mathbb{R}$. Note then that the tangent line at $(0,0)$ is $y = x/2$.



⁴"pathological" as in, "your clothes are so pathological, where'd you get them?"

You might be tempted to say then that this function is increasing at a rate of $1/2$ for x near zero. But this claim would be false since you can see that $f'(x)$ oscillates wildly without end near zero. We have a tangent line at $(0,0)$ with positive slope for a function which is not increasing at $(0,0)$ (recall that increasing is a concept we must define in an open interval to be careful). This sort of thing cannot happen if the derivative is continuous near the point in question.

The one-dimensional case is really quite special, even though we had discontinuity of the derivative we still had a well-defined tangent line to the point. However, many interesting theorems in calculus of one-variable require the function to be continuously differentiable near the point of interest. For example, to apply the 2nd-derivative test we need to find a point where the first derivative is zero and the second derivative exists. We cannot hope to compute $f''(x_o)$ unless f' is continuous at x_o . The next example is *sick*.

Example 3.2.10. Let us define $f(0,0) = 0$ and

$$f(x,y) = \frac{x^2y}{x^2 + y^2}$$

for all $(x,y) \neq (0,0)$ in \mathbb{R}^2 . It can be shown that f is continuous at $(0,0)$. Moreover, since $f(x,0) = f(0,y) = 0$ for all x and all y it follows that f vanishes identically along the coordinate axis. Thus the rate of change in the e_1 or e_2 directions is zero. We can calculate that

$$\frac{\partial f}{\partial x} = \frac{2xy^3}{(x^2 + y^2)^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x^4 - x^2y^2}{(x^2 + y^2)^2}$$

Consider the path to the origin $t \mapsto (t,t)$ gives $f_x(t,t) = 2t^4/(t^2 + t^2)^2 = 1/2$ hence $f_x(x,y) \rightarrow 1/2$ along the path $t \mapsto (t,t)$, but $f_x(0,0) = 0$ hence the partial derivative f_x is not continuous at $(0,0)$. In this example, the discontinuity of the partial derivatives makes the tangent plane fail to exist.

XXX— need to include graph of this thing.

Definition 3.2.11.

A mapping $F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **continuously differentiable** at $a \in U$ iff the partial derivative mappings $D_j F$ exist on an open set containing a and are continuous at a .

The definition above is interesting because of the proposition below. The import of the proposition is that we can build the tangent plane from the Jacobian matrix provided the partial derivatives are all continuous. This is a very nice result because the concept of the linear mapping is quite abstract but partial differentiation of a given mapping is easy.

Proposition 3.2.12.

If F is continuously differentiable at a then F is differentiable at a

We'll follow the proof in Edwards on pages 72-73.

- need to include the continuously diff. discuss here then do the examples.
- need to eliminate discussion of directional derivative except where absolutely necessary
- think about where to define the differential as a form. Consider peppering "differential at a point" many places...

3.2.3 gallery of derivatives

Our goal here is simply to exhibit the Jacobian matrix and partial derivatives for a few mappings. At the base of all these calculations is the observation that partial differentiation is just ordinary differentiation where we treat all the independent variable not being differentiated as constants. The criteria of independence is important. We'll study the case the variables are not independent in a later section.

Remark 3.2.13.

I have put remarks about the rank of the derivative in the examples below. Of course this has nothing to do with the process of calculating Jacobians. It's something to think about once we master the process of calculating the Jacobian. Ignore the red comments for now if you wish

Example 3.2.14. Let $f(t) = (t, t^2, t^3)$ then $f'(t) = (1, 2t, 3t^2)$. In this case we have

$$f'(t) = [df_t] = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}$$

The Jacobian here is a single column vector. It has rank 1 provided the vector is nonzero. We see that $f'(t) \neq (0, 0, 0)$ for all $t \in \mathbb{R}$. This corresponds to the fact that this space curve has a well-defined tangent line for each point on the path.

Example 3.2.15. Let $f(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$ be a mapping from $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$. I'll denote the coordinates in the domain by $(x_1, x_2, x_3, y_1, y_2, y_3)$ thus $f(\vec{x}, \vec{y}) = x_1y_1 + x_2y_2 + x_3y_3$. Calculate,

$$[df_{(\vec{x}, \vec{y})}] = \nabla f(\vec{x}, \vec{y})^T = [y_1, y_2, y_3, x_1, x_2, x_3]$$

The Jacobian here is a single row vector. It has rank 6 provided all entries of the input vectors are nonzero.

Example 3.2.16. Let $f(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$ be a mapping from $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. I'll denote the coordinates in the domain by $(x_1, \dots, x_n, y_1, \dots, y_n)$ thus $f(\vec{x}, \vec{y}) = \sum_{i=1}^n x_i y_i$. Calculate,

$$\frac{\partial}{\partial x_j} \left[\sum_{i=1}^n x_i y_i \right] = \sum_{i=1}^n \frac{\partial x_i}{\partial x_j} y_i = \sum_{i=1}^n \delta_{ij} y_i = y_j$$

Likewise,

$$\frac{\partial}{\partial y_j} \left[\sum_{i=1}^n x_i y_i \right] = \sum_{i=1}^n x_i \frac{\partial y_i}{\partial y_j} = \sum_{i=1}^n x_i \delta_{ij} = x_j$$

Therefore, noting that $\nabla f = (\partial_{x_1} f, \dots, \partial_{x_n} f, \partial_{y_1} f, \dots, \partial_{y_n} f)$,

$$[df_{(\vec{x}, \vec{y})}]^T = (\nabla f)(\vec{x}, \vec{y}) = \vec{y} \times \vec{x} = (y_1, \dots, y_n, x_1, \dots, x_n)$$

The Jacobian here is a single row vector. It has rank $2n$ provided all entries of the input vectors are nonzero.

Example 3.2.17. Suppose $F(x, y, z) = (xyz, y, z)$ we calculate,

$$\frac{\partial F}{\partial x} = (yz, 0, 0) \quad \frac{\partial F}{\partial y} = (xz, 1, 0) \quad \frac{\partial F}{\partial z} = (xy, 0, 1)$$

Remember these are actually column vectors in my sneaky notation; $(v_1, \dots, v_n) = [v_1, \dots, v_n]^T$. This means the **derivative** or **Jacobian matrix** of F at (x, y, z) is

$$F'(x, y, z) = [dF_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note, $\text{rank}(F'(x, y, z)) = 3$ for all $(x, y, z) \in \mathbb{R}^3$ such that $y, z \neq 0$. There are a variety of ways to see that claim, one way is to observe $\det[F'(x, y, z)] = yz$ and this determinant is nonzero so long as neither y nor z is zero. In linear algebra we learn that a square matrix is invertible iff it has nonzero determinant iff it has linearly independent column vectors.

Example 3.2.18. Suppose $F(x, y, z) = (x^2 + z^2, yz)$ we calculate,

$$\frac{\partial F}{\partial x} = (2x, 0) \quad \frac{\partial F}{\partial y} = (0, z) \quad \frac{\partial F}{\partial z} = (2z, y)$$

The derivative is a 2×3 matrix in this example,

$$F'(x, y, z) = [dF_{(x,y,z)}] = \begin{bmatrix} 2x & 0 & 2z \\ 0 & z & y \end{bmatrix}$$

The maximum rank for F' is 2 at a particular point (x, y, z) because there are at most two linearly independent vectors in \mathbb{R}^2 . You can consider the three square submatrices to analyze the rank for a given point. If any one of these is nonzero then the rank (dimension of the column space) is two.

$$M_1 = \begin{bmatrix} 2x & 0 \\ 0 & z \end{bmatrix} \quad M_2 = \begin{bmatrix} 2x & 2z \\ 0 & y \end{bmatrix} \quad M_3 = \begin{bmatrix} 0 & 2z \\ z & y \end{bmatrix}$$

We'll need either $\det(M_1) = 2xz \neq 0$ or $\det(M_2) = 2xy \neq 0$ or $\det(M_3) = -2z^2 \neq 0$. I believe the only point where all three of these fail to be true simultaneously is when $x = y = z = 0$. This mapping has maximal rank at all points except the origin.

Example 3.2.19. Suppose $F(x, y) = (x^2 + y^2, xy, x + y)$ we calculate,

$$\frac{\partial F}{\partial x} = (2x, y, 1) \quad \frac{\partial F}{\partial y} = (2y, x, 1)$$

The derivative is a 3×2 matrix in this example,

$$F'(x, y) = [dF_{(x,y)}] = \begin{bmatrix} 2x & 2y \\ y & x \\ 1 & 1 \end{bmatrix}$$

The maximum rank is again 2, this time because we only have two columns. The rank will be two if the columns are not linearly dependent. We can analyze the question of rank a number of ways but I find determinants of submatrices a comforting tool in these sort of questions. If the columns are linearly dependent then all three sub-square-matrices of F' will be zero. Conversely, if even one of them is nonvanishing then it follows the columns must be linearly independent. The submatrices for this problem are:

$$M_1 = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix} \quad M_2 = \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix} \quad M_3 = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$$

You can see $\det(M_1) = 2(x^2 - y^2)$, $\det(M_2) = 2(x - y)$ and $\det(M_3) = y - x$. Apparently we have $\text{rank}(F'(x, y, z)) = 2$ for all $(x, y) \in \mathbb{R}^2$ with $y \neq x$. In retrospect this is not surprising.

Example 3.2.20. Suppose $P(x, v, m) = (P_o, P_1) = (\frac{1}{2}mv^2 + \frac{1}{2}kx^2, mv)$ for some constant k . Let's calculate the derivative via gradients this time,

$$\nabla P_o = (\partial P_o / \partial x, \partial P_o / \partial v, \partial P_o / \partial m) = (kx, mv, \frac{1}{2}v^2)$$

$$\nabla P_1 = (\partial P_1 / \partial x, \partial P_1 / \partial v, \partial P_1 / \partial m) = (0, m, v)$$

Therefore,

$$P'(x, v, m) = \begin{bmatrix} kx & mv & \frac{1}{2}v^2 \\ 0 & m & v \end{bmatrix}$$

Example 3.2.21. Let $F(r, \theta) = (r \cos \theta, r \sin \theta)$. We calculate,

$$\partial_r F = (\cos \theta, \sin \theta) \quad \text{and} \quad \partial_\theta F = (-r \sin \theta, r \cos \theta)$$

Hence,

$$F'(r, \theta) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

We calculate $\det(F'(r, \theta)) = r$ thus this mapping has full rank everywhere except the origin.

Example 3.2.22. Let $G(x, y) = (\sqrt{x^2 + y^2}, \tan^{-1}(y/x))$. We calculate,

$$\partial_x G = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{-y}{x^2 + y^2} \right) \quad \text{and} \quad \partial_y G = \left(\frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{x^2 + y^2} \right)$$

Hence,

$$G'(x, y) = \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{bmatrix} = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{bmatrix} \quad \left(\text{using } r = \sqrt{x^2 + y^2} \right)$$

We calculate $\det(G'(x, y)) = 1/r$ thus this mapping has full rank everywhere except the origin.

Example 3.2.23. Let $F(x, y) = (x, y, \sqrt{R^2 - x^2 - y^2})$ for a constant R . We calculate,

$$\nabla \sqrt{R^2 - x^2 - y^2} = \left(\frac{-x}{\sqrt{R^2 - x^2 - y^2}}, \frac{-y}{\sqrt{R^2 - x^2 - y^2}} \right)$$

Also, $\nabla x = (1, 0)$ and $\nabla y = (0, 1)$ thus

$$F'(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{R^2 - x^2 - y^2}} & \frac{-y}{\sqrt{R^2 - x^2 - y^2}} \end{bmatrix}$$

This matrix clearly has rank 2 where it is well-defined. Note that we need $R^2 - x^2 - y^2 > 0$ for the derivative to exist. Moreover, we could define $G(y, z) = (\sqrt{R^2 - y^2 - z^2}, y, z)$ and calculate,

$$G'(y, z) = \begin{bmatrix} 1 & 0 \\ \frac{-y}{\sqrt{R^2 - y^2 - z^2}} & \frac{-z}{\sqrt{R^2 - y^2 - z^2}} \\ 0 & 1 \end{bmatrix}.$$

Observe that $G'(y, z)$ exists when $R^2 - y^2 - z^2 > 0$. Geometrically, F parametrizes the sphere above the equator at $z = 0$ whereas G parametrizes the right-half of the sphere with $x > 0$. These parametrizations overlap in the first octant where both x and z are positive. In particular, $\text{dom}(F') \cap \text{dom}(G') = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0 \text{ and } x^2 + y^2 < R^2\}$

Example 3.2.24. Let $F(x, y, z) = (x, y, z, \sqrt{R^2 - x^2 - y^2 - z^2})$ for a constant R . We calculate,

$$\nabla \sqrt{R^2 - x^2 - y^2 - z^2} = \left(\frac{-x}{\sqrt{R^2 - x^2 - y^2 - z^2}}, \frac{-y}{\sqrt{R^2 - x^2 - y^2 - z^2}}, \frac{-z}{\sqrt{R^2 - x^2 - y^2 - z^2}} \right)$$

Also, $\nabla x = (1, 0, 0)$, $\nabla y = (0, 1, 0)$ and $\nabla z = (0, 0, 1)$ thus

$$F'(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-x}{\sqrt{R^2 - x^2 - y^2 - z^2}} & \frac{-y}{\sqrt{R^2 - x^2 - y^2 - z^2}} & \frac{-z}{\sqrt{R^2 - x^2 - y^2 - z^2}} \end{bmatrix}$$

This matrix clearly has rank 3 where it is well-defined. Note that we need $R^2 - x^2 - y^2 - z^2 > 0$ for the derivative to exist. This mapping gives us a parametrization of the 3-sphere $x^2 + y^2 + z^2 + w^2 = R^2$ for $w > 0$. (drawing this is a little trickier)

Example 3.2.25. Let $f(x, y, z) = (x + y, y + z, x + z, xyz)$. You can calculate,

$$[df_{(x,y,z)}] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ yz & xz & xy \end{bmatrix}$$

This matrix clearly has rank 3 and is well-defined for all of \mathbb{R}^3 .

Example 3.2.26. Let $f(x, y, z) = xyz$. You can calculate,

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \end{bmatrix}$$

This matrix fails to have rank 3 if x, y or z are zero. In other words, $f'(x, y, z)$ has rank 3 in \mathbb{R}^3 provided we are at a point which is not on some coordinate plane. (the coordinate planes are $x = 0, y = 0$ and $z = 0$ for the yz, zx and xy coordinate planes respective)

Example 3.2.27. Let $f(x, y, z) = (xyz, 1 - x - y)$. You can calculate,

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \\ -1 & -1 & 0 \end{bmatrix}$$

This matrix has rank 3 if either $xy \neq 0$ or $(x - y)z \neq 0$. In contrast to the preceding example, the derivative does have rank 3 on certain points of the coordinate planes. For example, $f'(1, 1, 0)$ and $f'(0, 1, 1)$ both give $\text{rank}(f') = 3$.

Example 3.2.28. Let $f : \mathbb{R}^3 \times \mathbb{R}^3$ be defined by $f(x) = x \times v$ for a fixed vector $v \neq 0$. We denote $x = (x_1, x_2, x_3)$ and calculate,

$$\frac{\partial}{\partial x_a}(x \times v) = \frac{\partial}{\partial x_a} \left(\sum_{i,j,k} \epsilon_{ijk} x_i v_j e_k \right) = \sum_{i,j,k} \epsilon_{ijk} \frac{\partial x_i}{\partial x_a} v_j e_k = \sum_{i,j,k} \epsilon_{ijk} \delta_{ia} v_j e_k = \sum_{j,k} \epsilon_{ajk} v_j e_k$$

It follows,

$$\begin{aligned} \frac{\partial}{\partial x_1}(x \times v) &= \sum_{j,k} \epsilon_{1jk} v_j e_k = v_2 e_3 - v_3 e_2 = (0, -v_3, v_2) \\ \frac{\partial}{\partial x_2}(x \times v) &= \sum_{j,k} \epsilon_{2jk} v_j e_k = v_3 e_1 - v_1 e_3 = (v_3, 0, -v_1) \\ \frac{\partial}{\partial x_3}(x \times v) &= \sum_{j,k} \epsilon_{3jk} v_j e_k = v_1 e_2 - v_2 e_1 = (-v_2, v_1, 0) \end{aligned}$$

Thus the Jacobian is simply,

$$[df_{(x,y)}] = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & -v_1 \\ v_2 & v_1 & 0 \end{bmatrix}$$

In fact, $df_p(h) = f(h) = h \times v$ for each $p \in \mathbb{R}^3$. The given mapping is linear so the differential of the mapping is precisely the mapping itself.

Example 3.2.29. Let $f(x, y) = (x, y, 1 - x - y)$. You can calculate,

$$[df_{(x,y,z)}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Example 3.2.30. Let $X(u, v) = (x, y, z)$ where x, y, z denote functions of u, v and I prefer to omit the explicit dependence to reduce clutter in the equations to follow.

$$\frac{\partial X}{\partial u} = X_u = (x_u, y_u, z_u) \quad \text{and} \quad \frac{\partial X}{\partial v} = X_v = (x_v, y_v, z_v)$$

Then the Jacobian is the 3×2 matrix

$$[dX_{(u,v)}] = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$$

The matrix $[dX_{(u,v)}]$ has rank 2 if at least one of the determinants below is nonzero,

$$\det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \quad \det \begin{bmatrix} x_u & x_v \\ z_u & z_v \end{bmatrix} \quad \det \begin{bmatrix} y_u & y_v \\ z_u & z_v \end{bmatrix}$$

Example 3.2.31. . .

Example 3.2.32. . .

3.3 additivity and homogeneity of the derivative

Suppose $F_1 : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $F_2 : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$. Furthermore, suppose both of these are differentiable at $a \in \mathbb{R}^n$. It follows that $(dF_1)_a = L_1$ and $(dF_2)_a = L_2$ are linear operators from \mathbb{R}^n to \mathbb{R}^m which approximate the change in F_1 and F_2 near a , in particular,

$$\lim_{h \rightarrow 0} \frac{F_1(a+h) - F_1(a) - L_1(h)}{\|h\|} = 0 \quad \lim_{h \rightarrow 0} \frac{F_2(a+h) - F_2(a) - L_2(h)}{\|h\|} = 0$$

To prove that $H = F_1 + F_2$ is differentiable at $a \in \mathbb{R}^n$ we need to find a differential at a for H . Naturally, we expect $dH_a = d(F_1 + F_2)_a = (dF_1)_a + (dF_2)_a$. Let $L = (dF_1)_a + (dF_2)_a$ and consider,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{H(a+h) - H(a) - L(h)}{\|h\|} &= \lim_{h \rightarrow 0} \frac{F_1(a+h) + F_2(a+h) - F_1(a) - F_2(a) - L_1(h) - L_2(h)}{\|h\|} \\ &= \lim_{h \rightarrow 0} \frac{F_1(a+h) - F_1(a) - L_1(h)}{\|h\|} + \lim_{h \rightarrow 0} \frac{F_2(a+h) - F_2(a) - L_2(h)}{\|h\|} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Note that breaking up the limit was legal because we knew the subsequent limits existed and were zero by the assumption of differentiability of F_1 and F_2 at a . Finally, since $L = L_1 + L_2$ we know L is a linear transformation since the sum of linear transformations is a linear transformation. Moreover, the matrix of L is the sum of the matrices for L_1 and L_2 . Let $c \in \mathbb{R}$ and suppose $G = cF_1$ then we can also show that $dG_a = d(cF_1)_a = c(dF_1)_a$, the calculation is very similar except we just pull the constant c out of the limit. I'll let you write it out. Collecting our observations:

Proposition 3.3.1.

Suppose $F_1 : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $F_2 : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ are differentiable at $a \in U$ then $F_1 + F_2$ is differentiable at a and

$$\boxed{d(F_1 + F_2)_a = (dF_1)_a + (dF_2)_a} \quad \text{or} \quad \boxed{(F_1 + F_2)'(a) = F_1'(a) + F_2'(a)}$$

Likewise, if $c \in \mathbb{R}$ then

$$\boxed{d(cF_1)_a = c(dF_1)_a} \quad \text{or} \quad \boxed{(cF_1)'(a) = c(F_1'(a))}$$

These results suggest that the differential of a function is a new object which has a vector space structure. There is much more to say here later.

3.4 chain rule

The proof in Edwards is on 77-78. I'll give a heuristic proof here which captures the essence of the argument. The simplicity of this rule continues to amaze me.

Proposition 3.4.1.

If $F : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^p$ is differentiable at a and $G : V \subseteq \mathbb{R}^p \rightarrow \mathbb{R}^m$ is differentiable at $F(a) \in V$ then $G \circ F$ is differentiable at a and

$$d(G \circ F)_a = (dG)_{F(a)} \circ dF_a$$

or, in matrix notation,

$$(G \circ F)'(a) = G'(F(a))F'(a)$$

Proof Sketch:

In calculus III you may have learned how to calculate partial derivatives in terms of tree-diagrams and intermediate variable etc... We now have a way of understanding those rules and all the other chain rules in terms of one over-arching calculation: matrix multiplication of the constituent Jacobians in the composite function. Of course once we have this rule for the composite of two functions we can generalize to n -functions by a simple induction argument. For example, for three suitably defined mappings F, G, H ,

$$(F \circ G \circ H)'(a) = F'(G(H(a)))G'(H(a))H'(a)$$

Example 3.4.2. . .

Example 3.4.3. . .

Example 3.4.4. . .

Example 3.4.5. . .

Example 3.4.6. . .

Example 3.4.7. . .

3.5 product rules?

What sort of product can we expect to find among mappings? Remember two mappings have vector outputs and there is no way to multiply vectors in general. Of course, in the case we have two mappings that have equal-dimensional outputs we could take their dot-product. There is a product rule for that case: if $\vec{A}, \vec{B} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ then

$$\partial_j(\vec{A} \cdot \vec{B}) = (\partial_j \vec{A}) \cdot \vec{B} + \vec{A} \cdot (\partial_j \vec{B})$$

Or in the special case of $m = 3$ we could even take their cross-product and there is another product rule in that case:

$$\partial_j(\vec{A} \times \vec{B}) = (\partial_j \vec{A}) \times \vec{B} + \vec{A} \times (\partial_j \vec{B})$$

What other case can we "multiply" vectors? One very important case is $\mathbb{R}^2 = \mathbb{C}$ where it is customary to use the notation $(x, y) = x + iy$ and $f = u + iv$. If our range is complex numbers then we again have a product rule: if $f : \mathbb{R}^n \rightarrow \mathbb{C}$ and $g : \mathbb{R}^n \rightarrow \mathbb{C}$ then

$$\partial_j(fg) = (\partial_j f)g + f(\partial_j g)$$

I have relegated the proof of these product rules to the end of this chapter. One other object worth differentiating is a matrix-valued function of \mathbb{R}^n . If we **define** the partial derivative of a matrix to be the matrix of partial derivatives then partial differentiation will respect the sum and product of matrices (we may return to this in depth if need be later on):

$$\partial_j(A + B) = \partial_j A + \partial_j B$$

$$\partial_j(AB) = (\partial_j A)B + A(\partial_j B)$$

Moral of this story? If you have a pair mappings whose ranges allow some sort of product then it is entirely likely that there is a corresponding product rule ⁵.

3.5.1 scalar-vector product rule

There is one product rule which we can state for arbitrary mappings, note that we can always sensibly multiply a mapping by a function. Suppose then that $G : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable at $a \in U$. It follows that there exist linear transformations $L_G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $L_f : \mathbb{R}^n \rightarrow \mathbb{R}$ where

$$\lim_{h \rightarrow 0} \frac{G(a+h) - G(a) - L_G(h)}{\|h\|} = 0 \quad \lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - L_f(h)}{h} = 0$$

Since $G(a+h) \approx G(a) + L_G(h)$ and $f(a+h) \approx f(a) + L_f(h)$ we expect

$$\begin{aligned} fG(a+h) &\approx (f(a) + L_f(h))(G(a) + L_G(h)) \\ &\approx (fG)(a) + \underbrace{G(a)L_f(h) + f(a)L_G(h)}_{\text{linear in } h} + \underbrace{L_f(h)L_G(h)}_{2^{nd} \text{ order in } h} \end{aligned}$$

⁵In my research I consider functions on supernumbers, these also can be multiplied. Naturally there is a product rule for super functions, the catch is that super numbers z, w do not necessarily commute. However, if they're homogeneous $zw = (-1)^{\epsilon_w \epsilon_z} wz$. Because of this the super product rule is $\partial_M(fg) = (\partial_M f)g + (-1)^{\epsilon_f \epsilon_M} f(\partial_M g)$

Thus we propose: $L(h) = G(a)L_f(h) + f(a)L_G(h)$ is the best linear approximation of fG .

$$\begin{aligned}
& \lim_{h \rightarrow 0} \frac{(fG)(a+h) - (fG)(a) - L(h)}{\|h\|} = \\
&= \lim_{h \rightarrow 0} \frac{f(a+h)G(a+h) - f(a)G(a) - G(a)L_f(h) - f(a)L_G(h)}{\|h\|} \\
&= \lim_{h \rightarrow 0} \frac{f(a+h)G(a+h) - f(a)G(a) - G(a)L_f(h) - f(a)L_G(h)}{\|h\|} + \\
&\quad + \lim_{h \rightarrow 0} \frac{f(a)G(a+h) - G(a+h)f(a)}{\|h\|} \\
&\quad + \lim_{h \rightarrow 0} \frac{f(a+h)G(a) - G(a)f(a+h)}{\|h\|} \\
&\quad + \lim_{h \rightarrow 0} \frac{f(a)G(a) - G(a)f(a)}{\|h\|} \\
&= \lim_{h \rightarrow 0} \left[f(a) \frac{G(a+h) - G(a) - L_G(h)}{\|h\|} + \frac{f(a+h) - f(a) - L_f(h)}{\|h\|} G(a) + \right. \\
&\quad \left. + \left(f(a+h) - f(a) \right) \frac{G(a+h) - G(a)}{\|h\|} \right] \\
&= f(a) \left[\lim_{h \rightarrow 0} \frac{G(a+h) - G(a) - L_G(h)}{\|h\|} \right] + \left[\lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - L_f(h)}{\|h\|} \right] G(a) \\
&= 0
\end{aligned}$$

Where we have made use of the differentiability and the consequent continuity of both f and G at a . Furthermore, note

$$\begin{aligned}
L(h+ck) &= G(a)L_f(h+ck) + f(a)L_G(h+ck) \\
&= G(a)(L_f(h) + cL_f(k)) + f(a)(L_G(h) + cL_G(k)) \\
&= G(a)L_f(h) + f(a)(L_G(h) + c(G(a)L_f(k) + f(a)L_G(k))) \\
&= L(h) + cL(k)
\end{aligned}$$

for all $h, k \in \mathbb{R}^n$ and $c \in \mathbb{R}$ hence $L = G(a)L_f + f(a)L_G$ is a linear transformation. We have proved (most of) the following proposition:

Proposition 3.5.1.

If $G : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ are differentiable at $a \in U$ then fG is differentiable at a and

$$d(fG)_a = (df)_a G(a) + f(a)dG_a$$

$$(fG)'(a) = f'(a)G(a) + f(a)G'(a)$$

The argument above covers the ordinary product rule and a host of other less common rules. Note again that $G(a)$ and $G'(a)$ are vectors.

3.5.2 calculus of paths in \mathbb{R}^3

A **path** is a mapping from \mathbb{R} to \mathbb{R}^m . We use such mappings to model position, velocity and acceleration of particles in the case $m = 3$. Some of these things were proved in previous sections of this chapter but I intend for this section to be self-contained so that you can read it without digging through the rest of this chapter.

Proposition 3.5.2.

If $F, G : U \subseteq \mathbb{R} \rightarrow \mathbb{R}^m$ are differentiable vector-valued functions and $\phi : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable real-valued function then for each $t \in U$,

1. $(F + G)'(t) = F'(t) + G'(t)$.
2. $(cF)'(t) = cF'(t)$.
3. $(\phi F)'(t) = \phi'(t)F(t) + \phi(t)F'(t)$.
4. $(F \cdot G)'(t) = F'(t) \cdot G(t) + F(t) \cdot G'(t)$.
5. provided $m = 3$, $(F \times G)'(t) = F'(t) \times G(t) + F(t) \times G'(t)$.
6. provided $\phi(U) \subset \text{dom}(F')$, $(F \circ \phi)'(t) = \phi'(t)F'(\phi(t))$.

We have to insist that $m = 3$ for the statement with cross-products since we only have a standard cross-product in \mathbb{R}^3 . We prepare for the proof of the proposition with a useful lemma. Notice this lemma tells us how to actually calculate the derivative of paths in examples. The derivative of component functions is nothing more than calculus I and one of our goals is to reduce things to those sort of calculations whenever possible.

Lemma 3.5.3.

If $F : U \subseteq \mathbb{R} \rightarrow \mathbb{R}^m$ is differentiable vector-valued function then for all $t \in U$,

$$F'(t) = (F'_1(t), F'_2(t), \dots, F'_m(t))$$

We are given that the following vector limit exists and is equal to $F'(t)$,

$$F'(t) = \lim_{h \rightarrow 0} \frac{F(t+h) - F(t)}{h}$$

then by Proposition 1.4.10 the limit of a vector is related to the limits of its components as follows:

$$F'(t) \cdot e_j = \lim_{h \rightarrow 0} \frac{F_j(t+h) - F_j(t)}{h}.$$

Thus $(F'(t))_j = F'_j(t)$ and the lemma follows⁶. ∇

⁶this notation I first saw in a text by Marsden, it means the proof is partially completed but you should read on to finish the proof

Proof of proposition: We use the notation $F = \sum F_j e_j = (F_1, \dots, F_m)$ and $G = \sum_i G_i e_i = (G_1, \dots, G_m)$ throughout the proofs below. The \sum is understood to range over $1, 2, \dots, m$. Begin with (1.),

$$\begin{aligned}
 [(F + G)']_j &= \frac{d}{dt}[(F + G)_j] && \text{using the lemma} \\
 &= \frac{d}{dt}[F_j + G_j] && \text{using def. } (F + G)_j = F_j + G_j \\
 &= \frac{d}{dt}[F_j] + \frac{d}{dt}[G_j] && \text{by calculus I, } (f + g)' = f' + g'. \\
 &= [F' + G']_j && \text{def. of vector addition for } F' \text{ and } G'
 \end{aligned}$$

Hence $(F \times G)' = F' \times G + F \times G'$. The proofs of 2,3,5 and 6 are similar. I'll prove (5.),

$$\begin{aligned}
 [(F \times G)']_k &= \frac{d}{dt}[(F \times G)_k] && \text{using the lemma} \\
 &= \frac{d}{dt}[\sum \epsilon_{ijk} F_i G_j] && \text{using def. } F \times G \\
 &= \sum \epsilon_{ijk} \frac{d}{dt}[F_i G_j] && \text{repeatedly using, } (f + g)' = f' + g' \\
 &= \sum \epsilon_{ijk} [\frac{dF_i}{dt} G_j + F_i \frac{dG_j}{dt}] && \text{repeatedly using, } (fg)' = f'g + fg' \\
 &= \sum \epsilon_{ijk} \frac{dF_i}{dt} G_j \sum \epsilon_{ijk} F_i \frac{dG_j}{dt} && \text{property of finite sum } \sum \\
 &= (\frac{dF}{dt} \times G)_k + (F \times \frac{dG}{dt})_k && \text{def. of cross product} \\
 &= (\frac{dF}{dt} \times G + F \times \frac{dG}{dt})_k && \text{def. of vector addition}
 \end{aligned}$$

Notice that the calculus step really just involves calculus I applied to the components. The ordinary product rule was the crucial factor to prove the product rule for cross-products. We'll see the same for the dot product of mappings. Prove (4.)

$$\begin{aligned}
 (F \cdot G)'(t) &= \frac{d}{dt}[\sum F_k G_k] && \text{using def. } F \cdot G \\
 &= \sum \frac{d}{dt}[F_k G_k] && \text{repeatedly using, } (f + g)' = f' + g' \\
 &= \sum [\frac{dF_k}{dt} G_k + F_k \frac{dG_k}{dt}] && \text{repeatedly using, } (fg)' = f'g + fg' \\
 &= \frac{dF}{dt} \cdot G + F \cdot \frac{dG}{dt}. && \text{def. of dot product}
 \end{aligned}$$

The proof of (3.) follows from applying the product rule to each component of $\phi(t)F(t)$. The proof of (2.) follow from (3.) in the case that $\phi(t) = c$ so $\phi'(t) = 0$. Finally the proof of (6.) follows from applying the chain-rule to each component. \square

3.5.3 calculus of matrix-valued functions of a real variable

Definition 3.5.4.

A matrix-valued function of a real variable is a function from $I \subseteq \mathbb{R}$ to $\mathbb{R}^{m \times n}$. Suppose $A : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ is such that $A_{ij} : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is differentiable for each i, j then we define

$$\frac{dA}{dt} = \left[\frac{dA_{ij}}{dt} \right]$$

which can also be denoted $(A')_{ij} = A'_{ij}$. We likewise define $\int A dt = [\int A_{ij} dt]$ for A with integrable components. Definite integrals and higher derivatives are also defined component-wise.

Example 3.5.5. Suppose $A(t) = \begin{bmatrix} 2t & 3t^2 \\ 4t^3 & 5t^4 \end{bmatrix}$. I'll calculate a few items just to illustrate the definition above. calculate; to differentiate a matrix we differentiate each component one at a time:

$$A'(t) = \begin{bmatrix} 2 & 6t \\ 12t^2 & 20t^3 \end{bmatrix} \quad A''(t) = \begin{bmatrix} 0 & 6 \\ 24t & 60t^2 \end{bmatrix} \quad A'(0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Integrate by integrating each component:

$$\int A(t) dt = \begin{bmatrix} t^2 + c_1 & t^3 + c_2 \\ t^4 + c_3 & t^5 + c_4 \end{bmatrix} \quad \int_0^2 A(t) dt = \begin{bmatrix} t^2|_0^2 & t^3|_0^2 \\ t^4|_0^2 & t^5|_0^2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 16 & 32 \end{bmatrix}$$

Proposition 3.5.6.

Suppose A, B are matrix-valued functions of a real variable, f is a function of a real variable, c is a constant, and C is a constant matrix then

1. $(AB)' = A'B + AB'$ (product rule for matrices)
2. $(AC)' = A'C$
3. $(CA)' = CA'$
4. $(fA)' = f'A + fA'$
5. $(cA)' = cA'$
6. $(A + B)' = A' + B'$

where each of the functions is evaluated at the same time t and I assume that the functions and matrices are differentiable at that value of t and of course the matrices A, B, C are such that the multiplications are well-defined.

Proof: Suppose $A(t) \in \mathbb{R}^{m \times n}$ and $B(t) \in \mathbb{R}^{n \times p}$ consider,

$$\begin{aligned}
 (AB)'_{ij} &= \frac{d}{dt}((AB)_{ij}) && \text{defn. derivative of matrix} \\
 &= \frac{d}{dt}(\sum_k A_{ik} B_{kj}) && \text{defn. of matrix multiplication} \\
 &= \sum_k \frac{d}{dt}(A_{ik} B_{kj}) && \text{linearity of derivative} \\
 &= \sum_k \left[\frac{dA_{ik}}{dt} B_{kj} + A_{ik} \frac{dB_{kj}}{dt} \right] && \text{ordinary product rules} \\
 &= \sum_k \frac{dA_{ik}}{dt} B_{kj} + \sum_k A_{ik} \frac{dB_{kj}}{dt} && \text{algebra} \\
 &= (A'B)_{ij} + (AB')_{ij} && \text{defn. of matrix multiplication} \\
 &= (A'B + AB')_{ij} && \text{defn. matrix addition}
 \end{aligned}$$

this proves (1.) as i, j were arbitrary in the calculation above. The proof of (2.) and (3.) follow quickly from (1.) since C constant means $C' = 0$. Proof of (4.) is similar to (1.):

$$\begin{aligned}
 (fA)'_{ij} &= \frac{d}{dt}((fA)_{ij}) && \text{defn. derivative of matrix} \\
 &= \frac{d}{dt}(fA_{ij}) && \text{defn. of scalar multiplication} \\
 &= \frac{df}{dt}A_{ij} + f \frac{dA_{ij}}{dt} && \text{ordinary product rule} \\
 &= \left(\frac{df}{dt}A + f \frac{dA}{dt} \right)_{ij} && \text{defn. matrix addition} \\
 &= \left(\frac{df}{dt}A + f \frac{dA}{dt} \right)_{ij} && \text{defn. scalar multiplication.}
 \end{aligned}$$

The proof of (5.) follows from taking $f(t) = c$ which has $f' = 0$. I leave the proof of (6.) as an exercise for the reader. \square .

To summarize: the calculus of matrices is the same as the calculus of functions with the small qualifier that we must respect the rules of matrix algebra. The noncommutativity of matrix multiplication is the main distinguishing feature.

3.5.4 calculus of complex-valued functions of a real variable

Differentiation of functions from \mathbb{R} to \mathbb{C} is defined by splitting a given function into its real and imaginary parts then we just differentiate with respect to the real variable one component at a time. For example:

$$\begin{aligned}
 \frac{d}{dt}(e^{2t} \cos(t) + ie^{2t} \sin(t)) &= \frac{d}{dt}(e^{2t} \cos(t)) + i \frac{d}{dt}(e^{2t} \sin(t)) \\
 &= (2e^{2t} \cos(t) - e^{2t} \sin(t)) + i(2e^{2t} \sin(t) + e^{2t} \cos(t)) \\
 &= e^{2t}(2 + i)(\cos(t) + i \sin(t)) \\
 &= (2 + i)e^{(2+i)t}
 \end{aligned} \tag{3.1}$$

where I have made use of the identity⁷ $e^{x+iy} = e^x(\cos(y) + i \sin(y))$. We just saw that $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$ which seems obvious enough until you appreciate that we just proved it for $\lambda = 2 + i$.

⁷or definition, depending on how you choose to set-up the complex exponential, I take this as the definition in calculus II

3.6 complex analysis in a nutshell

Differentiation with respect to a real variable can be reduced to the slogan that we differentiate componentwise. Differentiation with respect to a complex variable requires additional structure. The key distinguishing ingredient is complex linearity:

Definition 3.6.1.

If we have some function $T : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$(1.) T(v + w) = T(v) + T(w) \text{ for all } v, w \in \mathbb{C} \quad (2.) T(cv) = cT(v) \text{ for all } c, v \in \mathbb{C}$$

then we would say that T is **complex-linear**.

Condition (1.) is **additivity** whereas condition (2.) is **homogeneity**. Note that complex linearity implies real linearity however the converse is not true.

Example 3.6.2. Suppose $T(z) = \bar{z}$ where if $z = x + iy$ for $x, y \in \mathbb{R}$ then $\bar{z} = x - iy$ is the **complex conjugate** of z . Consider for $c = a + ib$ where $a, b \in \mathbb{R}$,

$$\begin{aligned} T(cz) &= T((a + ib)(x + iy)) \\ &= T(ax - by + i(ay + bx)) \\ &= ax - by - i(ay + bx) \\ &= (a - ib)(x - iy) \\ &= \bar{c} T(z) \end{aligned}$$

hence this map is not complex linear. On the other hand, if we study multiplication by just $a \in \mathbb{R}$,

$$T(az) = T(a(x + iy)) = T(ax + iay) = ax - iay = a(x - iy) = aT(x + iy) = aT(z)$$

thus T is homogeneous with respect to real-number multiplication and it is also additive hence T is real linear.

Suppose that L is a linear mapping from \mathbb{R}^2 to \mathbb{R}^2 . It is known from linear algebra that there exists a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $L(v) = Av$ for all $v \in \mathbb{R}^2$. In this section we use the notation $a + ib = (a, b)$ and

$$(a, b) * (c, d) = (a + ib)(c + id) = ac - bd + i(ad + bc) = (ac - bd, ad + bc).$$

This construction is due to Gauss in the early nineteenth century, the idea is to use two component vectors to *construct* complex numbers. There are other ways to construct complex numbers⁸. Notice that $L(x + iy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax + by, cx + dy) = ax + by + i(cx + dy)$ defines a real linear mapping on \mathbb{C} for any choice of the real constants a, b, c, d . In contrast, complex linearity puts strict conditions on these constants:

⁸the same is true for real numbers, you can construct them in more than one way, however all constructions agree on the basic properties and as such it is the properties of real or complex numbers which truly defined them. That said, we choose Gauss' representation for convenience.

Theorem 3.6.3.

The linear mapping $L(v) = Av$ is complex linear iff the matrix A will have the special form below:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

To be clear, we mean to identify \mathbb{R}^2 with \mathbb{C} as before. Thus the condition of complex homogeneity reads $L((a, b) * (x, y)) = (a, b) * L(x, y)$

Proof: assume L is complex linear. Define the matrix of L as before:

$$L(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This yields,

$$L(x + iy) = ax + by + i(cx + dy)$$

We can gain conditions on the matrix by examining the special points $1 = (1, 0)$ and $i = (0, 1)$

$$L(1, 0) = (a, c) \quad L(0, 1) = (b, d)$$

Note that $(c_1, c_2) * (1, 0) = (c_1, c_2)$ hence $L((c_1 + ic_2)1) = (c_1 + ic_2)L(1)$ yields

$$(ac_1 + bc_2) + i(cc_1 + dc_2) = (c_1 + ic_2)(a + ic) = c_1a - c_2c + i(c_1c + c_2a)$$

We find two equations by equating the real and imaginary parts:

$$ac_1 + bc_2 = c_1a - c_2c \quad cc_1 + dc_2 = c_1c + c_2a$$

Therefore, $bc_2 = -c_2c$ and $dc_2 = c_2a$ for all $(c_1, c_2) \in \mathbb{C}$. Suppose $c_1 = 0$ and $c_2 = 1$. We find $b = -c$ and $d = a$. We leave the converse proof to the reader. The proposition follows. \square

In analogy with the real case we define $f'(z)$ as follows:

Definition 3.6.4.

Suppose $f : \text{dom}(f) \subseteq \mathbb{C} \rightarrow \mathbb{C}$ and $z \in \text{dom}(f)$ then we define $f'(z)$ by the limit below:

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}.$$

The derivative function f' is defined pointwise for all such $z \in \text{dom}(f)$ that the limit above exists.

Note that $f'(z) = \lim_{h \rightarrow 0} \frac{f'(z)h}{h}$ hence

$$\lim_{h \rightarrow 0} \frac{f'(z)h}{h} = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} \Rightarrow \lim_{h \rightarrow 0} \frac{f(z+h) - f(z) - f'(z)h}{h} = 0$$

Note that the limit above simply says that $L(v) = f'(z)v$ gives the is the best **complex-linear** approximation of $\Delta f = f(z+h) - f(z)$.

Proposition 3.6.5.

If f is a complex differentiable at z_o then linearization $L(h) = f'(z_o)h$ is a complex linear mapping.

Proof: let $c, h \in \mathbb{C}$ and note $L(ch) = f'(z_o)(ch) = cf'(z_o)h = cL(h)$. \square

It turns out that complex differentiability automatically induces real differentiability:

Proposition 3.6.6.

If f is a complex differentiable at z_o then f is (real) differentiable at z_o with $L(h) = f'(z_o)h$.

Proof: note that $\lim_{h \rightarrow 0} \frac{f(z+h) - f(z) - f'(z)h}{h} = 0$ implies

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z) - f'(z)h}{|h|} = 0$$

but then $|h| = ||h||$ and we know $L(h) = f'(z_o)h$ is real-linear hence L is the best linear approximation to Δf at z_o and the proposition follows. \square

Let's summarize what we've learned: if $f : \text{dom}(f) \rightarrow \mathbb{C}$ is complex differentiable at z_o and $f = u + iv$ then,

1. $L(h) = f'(z_o)h$ is complex linear.
2. $L(h) = f'(z_o)h$ is the best real linear approximation to f viewed as a mapping on \mathbb{R}^2 .

The Jacobian matrix for $f = (u, v)$ has the form

$$J_f(p_o) = \begin{bmatrix} u_x(p_o) & u_y(p_o) \\ v_x(p_o) & v_y(p_o) \end{bmatrix}$$

Theorem 3.6.3 applies to $J_f(p_o)$ since L is a complex linear mapping. Therefore we find the Cauchy Riemann equations: $u_x = v_y$ and $u_y = -v_x$. We have proved the following theorem:

Theorem 3.6.7.

If $f = u + iv$ is a complex function which is **complex-differentiable** at z_o then the partial derivatives of u and v exist at z_o and satisfy the Cauchy-Riemann equations at z_o

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Example 3.6.8. Let $f(z) = e^z$ where the definition of the complex exponential function is given by the following, for each $x, y \in \mathbb{R}$ and $z = x + iy$

$$f(x + iy) = e^{x+iy} = e^x(\cos(y) + i \sin(y)) = e^x \cos(y) + ie^x \sin(y)$$

Identify for $f = u + iv$ we have $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$. Calculate:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} [e^x \cos(y)] = e^x \cos(y) \quad \& \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} [e^x \cos(y)] = -e^x \sin(y),$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} [e^x \sin(y)] = e^x \sin(y) \quad \& \quad \frac{\partial v}{\partial y} = \frac{\partial}{\partial y} [e^x \sin(y)] = e^x \cos(y).$$

Thus f satisfies the CR-equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. The complex exponential function is complex differentiable.

The converse of Theorem 3.6.7 is not true in general. It is possible to have functions $u, v : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ that satisfy the CR-equations at $z_o \in U$ and yet $f = u + iv$ fails to be complex differentiable at z_o .

Example 3.6.9. Counter-example to converse of Theorem 3.6.7. Suppose $f(x+iy) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$.

Clearly f is identically zero on the coordinate axes thus along the x -axis we can calculate the partial derivatives for u and v and they are both zero. Likewise, along the y -axis we find u_y and v_y exist and are zero. At the origin we find u_x, u_y, v_x, v_y all exist and are zero. Therefore, the Cauchy-Riemann equations hold true at the origin. However, this function is not even continuous at the origin, thus it is not real differentiable!

The example above equally well serves as an example for a point where a function has partial derivatives which exist at all orders and yet the differential fails to exist. It's not a problem of complex variables in my opinion, it's a problem of advanced calculus. The key concept to reverse the theorem is continuous differentiability.

Theorem 3.6.10.

If u, v, u_x, u_y, v_x, v_y are continuous functions in some open disk of z_o and $u_x(z_o) = v_y(z_o)$ and $u_y(z_o) = -v_x(z_o)$ then $f = u + iv$ is complex differentiable at z_o .

Proof: we are given that a function $f : D_\epsilon(z_o) \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is continuous with continuous partial derivatives of its component functions u and v . Therefore, by Theorem ?? we know f is (real) differentiable at z_o . Therefore, we have a best linear approximation to the change in f near z_o which can be induced via multiplication of the Jacobian matrix:

$$L(v_1, v_2) = \begin{bmatrix} u_x(z_o) & u_y(z_o) \\ v_x(z_o) & v_y(z_o) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Note then that the given CR-equations show the matrix of L has the form

$$[L] = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $a = u_x(z_o)$ and $b = v_x(z_o)$. Consequently we find L is complex linear and it follows that f is complex differentiable at z_o since we have a complex linear map L such that

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z) - L(h)}{\|h\|} = 0$$

note that the limit with h in the denominator is equivalent to the limit above which followed directly from the (real) differentiability at z_o . (the following is not needed for the proof of the theorem, but perhaps it is interesting anyway) Moreover, we can write

$$\begin{aligned} L(h_1, h_2) &= \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &= \begin{bmatrix} u_x h_1 + u_y h_2 \\ -u_y h_1 + u_x h_2 \end{bmatrix} \\ &= u_x h_1 + u_y h_2 + i(-u_y h_1 + u_x h_2) \\ &= (u_x - iu_y)(h_1 + ih_2) \end{aligned}$$

Therefore we find $f'(z_o) = u_x - iu_y$ gives $L(h) = f'(z_o)z$. \square

In the preceding section we found necessary and sufficient conditions for the component functions u, v to construct an complex differentiable function $f = u + iv$. The definition that follows is the next logical step: we say a function is analytic⁹ at z_o if it is complex differentiable at each point in some open disk about z_o .

Definition 3.6.11.

Let $f = u + iv$ be a complex function. If there exists $\epsilon > 0$ such that f is complex differentiable for each $z \in D_\epsilon(z_o)$ then we say that f is **analytic** at z_o . If f is analytic for each $z_o \in U$ then we say f is analytic on U . If f is not analytic at z_o then we say that z_o is a **singular point**. Singular points may be outside the domain of the function. If f is analytic on the entire complex plane then we say f is **entire**. **Analytic functions are also called holomorphic functions**

If you look in my complex variables notes you can find proof of the following theorem (well, partial proof perhaps, but this result is shown in every good complex variables text)

Theorem 3.6.12.

If $f : \mathbb{R} \rightarrow \mathbb{R} \subset \mathbb{C}$ is a function and $\tilde{f} : \mathbb{C} \rightarrow \mathbb{C}$ is an extension of f which is analytic then \tilde{f} is unique. In particular, if there is an analytic extension of sine, cosine, hyperbolic sine or hyperbolic cosine then those extensions are unique.

This means if we demand analyticity then we actually had no freedom in our choice of the exponential. If we find a complex function which matches the exponential function on a line-segment (in

⁹you may recall that a function on \mathbb{R} was analytic at x_o if its Taylor series at x_o converged to the function in some neighborhood of x_o . This terminology is consistent but I leave the details for your complex analysis course

particular a closed interval in \mathbb{R} viewed as a subset of \mathbb{C} is a line-segment) then there is just one complex function which agrees with the real exponential and is complex differentiable everywhere.

$$f(x) = e^x \quad \text{extends uniquely to} \quad \tilde{f}(z) = e^{\operatorname{Re}(z)}(\cos(\operatorname{Im}(z)) + i \sin(\operatorname{Im}(z))).$$

Note $\tilde{f}(x + 0i) = e^x(\cos(0) + i \sin(0)) = e^x$ thus $\tilde{f}|_{\mathbb{R}} = f$. Naturally, analyticity is a desirable property for the complex-extension of known functions so this concept of analytic continuation is very nice. Existence aside, we should first construct sine, cosine etc... then we have to check they are both analytic and also that they actually agree with the real sine or cosine etc... If a function on \mathbb{R} has vertical asymptotes, points of discontinuity or points where it is not smooth then the story is more complicated.

3.6.1 harmonic functions

We've discussed in some depth how to determine if a given function $f = u + iv$ is in fact analytic. In this section we study another angle on the story. We learn that the component functions u, v of an analytic function $f = u + iv$ are *harmonic conjugates* and they satisfy the physically significant Laplace's equation $\nabla^2 \phi = 0$ where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. In addition we'll learn that if we have one solution of Laplace's equation then we can consider it to be the "u" of some yet undetermined analytic function $f = u + iv$. The remaining function v is then constructed through some integration guided by the CR-equations. The construction is similar to the problem of construction of a potential function for a given conservative force in calculus III.

Proposition 3.6.13.

If $f = u + iv$ is analytic on some domain $D \subseteq \mathbb{C}$ then u and v are solutions of Laplace's equation $\phi_{xx} + \phi_{yy} = 0$ on D .

Proof: since $f = u + iv$ is analytic we know the CR-equations hold true; $u_x = v_y$ and $u_y = -v_x$. Moreover, f is continuously differentiable so we may commute partial derivatives by a theorem from multivariate calculus. Consider

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

Likewise,

$$v_{xx} + v_{yy} = (v_x)_x + (v_y)_y = (-u_y)_x + (u_x)_y = -u_{yx} + u_{xy} = 0$$

Of course these relations hold for all points inside D and the proposition follows. \square

Example 3.6.14. Note $f(z) = z^2$ is analytic with $u = x^2 - y^2$ and $v = 2xy$. We calculate,

$$u_{xx} = 2, \quad u_{yy} = -2 \quad \Rightarrow \quad u_{xx} + u_{yy} = 0$$

Note $v_{xx} = v_{yy} = 0$ so v is also a solution to Laplace's equation.

Now let's see if we can reverse this idea.

Example 3.6.15. Let $u(x, y) = x + c_1$ notice that u solves Laplace's equation. We seek to find a harmonic conjugate of u . We need to find v such that,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$$

Integrate these equations to deduce $v(x, y) = y + c_2$ for some constant $c_2 \in \mathbb{R}$. We thus construct an analytic function $f(x, y) = x + c_1 + i(y + c_2) = x + iy + c_1 + ic_2$. This is just $f(z) = z + c$ for $c = c_1 + ic_2$.

Example 3.6.16. Suppose $u(x, y) = e^x \cos(y)$. Note that $u_{xx} = u$ whereas $u_{yy} = -u$ hence $u_{xx} + u_{yy} = 0$. We seek to find v such that

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos(y) \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin(y)$$

Integrating $v_y = e^x \cos(y)$ with respect to y and $v_x = e^x \sin(y)$ with respect to x yields $v(x, y) = e^x \sin(y)$. We thus construct an analytic function $f(x, y) = e^x \cos(y) + ie^x \sin(y)$. Of course we should recognize the function we just constructed, it's just the complex exponential $f(z) = e^z$.

Notice we cannot just construct an analytic function from any given function of two variables. We have to start with a solution to Laplace's equation. This condition is rather restrictive. There is much more to say about harmonic functions, especially where applications are concerned. My goal here was just to give another perspective on analytic functions. Geometrically one thing we could see without further work at this point is that for an analytic function $f = u + iv$ the families of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal. Note $\text{grad}(u) = \langle u_x, u_y \rangle$ and $\text{grad}(v) = \langle v_x, v_y \rangle$ have

$$\text{grad}(u) \cdot \text{grad}(v) = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0$$

This means the normal lines to the level curves for u and v are orthogonal. Hence the level curves of u and v are orthogonal.

Chapter 4

inverse and implicit function theorems

It is tempting to give a complete and rigorous proof of these theorems here, but I will resist the temptation in lecture. I'm actually more interested that the student understand what the theorem claims. I will sketch the proof and show many applications. A nearly complete proof is found in Edwards where he uses an iterative approximation technique founded on the contraction mapping principle. All his arguments are in some sense in vain unless you have some working knowledge of uniform convergence. It's hidden in his proof, but we cannot conclude the limit of his sequence of function has the properties we desire unless the sequence of functions is uniformly convergent. Sadly that material has it's home in real analysis and I dare not trespass in lecture. That said, if you wish I'd be happy to show you the full proof if you have about 20 extra hours to develop the material outside of class. Alternatively, as a course of future study, return to the proof after you have completed Math 431 here at Liberty¹. Some other universities put advanced calculus after the real analysis course so that more analytical depth can be built into the course²

4.1 inverse function theorem

Consider the problem of finding a **local** inverse for $f : \text{dom}(f) \subseteq \mathbb{R} \rightarrow \mathbb{R}$. If we are given a point $p \in \text{dom}(f)$ such that there exists an open interval I containing p with $f|_I$ a one-one function then we can reasonably construct an inverse function by the simple rule $f^{-1}(y) = x$ iff $f(x) = y$ for $x \in I$ and $y \in f(I)$. A sufficient condition to insure the existence of a local inverse is that the derivative function is either strictly positive or strictly negative on some neighborhood of p . If we are give a continuously differentiable function at p then it has a derivative which is continuous on some neighborhood of p . For such a function if $f'(p) \neq 0$ then there exists some interval centered at p for which the derivative is strictly positive or negative. It follows that such a function is strictly monotonic and is hence one-one thus there is a local inverse at p . Therefore, even in calculus I we find the derivative informs us about the local invertibility of a function.

¹often read incorrectly as LU

²If you think this would be worthwhile then by all means say as much in your exit interview, I believe we should value the opinions of students, especially when they are geared towards academic excellence

The arguments I just made are supported by theorems that are developed in calculus I. Let me shift gears a bit and give a direct calculational explanation based on the linearization approximation. If $x \approx p$ then $f(x) \approx f(p) + f'(p)(x - p)$. To find the formula for the inverse we solve $y = f(x)$ for x :

$$y \approx f(p) + f'(p)(x - p) \Rightarrow x \approx p + \frac{1}{f'(p)}[y - f(p)]$$

Therefore, $\boxed{f^{-1}(y) \approx p + \frac{1}{f'(p)}[y - f(p)]}$ for y near $f(p)$.

Example 4.1.1. *Just to help you believe me, consider $f(x) = 3x - 2$ then $f'(x) = 3$ for all x . Suppose we want to find the inverse function near $p = 2$ then the discussion preceding this example suggests,*

$$f^{-1}(y) = 2 + \frac{1}{3}(y - 4).$$

I invite the reader to check that $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$ for all $x, y \in \mathbb{R}$.

In the example above we found a global inverse exactly, but this is thanks to the linearity of the function in the example. Generally, inverting the linearization just gives the first approximation to the inverse.

Consider $F : \text{dom}(F) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$. If F is differentiable at $p \in \mathbb{R}^n$ then we can write $F(x) \approx F(p) + F'(p)(x - p)$ for $x \approx p$. Set $y = F(x)$ and solve for x via matrix algebra. This time we need to assume $F'(p)$ is an invertible matrix in order to isolate x ,

$$y \approx F(p) + F'(p)(x - p) \Rightarrow x \approx p + (F'(p))^{-1}[y - f(p)]$$

Therefore, $\boxed{F^{-1}(y) \approx p + (F'(p))^{-1}[y - f(p)]}$ for y near $F(p)$. Apparently the condition to find a local inverse for a mapping on \mathbb{R}^n is that the derivative matrix is nonsingular³ in some neighborhood of the point. Experience has taught us from the one-dimensional case that we must insist the derivative is continuous near the point in order to maintain the validity of the approximation.

Recall from calculus II that as we attempt to approximate a function with a power series it takes an infinite series of power functions to recapture the formula exactly. Well, something similar is true here. However, the method of approximation is through an iterative approximation procedure which is built off the idea of Newton's method. The product of this iteration is a nested sequence of composite functions. To prove the theorem below one must actually provide proof the recursively generated sequence of functions converges. See pages 160-187 of Edwards for an in-depth exposition of the iterative approximation procedure. Then see pages 404-411 of Edwards for some material on uniform convergence⁴ The main analytical tool which is used to prove the convergence is called the **contraction mapping principle**. The proof of the principle is relatively easy to follow and

³nonsingular matrices are also called invertible matrices and a convenient test is that A is invertible iff $\det(A) \neq 0$.

⁴actually that later chapter is part of why I chose Edwards' text, he makes a point of proving things in \mathbb{R}^n in such a way that the proof naturally generalizes to function space. This is done by arguing with properties rather than formulas. The properties often extend to infinite dimensions whereas the formulas usually do not.

interestingly the main non-trivial step is an application of the geometric series. For the student of analysis this is an important topic which you should spend considerable time really trying to absorb as deeply as possible. The contraction mapping is at the base of a number of interesting and nontrivial theorems. Read Rosenlicht's *Introduction to Analysis* for a broader and better organized exposition of this analysis. In contrast, Edwards' uses analysis as a tool to obtain results for advanced calculus but his central goal is not a broad or well-framed treatment of analysis. Consequently, if analysis is your interest then you really need to read something else in parallel to get a better ideas about sequences of functions and uniform convergence. I have some notes from a series of conversations with a student about Rosenlicht, I'll post those for the interested student. These notes focus on the part of the material I require for this course. This is Theorem 3.3 on page 185 of Edwards' text:

Theorem 4.1.2. (*inverse function theorem*)

Suppose $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable in an open set W containing a and the derivative matrix $F'(a)$ is invertible. Then F is locally invertible at a . This means that there exists an open set $U \subseteq W$ containing a and V a open set containing $b = F(a)$ and a one-one, continuously differentiable mapping $G : V \rightarrow W$ such that $G(F(x)) = x$ for all $x \in U$ and $F(G(y)) = y$ for all $y \in V$. Moreover, the local inverse G can be obtained as the limit of the sequence of successive approximations defined by

$$G_0(y) = a \quad \text{and} \quad G_{n+1}(y) = G_n(y) - [F'(a)]^{-1}[F(G_n(y)) - y] \quad \text{for all } y \in V.$$

The qualifier local is important to note. If we seek a global inverse then other ideas are needed. If the function is everywhere injective then logically $F(x) = y$ defines $F^{-1}(y) = x$ and F^{-1} so constructed is single-valued by virtue of the injectivity of F . However, for differentiable mappings, one might wonder how can the criteria of global injectivity be tested via the differential. Even in the one-dimensional case a vanishing derivative does not indicate a lack of injectivity; $f(x) = x^3$ has $f^{-1}(y) = \sqrt[3]{y}$ and yet $f'(0) = 0$ (therefore $f'(0)$ is not invertible). On the other hand, we'll see in the examples that follow that even if the derivative is invertible over a set it is possible for the values of the mapping to double-up and once that happens we cannot find a single-valued inverse function⁵

Remark 4.1.3. *James R. Munkres' Analysis on Manifolds good for a different proof.*

Another good place to read the inverse function theorem is in James R. Munkres *Analysis on Manifolds*. That text is careful and has rather complete arguments which are not entirely the same as the ones given in Edwards. Munkres' text does not use the contraction mapping principle, instead the arguments are more topological in nature.

⁵there are scientists and engineers who work with multiply-valued functions with great success, however, as a point of style if nothing else, we try to use functions in math.

To give some idea of what I mean by topological let me give an example of such an argument. Suppose $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable and $F'(p)$ is invertible. Here's a sketch of the argument that $F'(x)$ is invertible for all x near p as follows:

1. the function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $g(x) = \det(F'(x))$ is formed by a multinomial in the component functions of $F'(x)$. This function is clearly continuous since we are given that the partial derivatives of the component functions of F are all continuous.
2. note we are given $F'(p)$ is invertible and hence $\det(F'(p)) \neq 0$ thus the continuous function g is nonzero at p . It follows there is some open set U containing p for which $0 \notin g(U)$
3. we have $\det(F'(x)) \neq 0$ for all $x \in U$ hence $F'(x)$ is invertible on U .

I would argue this is a topological argument because the key idea here is the continuity of g . Topology is the study of continuity in general.

Remark 4.1.4. *James J. Callahan's Advanced Calculus: a Geometric View, good reading.*

James J. Callahan recently authored *Advanced Calculus: a Geometric View*. This text has great merit in both visualization and well-thought use of linear algebraic techniques. In addition, many student will enjoy his staggered proofs where he first shows the proof for a simple low dimensional case and then proceeds to the general case. I almost used his text this semester.

Example 4.1.5. Suppose $F(x, y) = (\sin(y) + 1, \sin(x) + 2)$ for $(x, y) \in \mathbb{R}^2$. Clearly F is continuously differentiable as all its component functions have continuous partial derivatives. Observe,

$$F'(x, y) = [\partial_x F \mid \partial_y F] = \begin{bmatrix} 0 & \cos(y) \\ \cos(x) & 0 \end{bmatrix}$$

Hence $F'(x, y)$ is invertible at points (x, y) such that $\det(F'(x, y)) = -\cos(x)\cos(y) \neq 0$. This means we **may** not be able to find local inverses at points (x, y) with $x = \frac{1}{2}(2n+1)\pi$ or $y = \frac{1}{2}(2m+1)\pi$ for some $m, n \in \mathbb{Z}$. Points where $F'(x, y)$ are singular are points where one or both of $\sin(y)$ and $\sin(x)$ reach extreme values thus the points where the Jacobian matrix are singular are in fact points where we cannot find a local inverse. Why? Because the function is clearly not 1-1 on any set which contains the points of singularity for dF . Continuing, recall from precalculus that sine has a standard inverse on $[-\pi/2, \pi/2]$. Suppose $(x, y) \in [-\pi/2, \pi/2]^2$ and seek to solve $F(x, y) = (a, b)$ for (x, y) :

$$F(x, y) = \begin{bmatrix} \sin(y) + 1 \\ \sin(x) + 2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{cases} \sin(y) + 1 = a \\ \sin(x) + 2 = b \end{cases} \Rightarrow \begin{cases} y = \sin^{-1}(a - 1) \\ x = \sin^{-1}(b - 2) \end{cases}$$

It follows that $F^{-1}(a, b) = (\sin^{-1}(b - 2), \sin^{-1}(a - 1))$ for $(a, b) \in [0, 2] \times [1, 3]$ where you should note $F([-\pi/2, \pi/2]^2) = [0, 2] \times [1, 3]$. We've found a local inverse for F on the region $[-\pi/2, \pi/2]^2$. In other words, we just found a global inverse for the restriction of F to $[-\pi/2, \pi/2]^2$. Technically we ought not write F^{-1} , to be more precise we should write:

$$(F|_{[-\pi/2, \pi/2]^2})^{-1}(a, b) = (\sin^{-1}(b - 2), \sin^{-1}(a - 1)).$$

It is customary to avoid such detail in many contexts. Inverse functions for sine, cosine, tangent etc... are good examples of this slight of language.

A **coordinate system** on \mathbb{R}^n is an invertible mapping of \mathbb{R}^n to \mathbb{R}^n . However, in practice the term coordinate system is used with less rigor. Often a coordinate system has various degeneracies. For example, in polar coordinates you could say $\theta = \pi/4$ or $\theta = 9\pi/4$ or generally $\theta = 2\pi k + \pi/4$ for any $k \in \mathbb{Z}$. Let's examine polar coordinates in view of the inverse function theorem.

Example 4.1.6. Let $T(r, \theta) = (r \cos(\theta), r \sin(\theta))$ for $(r, \theta) \in [0, \infty) \times (-\pi/2, \pi/2)$. Clearly T is continuously differentiable as all its component functions have continuous partial derivatives. To find the inverse we seek to solve $T(r, \theta) = (x, y)$ for (r, θ) . Hence, consider $x = r \cos(\theta)$ and $y = r \sin(\theta)$. Note that

$$x^2 + y^2 = r^2 \cos^2(\theta) + r^2 \sin^2(\theta) = r^2(\cos^2(\theta) + \sin^2(\theta)) = r^2$$

and

$$\frac{y}{x} = \frac{r \sin(\theta)}{r \cos(\theta)} = \tan(\theta).$$

It follows that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ for $(x, y) \in (0, \infty) \times \mathbb{R}$. We find

$$T^{-1}(x, y) = \left(\sqrt{x^2 + y^2}, \tan^{-1}(y/x) \right).$$

Let's see how the derivative fits with our results. Calculate,

$$T'(r, \theta) = [\partial_r T \mid \partial_\theta T] = \begin{bmatrix} \cos(\theta) & -r \sin(\theta) \\ \sin(\theta) & r \cos(\theta) \end{bmatrix}$$

note that $\det(T'(r, \theta)) = r$ hence we the inverse function theorem provides the existence of a local inverse around any point except the origin. Notice the derivative does not detect the defect in the angular coordinate. Challenge, find the inverse function for $T(r, \theta) = (r \cos(\theta), r \sin(\theta))$ with $\text{dom}(T) = [0, \infty) \times (\pi/2, 3\pi/2)$. Or, find the inverse for polar coordinates in a neighborhood of $(0, -1)$.

Example 4.1.7. Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is defined by $T(x, y, z) = (ax, by, cz)$ for constants $a, b, c \in \mathbb{R}$ where $abc \neq 0$. Clearly T is continuously differentiable as all its component functions have continuous partial derivatives. We calculate $T'(x, y, z) = [\partial_x T \mid \partial_y T \mid \partial_z T] = [ae_1 \mid be_2 \mid ce_3]$. Thus $\det(T'(x, y, z)) = abc \neq 0$ for all $(x, y, z) \in \mathbb{R}^3$ hence this function is locally invertible everywhere. Moreover, we calculate the inverse mapping by solving $T(x, y, z) = (u, v, w)$ for (x, y, z) :

$$(ax, by, cz) = (u, v, w) \Rightarrow (x, y, z) = (u/a, v/b, w/c) \Rightarrow \boxed{T^{-1}(u, v, w) = (u/a, v/b, w/c)}.$$

Example 4.1.8. Suppose $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by $F(x) = Ax + b$ for some matrix $A \in \mathbb{R}^{n \times n}$ and vector $b \in \mathbb{R}^n$. **Under what conditions is such a function invertible ?**. Since the formula for this function gives each component function as a polynomial in the n -variables we can conclude the

function is continuously differentiable. You can calculate that $F'(x) = A$. It follows that a sufficient condition for local inversion is $\det(A) \neq 0$. It turns out that this is also a necessary condition as $\det(A) = 0$ implies the matrix A has nontrivial solutions for $Av = 0$. We say $v \in \text{Null}(A)$ iff $Av = 0$. Note if $v \in \text{Null}(A)$ then $F(v) = Av + b = b$. This is not a problem when $\det(A) \neq 0$ for in that case the null space contains just zero; $\text{Null}(A) = \{0\}$. However, when $\det(A) = 0$ we learn in linear algebra that $\text{Null}(A)$ contains infinitely many vectors so F is far from injective. For example, suppose $\text{Null}(A) = \text{span}\{e_1\}$ then you can show that $F(a_1, a_2, \dots, a_n) = F(x, a_2, \dots, a_n)$ for all $x \in \mathbb{R}$. Hence any point will have other points nearby which output the same value under F . Suppose $\det(A) \neq 0$, to calculate the inverse mapping formula we should solve $F(x) = y$ for x ,

$$y = Ax + b \Rightarrow x = A^{-1}(y - b) \Rightarrow \boxed{F^{-1}(y) = A^{-1}(y - b)}.$$

Remark 4.1.9. *inverse function theorem holds for higher derivatives.*

In Munkres the inverse function theorem is given for r -times differentiable functions. In short, a C^r function with invertible differential at a point has a C^r inverse function local to the point. Edwards also has arguments for $r > 1$, see page 202 and arguments and surrounding arguments.

4.2 implicit function theorem

Consider the problem of solving $x^2 + y^2 = 1$ for y as a function of x .

$$x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2 \Rightarrow y = \pm\sqrt{1 - x^2}.$$

A function cannot have two outputs for a single input, when we write \pm in the expression above it simply indicates our ignorance as to which is chosen. Once further information is given then we may be able to choose a $+$ or a $-$. For example:

1. if $x^2 + y^2 = 1$ and we want to solve for y near $(0, 1)$ then $y = \sqrt{1 - x^2}$ is the correct choice since $y > 0$ at the point of interest.
2. if $x^2 + y^2 = 1$ and we want to solve for y near $(0, -1)$ then $y = -\sqrt{1 - x^2}$ is the correct choice since $y < 0$ at the point of interest.
3. if $x^2 + y^2 = 1$ and we want to solve for y near $(1, 0)$ then it's impossible to find a single function which reproduces $x^2 + y^2 = 1$ on an **open disk** centered at $(1, 0)$.

What is the defect of case (3.) ? The trouble is that no matter how close we zoom in to the point there are always two y -values for each given x -value. Geometrically, this suggests either we have a discontinuity, a kink, or a vertical tangent in the graph. The given problem has a vertical tangent and hopefully you can picture this with ease since its just the unit-circle. In calculus I we studied

implicit differentiation, our starting point was to assume $y = y(x)$ and then we differentiated equations to work out implicit formulas for dy/dx . Take the unit-circle and differentiate both sides,

$$x^2 + y^2 = 1 \Rightarrow 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

Note $\frac{dy}{dx}$ is not defined for $y = 0$. It's no accident that those two points $(-1, 0)$ and $(1, 0)$ are precisely the points at which we cannot solve for y as a function of x . Apparently, the singularity in the derivative indicates where we may have trouble solving an equation for one variable as a function of the remaining variable.

We wish to study this problem in general. Given n -equations in $(m+n)$ -unknowns when can we solve for the last n -variables as functions of the first m -variables. Given a continuously differentiable mapping $G = (G_1, G_2, \dots, G_n) : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ study the level set: (here k_1, k_2, \dots, k_n are constants)

$$\begin{aligned} G_1(x_1, \dots, x_m, y_1, \dots, y_n) &= k_1 \\ G_2(x_1, \dots, x_m, y_1, \dots, y_n) &= k_2 \\ &\vdots \\ G_n(x_1, \dots, x_m, y_1, \dots, y_n) &= k_n \end{aligned}$$

We wish to locally solve for y_1, \dots, y_n as functions of x_1, \dots, x_m . That is, find a mapping $h : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that $G(x, y) = k$ iff $y = h(x)$ near some point $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $G(a, b) = k$. In this section we use the notation $x = (x_1, x_2, \dots, x_m)$ and $y = (y_1, y_2, \dots, y_n)$.

Before we turn to the general problem let's analyze the unit-circle problem in this notation. We are given $G(x, y) = x^2 + y^2$ and we wish to find $f(x)$ such that $y = f(x)$ solves $G(x, y) = 1$. Differentiate with respect to x and use the chain-rule:

$$\frac{\partial G}{\partial x} \frac{dx}{dx} + \frac{\partial G}{\partial y} \frac{dy}{dx} = 0$$

We find that $\boxed{dy/dx = -G_x/G_y} = -x/y$. Given this analysis we should suspect that if we are given some level curve $G(x, y) = k$ then we may be able to solve for y as a function of x near p if $G(p) = k$ and $G_y(p) \neq 0$. This suspicion is valid and it is one of the many consequences of the implicit function theorem.

We again turn to the linearization approximation. Suppose $G(x, y) = k$ where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ and suppose $G : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable. Suppose $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$ has $G(a, b) = k$. Replace G with its linearization based at (a, b) :

$$G(x, y) \approx k + G'(a, b)(x - a, y - b)$$

here we have the matrix multiplication of the $n \times (m + n)$ matrix $G'(a, b)$ with the $(m + n) \times 1$ column vector $(x - a, y - b)$ to yield an n -component column vector. It is convenient to define partial derivatives with respect to a whole vector of variables,

$$\frac{\partial G}{\partial x} = \begin{bmatrix} \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial G_n}{\partial x_1} & \cdots & \frac{\partial G_n}{\partial x_m} \end{bmatrix} \quad \frac{\partial G}{\partial y} = \begin{bmatrix} \frac{\partial G_1}{\partial y_1} & \cdots & \frac{\partial G_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial G_n}{\partial y_1} & \cdots & \frac{\partial G_n}{\partial y_n} \end{bmatrix}$$

In this notation we can write the $n \times (m + n)$ matrix $G'(a, b)$ as the concatenation of the $n \times m$ matrix $\frac{\partial G}{\partial x}(a, b)$ and the $n \times n$ matrix $\frac{\partial G}{\partial y}(a, b)$

$$G'(a, b) = \left[\frac{\partial G}{\partial x}(a, b) \middle| \frac{\partial G}{\partial y}(a, b) \right]$$

With this notation we have

$$G(x, y) \approx k + \frac{\partial G}{\partial x}(a, b)(x - a) + \frac{\partial G}{\partial y}(a, b)(y - b)$$

If we are near (a, b) then $G(x, y) \approx k$ thus we are faced with the problem of solving the following equation for y :

$$k \approx k + \frac{\partial G}{\partial x}(a, b)(x - a) + \frac{\partial G}{\partial y}(a, b)(y - b)$$

Suppose the square matrix $\frac{\partial G}{\partial y}(a, b)$ is invertible at (a, b) then we find the following approximation for the implicit solution of $G(x, y) = 0$ for y as a function of x :

$$y = b - \left[\frac{\partial G}{\partial y}(a, b) \right]^{-1} \left[\frac{\partial G}{\partial x}(a, b)(x - a) \right].$$

Of course this is not a formal proof, but it does suggest that $\det \left[\frac{\partial G}{\partial y}(a, b) \right] \neq 0$ is a necessary condition for solving for the y variables.

As before suppose $G : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Suppose we have a continuously differentiable function $h : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that $h(a) = b$ and $G(x, h(x)) = k$. We seek to find the derivative of h in terms of the derivative of G . This is a generalization of the implicit differentiation calculation we perform in calculus I. I'm including this to help you understand the notation a bit more before I state the implicit function theorem. Differentiate with respect to x_l for $l \in \mathbb{N}_m$:

$$\frac{\partial}{\partial x_l} \left[G(x, h(x)) \right] = \sum_{i=1}^m \frac{\partial G}{\partial x_i} \frac{\partial x_i}{\partial x_l} + \sum_{j=1}^n \frac{\partial G}{\partial y_j} \frac{\partial h_j}{\partial x_l} = \frac{\partial G}{\partial x_l} + \sum_{j=1}^n \frac{\partial G}{\partial y_j} \frac{\partial h_j}{\partial x_l} = 0$$

we made use of the identity $\frac{\partial x_i}{\partial x_k} = \delta_{ik}$ to squash the sum of i to the single nontrivial term and the zero on the r.h.s follows from the fact that $\frac{\partial}{\partial x_l}(k) = 0$. Concatenate these derivatives from $k = 1$ up to $k = m$:

$$\left[\frac{\partial G}{\partial x_1} + \sum_{j=1}^n \frac{\partial G}{\partial y_j} \frac{\partial h_j}{\partial x_1} \middle| \frac{\partial G}{\partial x_2} + \sum_{j=1}^n \frac{\partial G}{\partial y_j} \frac{\partial h_j}{\partial x_2} \middle| \cdots \middle| \frac{\partial G}{\partial x_m} + \sum_{j=1}^n \frac{\partial G}{\partial y_j} \frac{\partial h_j}{\partial x_m} \right] = [0|0| \cdots |0]$$

Properties of matrix addition allow us to parse the expression above as follows:

$$\left[\frac{\partial G}{\partial x_1} \middle| \frac{\partial G}{\partial x_2} \right] \cdots \left[\frac{\partial G}{\partial x_m} \right] + \left[\sum_{j=1}^n \frac{\partial G}{\partial y_j} \frac{\partial h_j}{\partial x_1} \middle| \sum_{j=1}^n \frac{\partial G}{\partial y_j} \frac{\partial h_j}{\partial x_2} \right] \cdots \left[\sum_{j=1}^n \frac{\partial G}{\partial y_j} \frac{\partial h_j}{\partial x_m} \right] = [0|0| \cdots |0]$$

But, this reduces to

$$\frac{\partial G}{\partial x} + \left[\frac{\partial G}{\partial y} \frac{\partial h}{\partial x_1} \middle| \frac{\partial G}{\partial y} \frac{\partial h}{\partial x_2} \right] \cdots \left[\frac{\partial G}{\partial y} \frac{\partial h}{\partial x_m} \right] = 0 \in \mathbb{R}^{m \times n}$$

The concatenation property of matrix multiplication states $[Ab_1|Ab_2| \cdots |Ab_m] = A[b_1|b_2| \cdots |b_m]$ we use this to write the expression once more,

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \left[\frac{\partial h}{\partial x_1} \middle| \frac{\partial h}{\partial x_2} \right] \cdots \left[\frac{\partial h}{\partial x_m} \right] = 0 \Rightarrow \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{\partial h}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial h}{\partial x} = -\frac{\partial G^{-1}}{\partial y} \frac{\partial G}{\partial x}}$$

where in the last implication we made use of the assumption that $\frac{\partial G}{\partial y}$ is invertible.

Theorem 4.2.1. (*Theorem 3.4 in Edwards's Text see pg 190*)

Let $G : \text{dom}(G) \subseteq \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable in a open ball about the point (a, b) where $G(a, b) = k$ (a constant vector in \mathbb{R}^n). If the matrix $\frac{\partial G}{\partial y}(a, b)$ is invertible then there exists an open ball U containing a in \mathbb{R}^m and an open ball W containing (a, b) in $\mathbb{R}^m \times \mathbb{R}^n$ and a continuously differentiable mapping $h : U \rightarrow \mathbb{R}^n$ such that $G(x, y) = k$ iff $y = h(x)$ for all $(x, y) \in W$. Moreover, the mapping h is the limit of the sequence of successive approximations defined inductively below

$$h_0(x) = b, \quad h_{n+1} = h_n(x) - \left[\frac{\partial G}{\partial y}(a, b) \right]^{-1} G(x, h_n(x)) \quad \text{for all } x \in U.$$

We will not attempt a proof of the last sentence for the same reasons we did not pursue the details in the inverse function theorem. However, we have already derived the first step in the iteration in our study of the linearization solution.

Proof: Let $G : \text{dom}(G) \subseteq \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable in a open ball B about the point (a, b) where $G(a, b) = k$ ($k \in \mathbb{R}^n$ a constant). Furthermore, assume the matrix $\frac{\partial G}{\partial y}(a, b)$ is invertible. We seek to use the inverse function theorem to prove the implicit function theorem. Towards that end consider $F : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n$ defined by $F(x, y) = (x, G(x, y))$. To begin, observe that F is continuously differentiable in the open ball B which is centered at (a, b) since G and x have continuous partials of their components in B . Next, calculate the derivative of $F = (x, G)$,

$$F'(x, y) = [\partial_x F | \partial_y F] = \left[\begin{array}{c|c} \partial_x x & \partial_y x \\ \hline \partial_x G & \partial_y G \end{array} \right] = \left[\begin{array}{c|c} I_m & 0_{m \times n} \\ \hline \partial_x G & \partial_y G \end{array} \right]$$

The determinant of the matrix above is the product of the determinant of the blocks I_m and $\partial_y G$; $\det(F'(x, y)) = \det(I_m) \det(\partial_y G) = \det \partial_y G$. We are given that $\frac{\partial G}{\partial y}(a, b)$ is invertible and hence

$\det(\frac{\partial G}{\partial y}(a, b)) \neq 0$ thus $\det(F'(x, y)) \neq 0$ and we find $F'(a, b)$ is invertible. Consequently, the inverse function theorem applies to the function F at (a, b) . Therefore, there exists $F^{-1} : V \subseteq \mathbb{R}^m \times \mathbb{R}^n \rightarrow U \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that F^{-1} is continuously differentiable. Note $(a, b) \in U$ and V contains the point $F(a, b) = (a, G(a, b)) = (a, k)$.

Our goal is to find the implicit solution of $G(x, y) = k$. We know that

$$F^{-1}(F(x, y)) = (x, y) \quad \text{and} \quad F(F^{-1}(u, v)) = (u, v)$$

for all $(x, y) \in U$ and $(u, v) \in V$. As usual to find the formula for the inverse we can solve $F(x, y) = (u, v)$ for (x, y) this means we wish to solve $(x, G(x, y)) = (u, v)$ hence $x = u$. The formula for v is more elusive, but we know it exists by the inverse function theorem. Let's say $y = H(u, v)$ where $H : V \rightarrow \mathbb{R}^n$ and thus $F^{-1}(u, v) = (u, H(u, v))$. Consider then,

$$(u, v) = F(F^{-1}(u, v)) = F(u, H(u, v)) = (u, G(u, H(u, v)))$$

Let $v = k$ thus $(u, k) = (u, G(u, H(u, k)))$ for all $(u, v) \in V$. Finally, define $h(u) = H(u, k)$ for all $(u, k) \in V$ and note that $k = G(u, h(u))$. In particular, $(a, k) \in V$ and at that point we find $h(a) = H(a, k) = b$ by construction. It follows that $y = h(x)$ provides a continuously differentiable solution of $G(x, y) = k$ near (a, b) .

Uniqueness of the solution follows from the uniqueness for the limit of the sequence of functions described in Edwards' text on page 192. However, other arguments for uniqueness can be offered, independent of the iterative method, for instance: see page 75 of Munkres *Analysis on Manifolds*. \square

Remark 4.2.2. *notation and the implementation of the implicit function theorem.*

We assumed the variables y were to be written as functions of x variables to make explicit a local solution to the equation $G(x, y) = k$. This ordering of the variables is convenient to argue the proof, however the real theorem is far more general. We can select any subset of n input variables to make up the " y " so long as $\frac{\partial G}{\partial y}$ is invertible. I will use this generalization of the formal theorem in the applications that follow. Moreover, the notations x and y are unlikely to maintain the same interpretation as in the previous pages. Finally, we will for convenience make use of the notation $y = y(x)$ to express the existence of a function f such that $y = f(x)$ when appropriate. Also, $z = z(x, y)$ means there is some function h for which $z = h(x, y)$. If this notation confuses then invent names for the functions in your problem.

Example 4.2.3. Suppose $G(x, y, z) = x^2 + y^2 + z^2$. Suppose we are given a point (a, b, c) such that $G(a, b, c) = R^2$ for a constant R . **Problem:** For which variable can we solve? **What, if any, influence does the given point have on our answer?** *Solution:* to begin, we have one equation and three unknowns so we should expect to find one of the variables as functions of the remaining two variables. The implicit function theorem applies as G is continuously differentiable.

1. if we wish to solve $z = z(x, y)$ then we need $G_z(a, b, c) = 2c \neq 0$.

2. if we wish to solve $y = y(x, z)$ then we need $G_y(a, b, c) = 2b \neq 0$.

3. if we wish to solve $x = x(y, z)$ then we need $G_x(a, b, c) = 2a \neq 0$.

The point has no local solution for z if it is a point on the intersection of the xy -plane and the sphere $G(x, y, z) = R^2$. Likewise, we cannot solve for $y = y(x, z)$ on the $y = 0$ slice of the sphere and we cannot solve for $x = x(y, z)$ on the $x = 0$ slice of the sphere.

Notice, algebra verifies the conclusions we reached via the implicit function theorem:

$$z = \pm\sqrt{R^2 - x^2 - y^2} \quad y = \pm\sqrt{R^2 - x^2 - z^2} \quad x = \pm\sqrt{R^2 - y^2 - z^2}$$

When we are at zero for one of the coordinates then we cannot choose $+$ or $-$ since we need both on an open ball intersected with the sphere centered at such a point⁶. Remember, when I talk about local solutions I mean solutions which exist over the intersection of the solution set and an open ball in the ambient space (\mathbb{R}^3 in this context). The preceding example is the natural extension of the unit-circle example to \mathbb{R}^3 . A similar result is available for the n -sphere in \mathbb{R}^n . I hope you get the point of the example, if we have one equation then if we wish to solve for a particular variable in terms of the remaining variables then all we need is continuous differentiability of the level function and a nonzero partial derivative at the point where we wish to find the solution. Now, the implicit function theorem doesn't find the solution for us, but it does provide the existence. In the section that follows, existence is really all we need since focus our attention on rates of change rather than actually solutions to the level set equation.

Example 4.2.4. Consider the equation $e^{xy} + z^3 - xyz = 2$. Can we solve this equation for $z = z(x, y)$ near $(0, 0, 1)$? Let $G(x, y, z) = e^{xy} + z^3 - xyz$ and note $G(0, 0, 1) = e^0 + 1 + 0 = 2$ hence $(0, 0, 1)$ is a point on the solution set $G(x, y, z) = 2$. Note G is clearly continuously differentiable and

$$G_z(x, y, z) = 3z^2 - xy \Rightarrow G_z(0, 0, 1) = 3 \neq 0$$

therefore, there exists a continuously differentiable function $h : \text{dom}(h) \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ which solves $G(x, y, h(x, y)) = 2$ for (x, y) near $(0, 0)$ and $h(0, 0) = 1$.

I'll not attempt an explicit solution for the last example.

Example 4.2.5. Let $(x, y, z) \in S$ iff $x + y + z = 2$ and $y + z = 1$. **Problem:** For which variable(s) can we solve? **Solution:** define $G(x, y, z) = (x + y + z, y + z)$ we wish to study $G(x, y, z) = (2, 1)$. Notice the solution set is not empty since $G(1, 0, 1) = (1 + 0 + 1, 0 + 1) = (2, 1)$. Moreover, G is continuously differentiable. In this case we have two equations and three unknowns so we expect two variables can be written in terms of the remaining free variable. Let's examine the derivative of G :

$$G'(x, y, z) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

⁶if you consider $G(x, y, z) = R^2$ as a space then the open sets on the space are taken to be the intersection with the space and open balls in \mathbb{R}^3 . This is called the subspace topology in topology courses.

Suppose we wish to solve $x = x(z)$ and $y = y(z)$ then we should check invertibility of

$$\frac{\partial G}{\partial(x, y)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The matrix above is invertible hence the implicit function theorem applies and we can solve for x and y as functions of z . On the other hand, if we tried to solve for $y = y(x)$ and $z = z(x)$ then we'll get no help from the implicit function theorem as the matrix

$$\frac{\partial G}{\partial(y, z)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

is not invertible. Geometrically, we can understand these results from noting that $G(x, y, z) = (2, 1)$ is the intersection of the plane $x + y + z = 2$ and $y + z = 1$. Substituting $y + z = 1$ into $x + y + z = 2$ yields $x + 1 = 2$ hence $x = 1$ on the line of intersection. We can hardly use x as a free variable for the solution when the problem fixes x from the outset.

The method I just used to analyze the equations in the preceding example was a bit adhoc. In linear algebra we do much better for systems of linear equations. A procedure called Gaussian elimination naturally reduces a system of equations to a form in which it is manifestly obvious how to eliminate redundant variables in terms of a minimal set of basic free variables. The "y" of the implicit function proof discussions plays the role of the so-called **pivotal variables** whereas the "x" plays the role of the remaining **free variables**. These variables are generally intermingled in the list of total variables so to reproduce the pattern assumed for the implicit function theorem we would need to relabel variables from the outset of a calculation. The calculations in the examples that follow are not usually possible. Linear equations are particularly nice and basically what I'm doing is following the guide of the linearization derivation in the context of specific examples.

Example 4.2.6. XXX

Example 4.2.7. XXX

Example 4.2.8. XXX

4.3 implicit differentiation

Enough theory, let's calculate. In this section I apply previous theoretical constructions to specific problems. I also introduce standard notation for "constrained" partial differentiation which is also sometimes called "partial differentiation with a side condition". The typical problem is the following: given equations:

$$\begin{aligned} G_1(x_1, \dots, x_m, y_1, \dots, y_n) &= k_1 \\ G_2(x_1, \dots, x_m, y_1, \dots, y_n) &= k_2 \\ &\vdots \\ G_n(x_1, \dots, x_m, y_1, \dots, y_n) &= k_n \end{aligned}$$

calculate partial derivative of dependent variables with respect to independent variables. Continuing with the notation of the implicit function discussion we'll assume that y will be dependent on x . I want to recast some of our arguments via differentials⁷. Take the total differential of each equation above,

$$\begin{aligned} dG_1(x_1, \dots, x_m, y_1, \dots, y_n) &= 0 \\ dG_2(x_1, \dots, x_m, y_1, \dots, y_n) &= 0 \\ &\vdots \\ dG_n(x_1, \dots, x_m, y_1, \dots, y_n) &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} \partial_{x_1} G_1 dx_1 + \dots + \partial_{x_m} G_1 dx_m + \partial_{y_1} G_1 dy_1 + \dots + \partial_{y_n} G_1 dy_n &= 0 \\ \partial_{x_1} G_2 dx_1 + \dots + \partial_{x_m} G_2 dx_m + \partial_{y_1} G_2 dy_1 + \dots + \partial_{y_n} G_2 dy_n &= 0 \\ &\vdots \\ \partial_{x_1} G_n dx_1 + \dots + \partial_{x_m} G_n dx_m + \partial_{y_1} G_n dy_1 + \dots + \partial_{y_n} G_n dy_n &= 0 \end{aligned}$$

Notice, this can be nicely written in column vector notation as:

$$\partial_{x_1} G dx_1 + \dots + \partial_{x_m} G dx_m + \partial_{y_1} G dy_1 + \dots + \partial_{y_n} G dy_n = 0$$

Or, in matrix notation:

$$[\partial_{x_1} G | \dots | \partial_{x_m} G] \begin{bmatrix} dx_1 \\ \vdots \\ dx_m \end{bmatrix} + [\partial_{y_1} G | \dots | \partial_{y_n} G] \begin{bmatrix} dy_1 \\ \vdots \\ dy_n \end{bmatrix} = 0$$

⁷in contrast, In the previous section we mostly used derivative notation

Finally, solve for dy , we assume $[\partial_{y_1} G | \cdots | \partial_{y_n} G]^{-1}$ exists,

$$\begin{bmatrix} dy_1 \\ \vdots \\ dy_n \end{bmatrix} = -[\partial_{y_1} G | \cdots | \partial_{y_n} G]^{-1} [\partial_{x_1} G | \cdots | \partial_{x_m} G] \begin{bmatrix} dx_1 \\ \vdots \\ dx_m \end{bmatrix}$$

Given all of this we can calculate $\frac{\partial y_i}{\partial x_j}$ by simply reading the coefficient dx_j in the i -th row. I will make this idea quite explicit in the examples that follow.

Example 4.3.1. *Let's return to a common calculus III problem. Suppose $F(x, y, z) = k$ for some constant k . Find partial derivatives of x, y or z with respect to the remaining variables.*

Solution: I'll use the method of differentials once more:

$$dF = F_x dx + F_y dy + F_z dz = 0$$

We can solve for dx, dy or dz provided F_x, F_y or F_z is nonzero respective and these differential expressions reveal various partial derivatives of interest:

$$\begin{aligned} dx &= -\frac{F_y}{F_x} dy - \frac{F_z}{F_x} dz &\Rightarrow &\quad \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} \quad \& \quad \frac{\partial x}{\partial z} = -\frac{F_z}{F_x} \\ dy &= -\frac{F_x}{F_y} dx - \frac{F_z}{F_y} dz &\Rightarrow &\quad \frac{\partial y}{\partial x} = -\frac{F_x}{F_y} \quad \& \quad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y} \\ dz &= -\frac{F_x}{F_z} dx - \frac{F_y}{F_z} dy &\Rightarrow &\quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \& \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \end{aligned}$$

In each case above, the implicit function theorem allows us to solve for one variable in terms of the remaining two. If the partial derivative of F in the denominator are zero then the implicit function theorem does not apply and other thoughts are required. Often calculus text give the following as a homework problem:

$$\frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -\frac{F_y}{F_x} \frac{F_z}{F_y} \frac{F_x}{F_z} = -1.$$

In the equation above we have x appear as a dependent variable on y, z and also as an independent variable for the dependent variable z . These mixed expressions are actually of interest to engineering and physics. The less ambiguous notation below helps better handle such expressions:

$$\left(\frac{\partial x}{\partial y} \right)_z \left(\frac{\partial y}{\partial z} \right)_x \left(\frac{\partial z}{\partial x} \right)_y = -1.$$

In each part of the expression we have clearly denoted which variables are taken to depend on the others and in turn what sort of partial derivative we mean to indicate. Partial derivatives are not taken alone, they must be done in concert with an understanding of the totality of the independent variables for the problem. We hold all the remaining independent variables fixed as we take a partial derivative.

The explicit independent variable notation is more important for problems where we can choose more than one set of independent variables for a given dependent variables. In the example that follows we study $w = w(x, y)$ but we could just as well consider $w = w(x, z)$. Generally it will not be the case that $\left(\frac{\partial w}{\partial x}\right)_y$ is the same as $\left(\frac{\partial w}{\partial x}\right)_z$. In calculation of $\left(\frac{\partial w}{\partial x}\right)_y$ we hold y constant as we vary x whereas in $\left(\frac{\partial w}{\partial x}\right)_z$ we hold z constant as we vary x . There is no reason these ought to be the same⁸.

Example 4.3.2. Suppose $x+y+z+w = 3$ and $x^2-2xyz+w^3 = 5$. Calculate partial derivatives of z and w with respect to the independent variables x, y . Solution: we begin by calculation of the differentials of both equations:

$$\begin{aligned} dx + dy + dz + dw &= 0 \\ (2x - 2yz)dx - 2xzd y - 2xydz + 3w^2 dw &= 0 \end{aligned}$$

We can solve for (dz, dw) . In this calculation we can treat the differentials as formal variables.

$$\begin{aligned} dz + dw &= -dx - dy \\ -2xydz + 3w^2 dw &= -(2x - 2yz)dx + 2xzd y \end{aligned}$$

I find matrix notation is often helpful,

$$\begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix} \begin{bmatrix} dz \\ dw \end{bmatrix} = \begin{bmatrix} -dx - dy \\ -(2x - 2yz)dx + 2xzd y \end{bmatrix}$$

Use Kramer's rule, multiplication by inverse, substitution, adding/subtracting equations etc... whatever technique of solving linear equations you prefer. Our goal is to solve for dz and dw in terms of dx and dy . I'll use Kramer's rule this time:

$$dz = \frac{\det \begin{bmatrix} -dx - dy & 1 \\ -(2x - 2yz)dx + 2xzd y & 3w^2 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix}} = \frac{3w^2(-dx - dy) + (2x - 2yz)dx - 2xzd y}{3w^2 + 2xy}$$

Collecting terms,

$$dz = \left(\frac{-3w^2 + 2x - 2yz}{3w^2 + 2xy} \right) dx + \left(\frac{-3w^2 - 2xz}{3w^2 + 2xy} \right) dy$$

From the expression above we can read various implicit derivatives,

$$\boxed{\left(\frac{\partial z}{\partial x}\right)_y = \frac{-3w^2 + 2x - 2yz}{3w^2 + 2xy} \quad \& \quad \left(\frac{\partial z}{\partial y}\right)_x = \frac{-3w^2 - 2xz}{3w^2 + 2xy}}$$

The notation above indicates that z is understood to be a function of independent variables x, y . $\left(\frac{\partial z}{\partial x}\right)_y$ means we take the derivative of z with respect to x while holding y fixed. The appearance

⁸a good exercise would be to do the example over but instead aim to calculate partial derivatives for y, w with respect to independent variables x, z

of the dependent variable w can be removed by using the equations $G(x, y, z, w) = (3, 5)$. Similar ambiguities exist for implicit differentiation in calculus I. Apply Kramer's rule once more to solve for dw :

$$dw = \frac{\det \begin{bmatrix} 1 & -dx - dy \\ -2xy & -(2x - 2yz)dx + 2xzdy \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix}} = \frac{-(2x - 2yz)dx + 2xzdy - 2xy(dx + dy)}{3w^2 + 2xy}$$

Collecting terms,

$$dw = \left(\frac{-2x + 2yz - 2xy}{3w^2 + 2xy} \right) dx + \left(\frac{2xzdy - 2xydy}{3w^2 + 2xy} \right) dy$$

We can read the following from the differential above:

$$\boxed{\left(\frac{\partial w}{\partial x} \right)_y = \frac{-2x + 2yz - 2xy}{3w^2 + 2xy} \quad \& \quad \left(\frac{\partial w}{\partial y} \right)_x = \frac{2xzdy - 2xydy}{3w^2 + 2xy}}$$

You should ask: where did we use the implicit function theorem in the preceding example? Notice our underlying hope is that we can solve for $z = z(x, y)$ and $w = w(x, y)$. The implicit function theorem states this is possible precisely when $\frac{\partial G}{\partial(z, w)} = \begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix}$ is non singular. Interestingly this is the same matrix we must consider to isolate dz and dw . The calculations of the example are only meaningful if the $\det \begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix} \neq 0$. In such a case the implicit function theorem applies and it is reasonable to suppose z, w can be written as functions of x, y .

Example 4.3.3. Suppose the temperature in a room is given by $T(x, y, z) = 70 + 10\exp(-x^2 - y^2)$. Find how the temperature varies on a sphere $x^2 + y^2 + z^2 = R^2$. We can choose any one variable from (x, y, z) and write it as a function of the remaining two on the sphere. However, we do need to a

Chapter 5

geometry of level sets

Our goal in this chapter is to develop a few tools to analyze the geometry of solution sets to equation(s) in \mathbb{R}^n . These solution sets are commonly called **level sets**. I assume the reader is already familiar with the concept of level curves and surfaces from multivariate calculus. We go much further in this chapter. Our goal is to describe the tangent and normal spaces for a p -dimensional level set in \mathbb{R}^n . The dimension of the level set is revealed by its tangent space and we discuss conditions which are sufficient to insure the invariance of this dimension over the entirety of the level set. In contrast, the dimension of the normal space to a p -dimensional level set in \mathbb{R}^n is $n - p$. The theory of orthogonal complements is borrowed from linear algebra to help understand how all of this fits together at a given point on the level set. Finally, we use this geometry and a few simple lemmas to justify the method of Lagrange multipliers. Lagrange's technique and the theory of multivariate Taylor polynomials form the basis for analyzing extrema for multivariate functions. In short, this chapter deals with the question of extrema on the edges of a set whereas the next chapter deals with the interior point via the theory of quadratic forms applied to the second-order approximation to a function of several variables. Finally, we should mention that p -dimensional level sets provide examples of p -dimensional manifolds, however, we defer careful discussion of manifolds for a later chapter.

5.1 definition of level set

A level set is the solution set of some equation or system of equations. We confine our interest to level sets of \mathbb{R}^n . For example, the set of all (x, y) that satisfy

$$G(x, y) = k$$

is called a **level curve** in \mathbb{R}^2 . Often we can use k to label the curve. You should also recall **level surfaces** in \mathbb{R}^3 are defined by an equation of the form

$$G(x, y, z) = k.$$

The set of all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ which solve $G(x_1, x_2, x_3, x_4) = k$ is a **level volume** in \mathbb{R}^4 . We can obtain lower dimensional objects by simultaneously imposing several equations at once. For example, suppose $G_1(x, y, z) = z = 1$ and $G_2(x, y, z) = x^2 + y^2 + z^2 = 5$, points (x, y, z) which solve both of these equations are on the intersection of the plane $z = 1$ and the sphere $x^2 + y^2 + z^2 = 5$. Let $G = (G_1, G_2)$, note that $G(x, y, z) = (1, 5)$ describes a circle in \mathbb{R}^3 . More generally:

Definition 5.1.1.

Suppose $G : \text{dom}(G) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^p$. Let k be a vector of constants in \mathbb{R}^p and suppose $S = \{x \in \mathbb{R}^n \mid G(x) = k\}$ is non-empty and G is continuously differentiable on an open set containing S . We say S is an **$(n - p)$ -dimensional level set** iff $G'(x)$ has p linearly independent rows at each $x \in S$.

The condition of linear independence of the rows is give to eliminate possible redundancy in the system of equations. In the case that $p = 1$ the criteria reduces to the conditon level function has $G'(x) \neq 0$ over the level set of dimension $n - 1$. Intuitively we think of each equation in $G(x) = k$ as removing one of the dimensions of the ambient space \mathbb{R}^n . It is worthwhile to cite a useful result from linear algebra at this point:

Proposition 5.1.2.

Let $A \in \mathbb{R}^{m \times n}$. The number of linearly independent columns in A is the same as the number of linearly independent rows in A . This invariant of A is called the **rank** of A .

Given the wisdom of linear algebra we see that we should require a $(n - p)$ -dimensional level set $S = G^{-1}(k)$ to have a level function $G : \mathbb{R}^n \rightarrow \mathbb{R}^p$ whose derivative is of rank p over all of S . We can either analyze linear independence of columns or rows.

Example 5.1.3. Consider $G(x, y, z) = x^2 + y^2 - z^2$ and suppose $S = G^{-1}\{0\}$. Calculate,

$$G'(x, y, z) = [2x, 2y, -2z]$$

Notice that $(0, 0, 0) \in S$ and $G'(0, 0, 0) = [0, 0, 0]$ hence S is not rank one at the origin. At all other points in S we have $G'(x, y, z) \neq 0$ which means this is almost a $3 - 1 = 2$ -dimensional level set. However, almost is not good enough in math. Under our definition the cone S is not a 2-dimensional level set since it fails to meet the full-rank criteria at the point of the cone.

Example 5.1.4. Let $G(x, y, z) = (x, y)$ and define $S = G^{-1}(a, b)$ for some fixed pair of constants $a, b \in \mathbb{R}$. We calculate that $G'(x, y, z) = I_2 \in \mathbb{R}^{2 \times 2}$. We clearly have rank two at all points in S hence S is a $3 - 2 = 1$ -dimensional level set. Perhaps you realize S is the vertical line which passes through $(a, b, 0)$ in the xy -plane.

5.2 tangents and normals to a level set

There are many ways to define a tangent space for some subset of \mathbb{R}^n . One natural definition is that the tangent space to $p \in S$ is simply the set of all tangent vectors to curves on S which pass through the point p . In this section we study the geometry of curves on a level-set. We'll see how the tangent space is naturally a vector space in the particular context of level-sets in \mathbb{R}^n .

Throughout this section we assume that S is a k -dimensional level set defined by $G : \mathbb{R}^k \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ where $G^{-1}(c) = S$. This means that we can apply the implicit function theorem to S and for any given point $p = (p_x, p_y) \in S$ where $p_x \in \mathbb{R}^k$ and $p_y \in \mathbb{R}^p$. There exists a local continuously differentiable solution $h : U \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^p$ such that $h(p_x) = p_y$ and for all $x \in U$ we have $G(x, h(x)) = c$. We can view $G(x, y) = c$ for x near p as the graph of $y = h(x)$ for $x \in U$. With the set-up above in mind, suppose that $\gamma : \mathbb{R} \rightarrow U \subseteq S$. If we write $\gamma = (\gamma_x, \gamma_y)$ then it follows $\gamma = (\gamma_x, h \circ \gamma_x)$ over the subset $U \times h(U)$ of S . More explicitly, for all $t \in \mathbb{R}$ such that $\gamma(t) \in U \times h(U)$ we have $\gamma(t) = (\gamma_x(t), h(\gamma_x(t)))$. Therefore, if $\gamma(0) = p$ then $\gamma(0) = (p_x, h(p_x))$. Differentiate, use the chain-rule in the second factor to obtain: $\gamma'(t) = (\gamma'_x(t), h'(\gamma_x(t))\gamma'_x(t))$. We find that the tangent vector to $p \in S$ of γ has a rather special form which was forced on us by the implicit function theorem: $\gamma'(0) = (\gamma'_x(0), h'(p_x)\gamma'_x(0))$. Or to cut through the notation a bit, if $\gamma'(0) = v = (v_x, v_y)$ then $v = (v_x, h'(p_x)v_x)$. The second component of the vector is not free of the first, it essentially redundant. This makes us suspect that the tangent space to S at p is k -dimensional.

Theorem 5.2.1.

Let $G : \mathbb{R}^k \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ be a level-mapping which defines a k -dimensional level set S by $G^{-1}(c) = S$. Suppose $\gamma_1, \gamma_2 : \mathbb{R} \rightarrow S$ are differentiable curves with $\gamma'_1(0) = v_1$ and $\gamma'_2(0) = v_2$ then there exists a differentiable curve $\gamma : \mathbb{R} \rightarrow S$ such that $\gamma'(0) = v_1 + v_2$ and $\gamma(0) = p$. Moreover, there exists a differentiable curve $\beta : \mathbb{R} \rightarrow S$ such that $\beta'(0) = cv_1$ and $\beta(0) = p$.

Proof: It is convenient to define a map which gives a **local parametrization** of S at p . Since we have a description of S locally as a graph $y = h(x)$ (near p) it is simple to construct the parameterization. Define $\Phi : U \subseteq \mathbb{R}^k \rightarrow S$ by $\Phi(x) = (x, h(x))$. Clearly $\Phi(U) = U \times h(U)$ and there is an inverse mapping $\Phi^{-1}(x, y) = x$ is well-defined since $y = h(x)$ for each $(x, y) \in U \times h(U)$. Let $w \in \mathbb{R}^k$ and observe that

$$\psi(t) = \Phi(\Phi^{-1}(p) + tw) = \Phi(p_x + tw) = (p_x + tw, h(p_x + tw))$$

is a curve from \mathbb{R} to $U \subseteq S$ such that $\psi(0) = (p_x, h(p_x)) = (p_x, p_y) = p$ and using the chain rule on the final form of $\psi(t)$:

$$\psi'(0) = (w, h'(p_x)w).$$

The construction above shows that any vector of the form $(v_x, h'(p_x)v_x)$ is the tangent vector of a particular differentiable curve in the level set (differentiability of ψ follows from the differentiability of h and the other maps which we used to construct ψ). In particular we can apply this to the case $w = v_{1x} + v_{2x}$ and we find $\gamma(t) = \Phi(\Phi^{-1}(p) + t(v_{1x} + v_{2x}))$ has $\gamma'(0) = v_1 + v_2$ and $\gamma(0) = p$.

Likewise, apply the construction to the case $w = cv_{1x}$ to write $\beta(t) = \Phi(\Phi^{-1}(p) + t(cv_{1x}))$ with $\beta'(0) = cv_1$ and $\beta(0) = p$. \square

The idea of the proof is encapsulated in the picture below. This idea of mapping lines in a flat domain to obtain standard curves in a curved domain is an idea which plays over and over as you study manifold theory. The particular redundancy of the x and y sub-vectors is special to the discussion level-sets, however anytime we have a local parametrization we'll be able to construct curves with tangents of our choosing by essentially the same construction. In fact, there are infinitely many curves which produce a particular tangent vector in the tangent space of a manifold.

XXX - read this section again for improper, premature use of the term "manifold"

Theorem 5.2.1 shows that the definition given below is logical. In particular, it is not at all obvious that the sum of two tangent vectors ought to again be a tangent vector. However, that is just what the Theorem 5.2.1 told us for level-sets¹.

Definition 5.2.2.

Suppose S is a k -dimensional level-set defined by $S = G^{-1}\{c\}$ for $G : \mathbb{R}^k \times \mathbb{R}^p \rightarrow \mathbb{R}^p$. We define the **tangent space at $p \in S$** to be the set of pairs:

$$T_p S = \{(p, v) \mid \text{there exists differentiable } \gamma : \mathbb{R} \rightarrow S \text{ and } \gamma(0) = p \text{ where } v = \gamma'(0)\}$$

Moreover, we define (i.) **addition** and (ii.) **scalar multiplication** of vectors by the rules

$$(i.) \quad (p, v_1) + (p, v_2) = (p, v_1 + v_2) \quad (ii.) \quad c(p, v_1) = (p, cv_1)$$

for all $(p, v_1), (p, v_2) \in T_p S$ and $c \in \mathbb{R}$.

When I picture $T_p S$ in my mind I think of vectors pointing out from the base-point p . To make an explicit connection between the pairs of the above definition and the classical geometric form of the tangent space we simply take the image of $T_p S$ under the mapping $\Psi(x, y) = x + y$ thus $\Psi(T_p S) = \{p + v \mid (p, v) \in T_p S\}$. I often picture $T_p S$ as $\psi(T_p S)$ ²

¹technically, there is another logical gap which I currently ignore. I wonder if you can find it.

²In truth, as you continue to study manifold theory you'll find at least three seemingly distinct objects which are all called "tangent vectors"; equivalence classes of curves, derivations, contravariant tensors.

We could set out to calculate tangent spaces in view of the definition above, but we are actually interested in more than just the tangent space for a level-set. In particular, we want a concrete description of all the vectors which are not in the tangent space.

Definition 5.2.3.

Suppose S is a k -dimensional level-set defined by $S = G^{-1}\{c\}$ for $G : \mathbb{R}^k \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $T_p S$ is the tangent space at p . Note that $T_p S \leq V_p$ where $V_p = \{p\} \times \mathbb{R}^k \times \mathbb{R}^p$ is given the natural vector space structure which we already exhibited on the subspace $T_p S$. We define the **inner product** on V_p as follows: for all $(p, v), (p, w) \in V_p$,

$$(p, v) \cdot (p, w) = v \cdot w.$$

The length of a vector (p, v) is naturally defined by $\|(p, v)\| = \|v\|$. Moreover, we say two vectors $(p, v), (p, w) \in V_p$ are **orthogonal** iff $v \cdot w = 0$. Given a set of vectors $R \subseteq V_p$ we define the **orthogonal complement** by

$$R^\perp = \{(p, v) \in V_p \mid (p, v) \cdot (p, r) = 0 \text{ for all } (p, r) \in R\}.$$

Suppose $W_1, W_2 \subseteq V_p$ then we say W_1 is **orthogonal** to W_2 iff $w_1 \cdot w_2 = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. We denote orthogonality by writing $W_1 \perp W_2$. If every $v \in V_p$ can be written as $v = w_1 + w_2$ for a pair of $w_1 \in W_1$ and $w_2 \in W_2$ where $W_1 \perp W_2$ then we say that V_p is the **direct sum** of W_1 and W_2 which is denoted by $V_p = W_1 \oplus W_2$.

There is much more to say about orthogonality, however, our focus is not in that vein. We just need the language to properly define the normal space. The calculation below is probably the most important calculation to understand for a level-set. Suppose we have a curve $\gamma : \mathbb{R} \rightarrow S$ where $S = G^{-1}(c)$ is a k -dimensional level-set in $\mathbb{R}^k \times \mathbb{R}^p$. Observe that for all $t \in \mathbb{R}$,

$$G(\gamma(t)) = c \Rightarrow G'(\gamma(t))\gamma'(t) = 0.$$

In particular, suppose for $t = 0$ we have $\gamma(0) = p$ and $v = \gamma'(0)$ which makes $(p, v) \in T_p S$ with

$$G'(p)v = 0.$$

Recall $G : \mathbb{R}^k \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ has an $p \times n$ derivative matrix where the j -th row is the gradient vector of the j -th component function. The equation $G'(p)v = 0$ gives us p -independent equations as we examine it componentwise. In particular, it reveals that (p, v) is orthogonal to $\nabla G_j(p)$ for $j = 1, 2, \dots, p$. We have derived the following theorem:

Theorem 5.2.4.

Let $G : \mathbb{R}^k \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ be a level-mapping which defines a k -dimensional level set S by $G^{-1}(c) = S$. The gradient vectors $\nabla G_j(p)$ are perpendicular to the tangent space at p ; for each $j \in \mathbb{N}_p$

$$(p, \nabla(G_j(p))^T) \in (T_p S)^\perp.$$

It's time to do some counting. Observe that the mapping $\phi : \mathbb{R}^k \rightarrow T_p S$ defined by $\phi(v) = (p, v)$ is an isomorphism of vector spaces hence $\dim(T_p S) = k$. But, by the same isomorphism we can see that $V_p = \phi(\mathbb{R}^k \times \mathbb{R}^p)$ hence $\dim(V_p) = p + k$. In linear algebra we learn that if we have a k -dimensional subspace W of an n -dimensional vector space V then the orthogonal complement W^\perp is a subspace of V with **codimension** k . The term **codimension** is used to indicate a loss of dimension from the ambient space, in particular $\dim(W^\perp) = n - k$. We should note that the direct sum of W and W^\perp covers the whole space; $W \oplus W^\perp = V$. In the case of the tangent space, the codimension of $T_p S \leq V_p$ is found to be $p + k - k = p$. Thus $\dim(T_p S)^\perp = p$. Any basis for this space must consist of p linearly independent vectors which are all orthogonal to the tangent space. Naturally, the subset of vectors $\{(p, (\nabla G_j(p))^T)_{j=1}^p\}$ forms just such a basis since it is given to be linearly independent by the $\text{rank}(G'(p)) = p$ condition. It follows that:

$$(T_p S)^\perp \approx \text{Row}(G'(p))$$

where equality can be obtained by the slightly tedious equation $(T_p S)^\perp = \phi(\text{Col}(G'(p)^T))$. That equation simply does the following:

1. transpose $G'(p)$ to swap rows to columns
2. construct column space by taking span of columns in $G'(p)^T$
3. adjoin p to make pairs of vectors which live in V_p

many wiser authors wouldn't bother. The comments above are primarily about notation. Certainly hiding these details would make this section prettier, however, would it make it better? Finally, I once more refer the reader to linear algebra where we learn that $(\text{Row}(A))^\perp = \text{Null}(A^T)$. Let me walk you through the proof: let $A \in \mathbb{R}^{m \times n}$. Observe $v \in \text{Null}(A^T)$ iff $A^T v = 0$ for $v \in \mathbb{R}^m$ iff $v^T A = 0$ iff $v^T \text{col}_j(A) = 0$ for $j = 1, 2, \dots, n$ iff $v \cdot \text{col}_j(A) = 0$ for $j = 1, 2, \dots, n$ iff $v \in \text{Col}(A)^\perp$. Another useful identity for the "perp" is that $(A^\perp)^\perp = A$. With those two gems in mind consider that:

$$(T_p S)^\perp \approx \text{Row}(G'(p)) \Rightarrow T_p S \approx \text{Row}(G'(p))^\perp = \text{Null}(G'(p)^T)$$

Let me once more replace \approx by a more tedious, but explicit, procedure:

$$T_p S = \phi(\text{Null}(G'(p)^T))$$

Theorem 5.2.5.

Let $G : \mathbb{R}^k \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ be a level-mapping which defines a k -dimensional level set S by $G^{-1}(c) = S$. The tangent space $T_p S$ and the **normal space** at $p \in S$ are given by

$$T_p S = \{p\} \times \text{Null}(G'(p)^T) \quad \& \quad T_p S^\perp = \{p\} \times \text{Col}(G'(p)^T).$$

Moreover, $V_p = T_p S \oplus T_p S^\perp$. Every vector can be uniquely written as the sum of a tangent vector and a normal vector.

The fact that there are only tangents and normals is the key to the method of Lagrange multipliers. It forces two seemingly distinct objects to be in the same direction as one another.

Example 5.2.6. Let $g : \mathbb{R}^4 \rightarrow \mathbb{R}$ be defined by $g(x, y, z, t) = t + x^2 + y^2 - 2z^2$ note that $g(x, y, z, t) = 0$ gives a three dimensional subset of \mathbb{R}^4 , let's call it M . Notice $\nabla g = \langle 2x, 2y, -4z, 1 \rangle$ is nonzero everywhere. Let's focus on the point $(2, 2, 1, 0)$ note that $g(2, 2, 1, 0) = 0$ thus the point is on M . The tangent plane at $(2, 2, 1, 0)$ is formed from the union of all tangent vectors to $g = 0$ at the point $(2, 2, 1, 0)$. To find the equation of the tangent plane we suppose $\gamma : \mathbb{R} \rightarrow M$ is a curve with $\gamma' \neq 0$ and $\gamma(0) = (2, 2, 1, 0)$. By assumption $g(\gamma(s)) = 0$ since $\gamma(s) \in M$ for all $s \in \mathbb{R}$. Define $\gamma'(0) = \langle a, b, c, d \rangle$, we find a condition from the chain-rule applied to $g \circ \gamma = 0$ at $s = 0$,

$$\begin{aligned} \frac{d}{ds}(g \circ \gamma(s)) &= (\nabla g)(\gamma(s)) \cdot \gamma'(s) = 0 &\Rightarrow &\nabla g(2, 2, 1, 0) \cdot \langle a, b, c, d \rangle = 0 \\ &&\Rightarrow &\langle 4, 4, -4, 1 \rangle \cdot \langle a, b, c, d \rangle = 0 \\ &&\Rightarrow &4a + 4b - 4c + d = 0 \end{aligned}$$

Thus the equation of the tangent plane is $4(x - 2) + 4(y - 2) - 4(z - 1) + t = 0$. In invite the reader to find a vector in the tangent plane and check it is orthogonal to $\nabla g(2, 2, 1, 0)$. However, this should not be surprising, the condition the chain rule just gave us is just the statement that $\langle a, b, c, d \rangle \in \text{Null}(\nabla g(2, 2, 1, 0)^T)$ and that is precisely the set of vector orthogonal to $\nabla g(2, 2, 1, 0)$.

Example 5.2.7. Let $G : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be defined by $G(x, y, z, t) = (z + x^2 + y^2 - 2, z + y^2 + t^2 - 2)$. In this case $G(x, y, z, t) = (0, 0)$ gives a two-dimensional manifold in \mathbb{R}^4 let's call it M . Notice that $G_1 = 0$ gives $z + x^2 + y^2 = 2$ and $G_2 = 0$ gives $z + y^2 + t^2 = 2$ thus $G = 0$ gives the intersection of both of these three dimensional manifolds in \mathbb{R}^4 (no I can't "see" it either). Note,

$$\nabla G_1 = \langle 2x, 2y, 1, 0 \rangle \quad \nabla G_2 = \langle 0, 2y, 1, 2t \rangle$$

It turns out that the inverse mapping theorem says $G = 0$ describes a manifold of dimension 2 if the gradient vectors above form a linearly independent set of vectors. For the example considered here the gradient vectors are linearly dependent at the origin since $\nabla G_1(0) = \nabla G_2(0) = (0, 0, 1, 0)$. In fact, these gradient vectors are colinear along along the plane $x = t = 0$ since $\nabla G_1(0, y, z, 0) = \nabla G_2(0, y, z, 0) = \langle 0, 2y, 1, 0 \rangle$. We again seek to contrast the tangent plane and its normal at some particular point. Choose $(1, 1, 0, 1)$ which is in M since $G(1, 1, 0, 1) = (0 + 1 + 1 - 2, 0 + 1 + 1 - 2) = (0, 0)$. Suppose that $\gamma : \mathbb{R} \rightarrow M$ is a path in M which has $\gamma(0) = (1, 1, 0, 1)$ whereas $\gamma'(0) = \langle a, b, c, d \rangle$. Note that $\nabla G_1(1, 1, 0, 1) = \langle 2, 2, 1, 0 \rangle$ and $\nabla G_2(1, 1, 0, 1) = \langle 0, 2, 1, 1 \rangle$. Applying the chain rule to both G_1 and G_2 yields:

$$\begin{aligned} (G_1 \circ \gamma)'(0) &= \nabla G_1(\gamma(0)) \cdot \langle a, b, c, d \rangle = 0 &\Rightarrow &\langle 2, 2, 1, 0 \rangle \cdot \langle a, b, c, d \rangle = 0 \\ (G_2 \circ \gamma)'(0) &= \nabla G_2(\gamma(0)) \cdot \langle a, b, c, d \rangle = 0 &\Rightarrow &\langle 0, 2, 1, 1 \rangle \cdot \langle a, b, c, d \rangle = 0 \end{aligned}$$

This is two equations and four unknowns, we can solve it and write the vector in terms of two free variables correspondant to the fact the tangent space is two-dimensional. Perhaps it's easier to use

matrix techniques to organize the calculation:

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We calculate, $\text{rref} \begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \end{bmatrix}$. It's natural to chose c, d as free variables then we can read that $a = d/2$ and $b = -c/2 - d/2$ hence

$$\langle a, b, c, d \rangle = \langle d/2, -c/2 - d/2, c, d \rangle = \frac{c}{2} \langle 0, -1, 2, 0 \rangle + \frac{d}{2} \langle 1, -1, 0, 2 \rangle$$

We can see a basis for the tangent space. In fact, I can give parametric equations for the tangent space as follows:

$$X(u, v) = (1, 1, 0, 1) + u \langle 0, -1, 2, 0 \rangle + v \langle 1, -1, 0, 2 \rangle$$

Not surprisingly the basis vectors of the tangent space are perpendicular to the gradient vectors $\nabla G_1(1, 1, 0, 1) = \langle 2, 2, 1, 0 \rangle$ and $\nabla G_2(1, 1, 0, 1) = \langle 0, 2, 1, 1 \rangle$ which span the **normal plane** N_p to the tangent plane T_p at $p = (1, 1, 0, 1)$. We find that T_p is orthogonal to N_p . In summary $T_p^\perp = N_p$ and $T_p \oplus N_p = \mathbb{R}^4$. This is just a fancy way of saying that the normal and the tangent plane only intersect at zero and they together span the entire ambient space.

5.3 method of Lagrange mulitpliers

Let us begin with a statement of the problem we wish to solve.

Problem: given an objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and continuously differentiable constraint function $G : \mathbb{R}^n \rightarrow \mathbb{R}^p$, find extreme values for the objective function f relative to the constraint $G(x) = c$.

Note that $G(x) = c$ is a vector notation for p -scalar equations. If we suppose $\text{rank}(G'(x)) = p$ then the constraint surface $G(x) = c$ will form an $(n - p)$ -dimensional level set. Let us make that supposition throughout the remainder of this section.

In order to solve a problem it is sometimes helpful to find necessary conditions by assuming an answer exists. Let us do that here. Suppose x_o maps to the local extrema of $f(x_o)$ on $S = G^{-1}\{c\}$. This means there exists an open ball around x_o for which $f(x_o)$ is either an upper or lower bound of all the values of f over the ball intersected with S . One clear implication of this data is that if we take any continuously differentiable curve on S which passes through x_o , say $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ with $\gamma(0) = x_o$ and $G(\gamma(t)) = c$ for all t , then the composite $f \circ \gamma$ is a function on \mathbb{R} which takes an extreme value at $t = 0$. Fermat's theorem from calculus I applies and as $f \circ \gamma$ is differentiable near $t = 0$ we find $(f \circ \gamma)'(0) = 0$ is a necessary condition. But, this means we have two necessary conditions on γ :

1. $G(\gamma(t)) = c$
2. $(f \circ \gamma)'(0) = 0$

Let us expand a bit on both of these conditions:

1. $G'(x_o)\gamma'(0) = 0$
2. $f'(x_o)\gamma'(0) = 0$

The first of these conditions places $\gamma'(0) \in T_{x_o}S$ but then the second condition says that $f'(x_o) = (\nabla f)(x_o)^T$ is orthogonal to $\gamma'(0)$ hence $(\nabla f)(x_o)^T \in N_{x_o}$. Now, recall from the last section that the gradient vectors of the component functions to G span the normal space, this means any vector in N_{x_o} can be written as a linear combination of the gradient vectors. In particular, this means there exist constants $\lambda_1, \lambda_2, \dots, \lambda_p$ such that

$$(\nabla f)(x_o)^T = \lambda_1(\nabla G_1)(x_o)^T + \lambda_2(\nabla G_2)(x_o)^T + \dots + \lambda_p(\nabla G_p)(x_o)^T$$

We may summarize the method of Lagrange multipliers as follows:

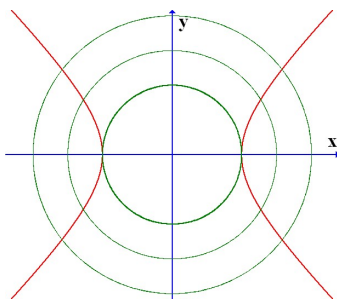
- 1. choose n -variables which aptly describe your problem.**
- 2. identify your objective function and write all constraints as level surfaces.**
- 3. solve $\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2 + \dots + \lambda_p \nabla G_p$ subject to the constraint $G(x) = c$.**
- 4. test the validity of your proposed extremal points.**

The obvious gap in the method is the supposition that an extrema exists for the restriction $f|_S$. We'll examine a few examples before I reveal a sufficient condition. We'll also see how absence of that sufficient condition does allow the method to fail.

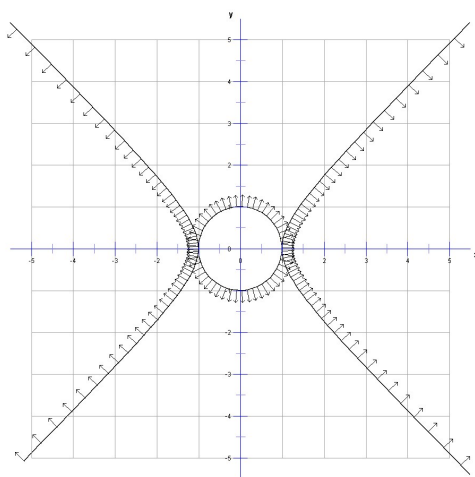
Example 5.3.1. Suppose we wish to find maximum and minimum distance to the origin for points on the curve $x^2 - y^2 = 1$. In this case we can use the distance-squared function as our objective $f(x, y) = x^2 + y^2$ and the single constraint function is $g(x, y) = x^2 - y^2$. Observe that $\nabla f = \langle 2x, 2y \rangle$ whereas $\nabla g = \langle 2x, -2y \rangle$. We seek solutions of $\nabla f = \lambda \nabla g$ which gives us $\langle 2x, 2y \rangle = \lambda \langle 2x, -2y \rangle$. Hence $2x = 2\lambda x$ and $2y = -2\lambda y$. We must solve these equations subject to the condition $x^2 - y^2 = 1$. Observe that $x = 0$ is not a solution since $0 - y^2 = 1$ has no real solution. On the other hand, $y = 0$ does fit the constraint and $x^2 - 0 = 1$ has solutions $x = \pm 1$. Consider then

$$2x = 2\lambda x \quad \text{and} \quad 2y = -2\lambda y \quad \Rightarrow \quad x(1 - \lambda) = 0 \quad \text{and} \quad y(1 + \lambda) = 0$$

Since $x \neq 0$ on the constraint curve it follows that $1 - \lambda = 0$ hence $\lambda = 1$ and we learn that $y(1 + 1) = 0$ hence $y = 0$. Consequently, $(1, 0)$ and $(-1, 0)$ are the two points where we expect to find extreme-values of f . In this case, the method of Lagrange multipliers served its purpose, as you can see in the graph. Below the green curves are level curves of the objective function whereas the particular red curve is the given constraint curve.



The picture below is a screen-shot of the Java applet created by David Lippman and Konrad Polthier to explore 2D and 3D graphs. Especially nice is the feature of adding vector fields to given objects, many other plotters require much more effort for similar visualization. See more at the website: <http://dlippman.imathas.com/g1/GrapherLaunch.html>.



Note how the gradient vectors to the objective function and constraint function line-up nicely at those points.

In the previous example, we actually got lucky. There are examples of this sort where we could get false maxima due to the nature of the constraint function.

Example 5.3.2. Suppose we wish to find the points on the unit circle $g(x, y) = x^2 + y^2 = 1$ which give extreme values for the objective function $f(x, y) = x^2 - y^2$. Apply the method of Lagrange multipliers and seek solutions to $\nabla f = \lambda \nabla g$:

$$\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$$

We must solve $2x = 2x\lambda$ which is better cast as $(1 - \lambda)x = 0$ and $-2y = 2\lambda y$ which is nicely written as $(1 + \lambda)y = 0$. On the basis of these equations alone we have several options:

1. if $\lambda = 1$ then $(1 + 1)y = 0$ hence $y = 0$

2. if $\lambda = -1$ then $(1 - (1))x = 0$ hence $x = 0$

But, we also must fit the constraint $x^2 + y^2 = 1$ hence we find four solutions:

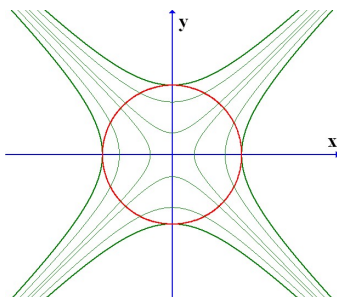
1. if $\lambda = 1$ then $y = 0$ thus $x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow (\pm 1, 0)$

2. if $\lambda = -1$ then $x = 0$ thus $y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow (0, \pm 1)$

We test the objective function at these points to ascertain which type of extrema we've located:

$$f(0, \pm 1) = 0^2 - (\pm 1)^2 = -1 \quad \& \quad f(\pm 1, 0) = (\pm 1)^2 - 0^2 = 1$$

When constrained to the unit circle we find the objective function attains a maximum value of 1 at the points $(1, 0)$ and $(-1, 0)$ and a minimum value of -1 at $(0, 1)$ and $(0, -1)$. Let's illustrate the answers as well as a few non-answers to get perspective. Below the green curves are level curves of the objective function whereas the particular red curve is the given constraint curve.



The success of the last example was no accident. The fact that the constraint curve was a circle which is a closed and bounded subset of \mathbb{R}^2 means that it is a **compact** subset of \mathbb{R}^2 . A well-known theorem of analysis states that any real-valued continuous function on a compact domain attains both maximum and minimum values. The objective function is continuous and the domain is compact hence the theorem applies and the method of Lagrange multipliers succeeds. In contrast, the constraint curve of the preceding example was a hyperbola which is not compact. We have no assurance of the existence of any extrema. Indeed, we only found minima but no maxima in Example 5.3.1.

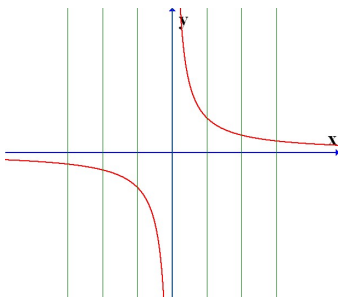
The generality of the method of Lagrange multipliers is naturally limited to smooth constraint curves and smooth objective functions. We must insist the gradient vectors exist at all points of inquiry. Otherwise, the method breaks down. If we had a constraint curve which has sharp corners then the method of Lagrange breaks down at those corners. In addition, if there are points of discontinuity in the constraint then the method need not apply. This is not terribly surprising, even in calculus I the main attack to analyze extrema of function on \mathbb{R} assumed continuity, differentiability and sometimes twice differentiability. Points of discontinuity require special attention in whatever context you meet them.

At this point it is doubtless the case that some of you are, to misquote an ex-student of mine, "not-impressed". Perhaps the following examples better illustrate the dangers of non-compact constraint curves.

Example 5.3.3. Suppose we wish to find extrema of $f(x, y) = x$ when constrained to $xy = 1$. Identify $g(x, y) = xy = 1$ and apply the method of Lagrange multipliers and seek solutions to $\nabla f = \lambda \nabla g$:

$$\langle 1, 0 \rangle = \lambda \langle y, x \rangle \Rightarrow 1 = \lambda y \text{ and } 0 = \lambda x$$

If $\lambda = 0$ then $1 = \lambda y$ is impossible to solve hence $\lambda \neq 0$ and we find $x = 0$. But, if $x = 0$ then $xy = 1$ is not solvable. Therefore, we find no solutions. Well, I suppose we have succeeded here in a way. We just learned there is no extreme value of x on the hyperbola $xy = 1$. Below the green curves are level curves of the objective function whereas the particular red curve is the given constraint curve.



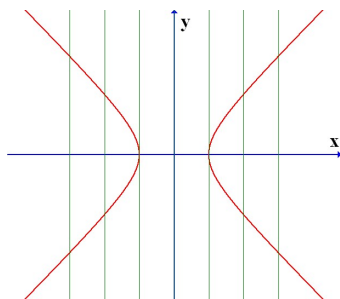
Example 5.3.4. Suppose we wish to find extrema of $f(x, y) = x$ when constrained to $x^2 - y^2 = 1$. Identify $g(x, y) = x^2 - y^2 = 1$ and apply the method of Lagrange multipliers and seek solutions to $\nabla f = \lambda \nabla g$:

$$\langle 1, 0 \rangle = \lambda \langle 2x, -2y \rangle \Rightarrow 1 = 2\lambda x \text{ and } 0 = -2\lambda y$$

If $\lambda = 0$ then $1 = 2\lambda x$ is impossible to solve hence $\lambda \neq 0$ and we find $y = 0$. If $y = 0$ and $x^2 - y^2 = 1$ then we must solve $x^2 = 1$ whence $x = \pm 1$. We are tempted to conclude that:

1. the objective function $f(x, y) = x$ attains a maximum on $x^2 - y^2 = 1$ at $(1, 0)$ since $f(1, 0) = 1$
2. the objective function $f(x, y) = x$ attains a minimum on $x^2 - y^2 = 1$ at $(-1, 0)$ since $f(-1, 0) = -1$

But, both conclusions are false. Note $\sqrt{2}^2 - 1^2 = 1$ hence $(\pm\sqrt{2}, 1)$ are points on the constraint curve and $f(\sqrt{2}, 1) = \sqrt{2}$ and $f(-\sqrt{2}, 1) = -\sqrt{2}$. The error of the method of Lagrange multipliers in this context is the supposition that there exists extrema to find, in this case there are no such points. It is possible for the gradient vectors to line-up at points where there are no extrema. Below the green curves are level curves of the objective function whereas the particular red curve is the given constraint curve.



Incidentally, if you want additional discussion of Lagrange multipliers for two-dimensional problems one very nice source I certainly profitted from was the YouTube video by Edward Frenkel of Berkley. See his website <http://math.berkeley.edu/frenkel/> for links.

Example 5.3.5. XXX- add example from last notes. (chapter 8)

Example 5.3.6. XXX- add example from last notes. (chapter 8)

Example 5.3.7. XXX- add example from last notes. (chapter 8)

XXX—need to polish the notation for normal space. XXX—add examples for level sets.

Chapter 6

critical point analysis for several variables

In the typical calculus sequence you learn the first and second derivative tests in calculus I. Then in calculus II you learn about power series and Taylor's Theorem. Finally, in calculus III, in many popular texts, you learn an essentially ad-hoc procedure for judging the nature of critical points as minimum, maximum or saddle. These topics are easily seen as disconnected events. In this chapter, we connect them. We learn that the geometry of quadratic forms is elegantly revealed by eigenvectors and more than that this geometry is precisely what elucidates the proper classifications of critical points of multivariate functions with real values.

6.1 multivariate power series

We set aside the issue of convergence for another course. We will suppose the series discussed in this section exist on and converge on some domain, but we do not seek to treat that topic here and now. Our focus is computational. How should we calculate the Taylor series for $f(x, y)$ at (a, b) ? Or, what about $f(x)$ at $x_o \in \mathbb{R}^n$?

6.1.1 taylor's polynomial for one-variable

If $f : U \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is analytic at $x_o \in U$ then we can write

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{1}{2}f''(x_o)(x - x_o)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!} (x - x_o)^n$$

We could write this in terms of the operator $D = \frac{d}{dt}$ and the evaluation of $t = x_o$

$$f(x) = \left[\sum_{n=0}^{\infty} \frac{1}{n!} (x - t)^n D^n f(t) \right]_{t=x_o} =$$

I remind the reader that a function is called **entire** if it is analytic on all of \mathbb{R} , for example e^x , $\cos(x)$ and $\sin(x)$ are all entire. In particular, you should know that:

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Since $e^x = \cosh(x) + \sinh(x)$ it also follows that

$$\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$$

The geometric series is often useful, for $a, r \in \mathbb{R}$ with $|r| < 1$ it is known

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

This generates a whole host of examples, for instance:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots$$

$$\frac{x^3}{1-2x} = x^3(1 + 2x + (2x)^2 + \cdots) = x^3 + 2x^4 + 4x^5 + \cdots$$

Moreover, the term-by-term integration and differentiation theorems yield additional results in conjunction with the geometric series:

$$\tan^{-1}(x) = \int \frac{dx}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$

$$\ln(1-x) = \int \frac{d}{dx} \ln(1-x) dx = \int \frac{-1}{1-x} dx = - \int \sum_{n=0}^{\infty} x^n dx = \sum_{n=0}^{\infty} \frac{-1}{n+1} x^{n+1}$$

Of course, these are just the basic building blocks. We also can twist things and make the student use algebra,

$$e^{x+2} = e^x e^2 = e^2(1 + x + \frac{1}{2}x^2 + \cdots)$$

or trigonometric identities,

$$\begin{aligned} \sin(x) &= \sin(x - 2 + 2) = \sin(x - 2) \cos(2) + \cos(x - 2) \sin(2) \\ \Rightarrow \sin(x) &= \cos(2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x-2)^{2n+1} + \sin(2) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x-2)^{2n}. \end{aligned}$$

Feel free to peruse my most recent calculus II materials to see a host of similarly sneaky calculations.

6.1.2 taylor's multinomial for two-variables

Suppose we wish to find the taylor polynomial centered at $(0,0)$ for $f(x,y) = e^x \sin(y)$. It is a simple as this:

$$f(x,y) = \left(1 + x + \frac{1}{2}x^2 + \cdots\right) \left(y - \frac{1}{6}y^3 + \cdots\right) = y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \cdots$$

the resulting expression is called a multinomial since it is a polynomial in multiple variables. If all functions $f(x,y)$ could be written as $f(x,y) = F(x)G(y)$ then multiplication of series known from calculus II would often suffice. However, many functions do not possess this very special form. For example, how should we expand $f(x,y) = \cos(xy)$ about $(0,0)$? We need to derive the two-dimensional Taylor's theorem.

We already know Taylor's theorem for functions on \mathbb{R} ,

$$g(x) = g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \cdots + \frac{1}{k!}g^{(k)}(a)(x-a)^k + R_k$$

and... If the remainder term vanishes as $k \rightarrow \infty$ then the function g is represented by the Taylor series given above and we write:

$$g(x) = \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(a)(x-a)^k.$$

Consider the function of two variables $f : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ which is smooth with smooth partial derivatives of all orders. Furthermore, let $(a,b) \in U$ and construct a line through (a,b) with direction vector (h_1, h_2) as usual:

$$\phi(t) = (a,b) + t(h_1, h_2) = (a + th_1, b + th_2)$$

for $t \in \mathbb{R}$. Note $\phi(0) = (a,b)$ and $\phi'(t) = (h_1, h_2) = \phi'(0)$. Construct $g = f \circ \phi : \mathbb{R} \rightarrow \mathbb{R}$ and choose $\text{dom}(g)$ such that $\phi(t) \in U$ for $t \in \text{dom}(g)$. This function g is a real-valued function of a

real variable and we will be able to apply Taylor's theorem from calculus II on g . However, to differentiate g we'll need tools from calculus III to sort out the derivatives. In particular, as we differentiate g , note we use the chain rule for functions of several variables:

$$\begin{aligned} g'(t) &= (f \circ \phi)'(t) = f'(\phi(t))\phi'(t) \\ &= \nabla f(\phi(t)) \cdot (h_1, h_2) \\ &= h_1 f_x(a + th_1, b + th_2) + h_2 f_y(a + th_1, b + th_2) \end{aligned}$$

Note $g'(0) = h_1 f_x(a, b) + h_2 f_y(a, b)$. Differentiate again (I omit $(\phi(t))$ dependence in the last steps),

$$\begin{aligned} g''(t) &= h_1 f'_x(a + th_1, b + th_2) + h_2 f'_y(a + th_1, b + th_2) \\ &= h_1 \nabla f_x(\phi(t)) \cdot (h_1, h_2) + h_2 \nabla f_y(\phi(t)) \cdot (h_1, h_2) \\ &= h_1^2 f_{xx} + h_1 h_2 f_{xy} + h_2 h_1 f_{xy} + h_2^2 f_{yy} \\ &= h_1^2 f_{xx} + 2h_1 h_2 f_{xy} + h_2^2 f_{yy} \end{aligned}$$

Thus, making explicit the point dependence, $g''(0) = h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)$. We may construct the Taylor series for g up to quadratic terms:

$$\begin{aligned} g(0 + t) &= g(0) + tg'(0) + \frac{1}{2}g''(0) + \cdots \\ &= f(a, b) + t[h_1 f_x(a, b) + h_2 f_y(a, b)] + \frac{t^2}{2}[h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)] + \cdots \end{aligned}$$

Note that $g(t) = f(a + th_1, b + th_2)$ hence $g(1) = f(a + h_1, b + h_2)$ and consequently,

$$\begin{aligned} f(a + h_1, b + h_2) &= f(a, b) + h_1 f_x(a, b) + h_2 f_y(a, b) + \\ &\quad + \frac{1}{2} \left[h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b) \right] + \cdots \end{aligned}$$

Omitting point dependence on the 2^{nd} derivatives,

$$\boxed{f(a + h_1, b + h_2) = f(a, b) + h_1 f_x(a, b) + h_2 f_y(a, b) + \frac{1}{2} [h_1^2 f_{xx} + 2h_1 h_2 f_{xy} + h_2^2 f_{yy}] + \cdots}$$

Sometimes we'd rather have an expansion about (x, y) . To obtain that formula simply substitute $x - a = h_1$ and $y - b = h_2$. Note that the point (a, b) is fixed in this discussion so the derivatives are not modified in this substitution,

$$\begin{aligned} f(x, y) &= f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b) + \\ &\quad + \frac{1}{2} \left[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b) \right] + \cdots \end{aligned}$$

At this point we ought to recognize the first three terms give the tangent plane to $z = f(x, y)$ at $(a, b, f(a, b))$. The higher order terms are nonlinear corrections to the linearization, these quadratic

terms form a *quadratic form*. If we computed third, fourth or higher order terms we will find that, using $a = a_1$ and $b = a_2$ as well as $x = x_1$ and $y = x_2$,

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{i_1=0}^2 \sum_{i_2=0}^2 \cdots \sum_{i_n=0}^2 \frac{1}{n!} \frac{\partial^{(n)} f(a_1, a_2)}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2}) \cdots (x_{i_n} - a_{i_n})$$

Example 6.1.1. Expand $f(x, y) = \cos(xy)$ about $(0, 0)$. We calculate derivatives,

$$f_x = -y \sin(xy) \quad f_y = -x \sin(xy)$$

$$f_{xx} = -y^2 \cos(xy) \quad f_{xy} = -\sin(xy) - xy \cos(xy) \quad f_{yy} = -x^2 \cos(xy)$$

$$f_{xxx} = y^3 \sin(xy) \quad f_{xxy} = -y \cos(xy) - y \cos(xy) + xy^2 \sin(xy)$$

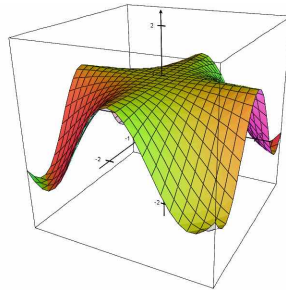
$$f_{xyy} = -x \cos(xy) - x \cos(xy) + x^2 y \sin(xy) \quad f_{yyy} = x^3 \sin(xy)$$

Next, evaluate at $x = 0$ and $y = 0$ to find $f(x, y) = 1 + \cdots$ to third order in x, y about $(0, 0)$. We can understand why these derivatives are all zero by approaching the expansion a different route: simply expand cosine directly in the variable (xy) ,

$$f(x, y) = 1 - \frac{1}{2}(xy)^2 + \frac{1}{4!}(xy)^4 + \cdots = 1 - \frac{1}{2}x^2y^2 + \frac{1}{4!}x^4y^4 + \cdots$$

Apparently the given function only has nontrivial derivatives at $(0, 0)$ at orders $0, 4, 8, \dots$. We can deduce that $f_{xxxxy}(0, 0) = 0$ without further calculation.

This is actually a very interesting function, I think it defies our analysis in the later portion of this chapter. The second order part of the expansion reveals nothing about the nature of the critical point $(0, 0)$. Of course, any student of trigonometry should recognize that $f(0, 0) = 1$ is likely a local maximum, it's certainly not a local minimum. The graph reveals that $f(0, 0)$ is a local maximum for f restricted to certain rays from the origin whereas it is constant on several special directions (the coordinate axes).



6.1.3 Taylor's multinomial for many-variables

Suppose $f : \text{dom}(f) \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is a function of n -variables and we seek to derive the Taylor series centered at $a = (a_1, a_2, \dots, a_n)$. Once more consider the composition of f with a line in $\text{dom}(f)$. In particular, let $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$ be defined by $\phi(t) = a + th$ where $h = (h_1, h_2, \dots, h_n)$ gives the direction of the line and clearly $\phi'(t) = h$. Let $g : \text{dom}(g) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g(t) = f(\phi(t))$ for all $t \in \mathbb{R}$ such that $\phi(t) \in \text{dom}(f)$. Differentiate, use the multivariate chain rule, recall here that $\nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + \dots + e_n \frac{\partial}{\partial x_n} = \sum_{i=1}^n e_i \partial_i$,

$$g'(t) = \nabla f(\phi(t)) \cdot \phi'(t) = \nabla f(\phi(t)) \cdot h = \sum_{i=1}^n h_i (\partial_i f)(\phi(t))$$

If we omit the explicit dependence on $\phi(t)$ then we find the simple formula $g'(t) = \sum_{i=1}^n h_i \partial_i f$. Differentiate a second time,

$$g''(t) = \frac{d}{dt} \left[\sum_{i=1}^n h_i \partial_i f(\phi(t)) \right] = \sum_{i=1}^n h_i \frac{d}{dt} \left[(\partial_i f)(\phi(t)) \right] = \sum_{i=1}^n h_i (\nabla \partial_i f)(\phi(t)) \cdot \phi'(t)$$

Omitting the $\phi(t)$ dependence and once more using $\phi'(t) = h$ we find

$$g''(t) = \sum_{i=1}^n h_i \nabla \partial_i f \cdot h$$

Recall that $\nabla = \sum_{j=1}^n e_j \partial_j$ and expand the expression above,

$$g''(t) = \sum_{i=1}^n h_i \left(\sum_{j=1}^n e_j \partial_j \partial_i f \right) \cdot h = \sum_{i=1}^n \sum_{j=1}^n h_i h_j \partial_j \partial_i f$$

where we should remember $\partial_j \partial_i f$ depends on $\phi(t)$. It should be clear that if we continue and take k -derivatives then we will obtain:

$$g^{(k)}(t) = \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_k=1}^n h_{i_1} h_{i_2} \dots h_{i_k} \partial_{i_1} \partial_{i_2} \dots \partial_{i_k} f$$

More explicitly,

$$g^{(k)}(t) = \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_k=1}^n h_{i_1} h_{i_2} \dots h_{i_k} (\partial_{i_1} \partial_{i_2} \dots \partial_{i_k} f)(\phi(t))$$

Hence, by Taylor's theorem, provided we are sufficiently close to $t = 0$ as to bound the remainder¹

$$g(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_k=1}^n h_{i_1} h_{i_2} \dots h_{i_k} (\partial_{i_1} \partial_{i_2} \dots \partial_{i_k} f)(\phi(t)) \right) t^k$$

¹there exist smooth examples for which no neighborhood is small enough, the bump function in one-variable has higher-dimensional analogues, we focus our attention to functions for which it is possible for the series below to converge

Recall that $g(t) = f(\phi(t)) = f(a + th)$. Put² $t = 1$ and bring in the $\frac{1}{k!}$ to derive

$$f(a + h) = \sum_{k=0}^{\infty} \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_k=1}^n \frac{1}{k!} (\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f)(a) h_{i_1} h_{i_2} \cdots h_{i_k}.$$

Naturally, we sometimes prefer to write the series expansion about a as an expression in $x = a + h$. With this substitution we have $h = x - a$ and $h_{i_j} = (x - a)_{i_j} = x_{i_j} - a_{i_j}$ thus

$$f(x) = \sum_{k=0}^{\infty} \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_k=1}^n \frac{1}{k!} (\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f)(a) (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2}) \cdots (x_{i_k} - a_{i_k}).$$

Example 6.1.2. Suppose $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ let's unravel the Taylor series centered at $(0, 0, 0)$ from the general formula boxed above. Utilize the notation $x = x_1, y = x_2$ and $z = x_3$ in this example.

$$f(x) = \sum_{k=0}^{\infty} \sum_{i_1=1}^3 \sum_{i_2=1}^3 \cdots \sum_{i_k=1}^3 \frac{1}{k!} (\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f)(0) x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The terms to order 2 are as follows:

$$\begin{aligned} f(x) &= f(0) + f_x(0)x + f_y(0)y + f_z(0)z \\ &\quad + \frac{1}{2} \left(f_{xx}(0)x^2 + f_{yy}(0)y^2 + f_{zz}(0)z^2 + \right. \\ &\quad \left. + f_{xy}(0)xy + f_{xz}(0)xz + f_{yz}(0)yz + f_{yx}(0)yx + f_{zx}(0)zx + f_{zy}(0)zy \right) + \cdots \end{aligned}$$

Partial derivatives commute for smooth functions hence,

$$\begin{aligned} f(x) &= f(0) + f_x(0)x + f_y(0)y + f_z(0)z \\ &\quad + \frac{1}{2} \left(f_{xx}(0)x^2 + f_{yy}(0)y^2 + f_{zz}(0)z^2 + 2f_{xy}(0)xy + 2f_{xz}(0)xz + 2f_{yz}(0)yz \right) \\ &\quad + \frac{1}{3!} \left(f_{xxx}(0)x^3 + f_{yyy}(0)y^3 + f_{zzz}(0)z^3 + 3f_{xxy}(0)x^2y + 3f_{xxz}(0)x^2z \right. \\ &\quad \left. + 3f_{yyz}(0)y^2z + 3f_{xyy}(0)xy^2 + 3f_{xzz}(0)xz^2 + 3f_{yzz}(0)yz^2 + 6f_{xyz}(0)xyz \right) + \cdots \end{aligned}$$

Example 6.1.3. Suppose $f(x, y, z) = e^{xyz}$. Find a quadratic approximation to f near $(0, 1, 2)$. Observe:

$$\begin{aligned} f_x &= yze^{xyz} & f_y &= xze^{xyz} & f_z &= xye^{xyz} \\ f_{xx} &= (yz)^2 e^{xyz} & f_{yy} &= (xz)^2 e^{xyz} & f_{zz} &= (xy)^2 e^{xyz} \\ f_{xy} &= ze^{xyz} + xyz^2 e^{xyz} & f_{yz} &= xe^{xyz} + x^2 yz e^{xyz} & f_{xz} &= ye^{xyz} + xy^2 z e^{xyz} \end{aligned}$$

²if $t = 1$ is not in the domain of g then we should rescale the vector h so that $t = 1$ places $\phi(1)$ in $\text{dom}(f)$, if f is smooth on some neighborhood of a then this is possible

Evaluating at $x = 0, y = 1$ and $z = 2$,

$$\begin{aligned} f_x(0, 1, 2) &= 2 & f_y(0, 1, 2) &= 0 & f_z(0, 1, 2) &= 0 \\ f_{xx}(0, 1, 2) &= 4 & f_{yy}(0, 1, 2) &= 0 & f_{zz}(0, 1, 2) &= 0 \\ f_{xy}(0, 1, 2) &= 2 & f_{yz}(0, 1, 2) &= 0 & f_{xz}(0, 1, 2) &= 1 \end{aligned}$$

Hence, as $f(0, 1, 2) = e^0 = 1$ we find

$$f(x, y, z) = 1 + 2x + 2x^2 + 2x(y - 1) + 2x(z - 2) + \dots$$

Another way to calculate this expansion is to make use of the adding zero trick,

$$f(x, y, z) = e^{x(y-1+1)(z-2+2)} = 1 + x(y - 1 + 1)(z - 2 + 2) + \frac{1}{2}[x(y - 1 + 1)(z - 2 + 2)]^2 + \dots$$

Keeping only terms with two or less of x , $(y - 1)$ and $(z - 2)$ variables,

$$f(x, y, z) = 1 + 2x + x(y - 1)(2) + x(1)(z - 2) + \frac{1}{2}x^2(1)^2(2)^2 + \dots$$

Which simplifies once more to $f(x, y, z) = 1 + 2x + 2x(y - 1) + x(z - 2) + 2x^2 + \dots$.

6.2 a brief introduction to the theory of quadratic forms

Definition 6.2.1.

Generally, a **quadratic form** Q is a function $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ whose formula can be written $Q(\vec{x}) = \vec{x}^T A \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ where $A \in \mathbb{R}^{n \times n}$ such that $A^T = A$. In particular, if $\vec{x} = (x, y)$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = ax^2 + bxy + byx + cy^2 = ax^2 + 2bxy + y^2.$$

The $n = 3$ case is similar, denote $A = [A_{ij}]$ and $\vec{x} = (x, y, z)$ so that

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = A_{11}x^2 + 2A_{12}xy + 2A_{13}xz + A_{22}y^2 + 2A_{23}yz + A_{33}z^2.$$

Generally, if $[A_{ij}] \in \mathbb{R}^{n \times n}$ and $\vec{x} = [x_i]^T$ then the associated quadratic form is

$$Q(\vec{x}) = \vec{x}^T A \vec{x} = \sum_{i,j} A_{ij} x_i x_j = \sum_{i=1}^n A_{ii} x_i^2 + \sum_{i < j} 2A_{ij} x_i x_j.$$

In case you wondering, yes you could write a given quadratic form with a different matrix which is not symmetric, but we will find it convenient to insist that our matrix is symmetric since that choice is always possible for a given quadratic form.

It is at times useful to use the dot-product to express a given quadratic form:

$$\vec{x}^T A \vec{x} = \vec{x} \cdot (A \vec{x}) = (A \vec{x}) \cdot \vec{x} = \vec{x}^T A^T \vec{x}$$

Some texts actually use the middle equality above to define a symmetric matrix.

Example 6.2.2.

$$2x^2 + 2xy + 2y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 6.2.3.

$$2x^2 + 2xy + 3xz - 2y^2 - z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 1 & 3/2 \\ 1 & -2 & 0 \\ 3/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proposition 6.2.4.

The values of a quadratic form on $\mathbb{R}^n - \{0\}$ is completely determined by it's values on the $(n-1)$ -sphere $S_{n-1} = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| = 1\}$. In particular, $Q(\vec{x}) = \|\vec{x}\|^2 Q(\hat{x})$ where $\hat{x} = \frac{1}{\|\vec{x}\|} \vec{x}$.

Proof: Let $Q(\vec{x}) = \vec{x}^T A \vec{x}$. Notice that we can write any nonzero vector as the product of its magnitude $\|\vec{x}\|$ and its direction $\hat{x} = \frac{1}{\|\vec{x}\|} \vec{x}$,

$$Q(\vec{x}) = Q(\|\vec{x}\|\hat{x}) = (\|\vec{x}\|\hat{x})^T A \|\vec{x}\|\hat{x} = \|\vec{x}\|^2 \hat{x}^T A \hat{x} = \|\vec{x}\|^2 Q(\hat{x}).$$

Therefore $Q(\vec{x})$ is simply proportional to $Q(\hat{x})$ with proportionality constant $\|\vec{x}\|^2$. \square

The proposition above is very interesting. It says that if we know how Q works on unit-vectors then we can extrapolate its action on the remainder of \mathbb{R}^n . If $f : S \rightarrow \mathbb{R}$ then we could say $f(S) > 0$ iff $f(s) > 0$ for all $s \in S$. Likewise, $f(S) < 0$ iff $f(s) < 0$ for all $s \in S$. The proposition below follows from the proposition above since $\|\vec{x}\|^2$ ranges over all nonzero positive real numbers in the equations above.

Proposition 6.2.5.

If Q is a quadratic form on \mathbb{R}^n and we denote $\mathbb{R}_*^n = \mathbb{R}^n - \{0\}$

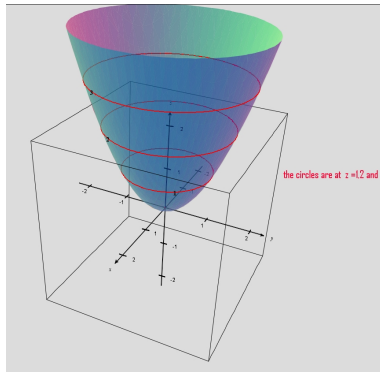
1.(negative definite) $Q(\mathbb{R}_*^n) < 0$ iff $Q(S_{n-1}) < 0$

2.(positive definite) $Q(\mathbb{R}_*^n) > 0$ iff $Q(S_{n-1}) > 0$

3.(non-definite) $Q(\mathbb{R}_*^n) = \mathbb{R} - \{0\}$ iff $Q(S_{n-1})$ has both positive and negative values.

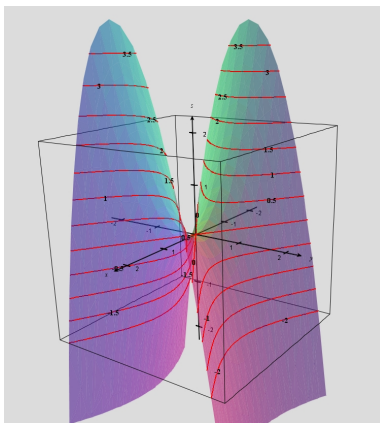
Before I get too carried away with the theory let's look at a couple examples.

Example 6.2.6. Consider the quadric form $Q(x, y) = x^2 + y^2$. You can check for yourself that $z = Q(x, y)$ is a cone and Q has positive outputs for all inputs except $(0, 0)$. Notice that $Q(v) = \|v\|^2$ so it is clear that $Q(S_1) = 1$. We find agreement with the preceding proposition. Next, think about the application of $Q(x, y)$ to level curves; $x^2 + y^2 = k$ is simply a circle of radius \sqrt{k} or just the origin. Here's a graph of $z = Q(x, y)$:



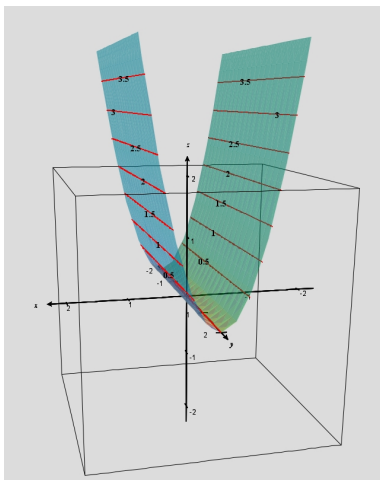
Notice that $Q(0, 0) = 0$ is the absolute minimum for Q . Finally, let's take a moment to write $Q(x, y) = [x, y] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = \lambda_2 = 1$.

Example 6.2.7. Consider the quadric form $Q(x, y) = x^2 - 2y^2$. You can check for yourself that $z = Q(x, y)$ is a hyperboloid and Q has non-definite outputs since sometimes the x^2 term dominates whereas other points have $-2y^2$ as the dominant term. Notice that $Q(1, 0) = 1$ whereas $Q(0, 1) = -2$ hence we find $Q(S_1)$ contains both positive and negative values and consequently we find agreement with the preceding proposition. Next, think about the application of $Q(x, y)$ to level curves; $x^2 - 2y^2 = k$ yields either hyperbolas which open vertically ($k > 0$) or horizontally ($k < 0$) or a pair of lines $y = \pm \frac{x}{2}$ in the $k = 0$ case. Here's a graph of $z = Q(x, y)$:



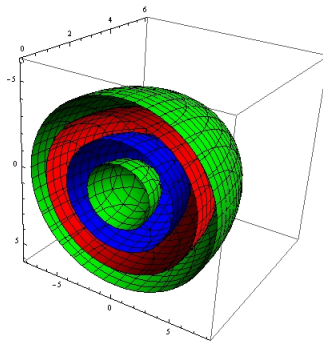
The origin is a **saddle point**. Finally, let's take a moment to write $Q(x, y) = [x, y] \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = -2$.

Example 6.2.8. Consider the quadric form $Q(x, y) = 3x^2$. You can check for yourself that $z = Q(x, y)$ is parabola-shaped trough along the y -axis. In this case Q has positive outputs for all inputs except $(0, y)$, we would call this form **positive semi-definite**. A short calculation reveals that $Q(S_1) = [0, 3]$ thus we again find agreement with the preceding proposition (case 3). Next, think about the application of $Q(x, y)$ to level curves; $3x^2 = k$ is a pair of vertical lines: $x = \pm \sqrt{k/3}$ or just the y -axis. Here's a graph of $z = Q(x, y)$:



Finally, let's take a moment to write $Q(x, y) = [x, y] \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 3$ and $\lambda_2 = 0$.

Example 6.2.9. Consider the quadric form $Q(x, y, z) = x^2 + 2y^2 + 3z^2$. Think about the application of $Q(x, y, z)$ to level surfaces; $x^2 + 2y^2 + 3z^2 = k$ is an ellipsoid. I can't graph a function of three variables, however, we can look at level surfaces of the function. I use Mathematica to plot several below:



Finally, let's take a moment to write $Q(x, y, z) = [x, y, z] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = 2$ and $\lambda_3 = 3$.

6.2.1 diagonalizing forms via eigenvectors

The examples given thus far are the simplest cases. We don't really need linear algebra to understand them. In contrast, e-vectors and e-values will prove a useful tool to unravel the later examples³

Definition 6.2.10.

Let $A \in \mathbb{R}^{n \times n}$. If $v \in \mathbb{R}^{n \times 1}$ is **nonzero** and $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ then we say v is an **eigenvector** with **eigenvalue** λ of the matrix A .

Proposition 6.2.11.

Let $A \in \mathbb{R}^{n \times n}$ then λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$. We say $P(\lambda) = \det(A - \lambda I)$ the **characteristic polynomial** and $\det(A - \lambda I) = 0$ is the **characteristic equation**.

Proof: Suppose λ is an eigenvalue of A then there exists a nonzero vector v such that $Av = \lambda v$ which is equivalent to $Av - \lambda v = 0$ which is precisely $(A - \lambda I)v = 0$. Notice that $(A - \lambda I)0 = 0$

³this is the one place in this course where we need eigenvalues and eigenvector calculations, I include these to illustrate the structure of quadratic forms in general, however, as linear algebra is not a prerequisite you may find some things in this section mysterious. The homework and study guide will elaborate on what is required this semester

thus the matrix $(A - \lambda I)$ is singular as the equation $(A - \lambda I)x = 0$ has more than one solution. Consequently $\det(A - \lambda I) = 0$.

Conversely, suppose $\det(A - \lambda I) = 0$. It follows that $(A - \lambda I)$ is singular. Clearly the system $(A - \lambda I)x = 0$ is consistent as $x = 0$ is a solution hence we know there are infinitely many solutions. In particular there exists at least one vector $v \neq 0$ such that $(A - \lambda I)v = 0$ which means the vector v satisfies $Av = \lambda v$. Thus v is an eigenvector with eigenvalue λ for A . \square

Example 6.2.12. Let $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ find the e -values and e -vectors of A .

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 1 \\ 3 & 1 - \lambda \end{bmatrix} = (3 - \lambda)(1 - \lambda) - 3 = \lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0$$

We find $\lambda_1 = 0$ and $\lambda_2 = 4$. Now find the e -vector with e -value $\lambda_1 = 0$, let $u_1 = [u, v]^T$ denote the e -vector we wish to find. Calculate,

$$(A - 0I)u_1 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3u + v \\ 3u + v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Obviously the equations above are redundant and we have infinitely many solutions of the form $3u + v = 0$ which means $v = -3u$ so we can write, $u_1 = \begin{bmatrix} u \\ -3u \end{bmatrix} = u \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. In applications we often make a choice to select a particular e -vector. Most modern graphing calculators can calculate e -vectors. It is customary for the e -vectors to be chosen to have length one. That is a useful choice for certain applications as we will later discuss. If you use a calculator it would likely give $u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ although the $\sqrt{10}$ would likely be approximated unless your calculator is smart.

Continuing we wish to find eigenvectors $u_2 = [u, v]^T$ such that $(A - 4I)u_2 = 0$. Notice that u, v are disposable variables in this context, I do not mean to connect the formulas from the $\lambda = 0$ case with the case considered now.

$$(A - 4I)u_1 = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u + v \\ 3u - 3v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Again the equations are redundant and we have infinitely many solutions of the form $v = u$. Hence, $u_2 = \begin{bmatrix} u \\ u \end{bmatrix} = u \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for any $u \in \mathbb{R}$ such that $u \neq 0$.

Theorem 6.2.13.

A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric iff there exists an orthonormal eigenbasis for A .

There is a geometric proof of this theorem in Edwards⁴ (see Theorem 8.6 pgs 146-147). I prove half of this theorem in my linear algebra notes by a non-geometric argument (full proof is in Appendix C of Insel, Spence and Friedberg). It might be very interesting to understand the connection between the geometric versus algebraic arguments. We'll content ourselves with an example here:

Example 6.2.14. Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. Observe that $\det(A - \lambda I) = -\lambda(\lambda + 1)(\lambda - 3)$ thus $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 3$. We can calculate orthonormal e-vectors of $v_1 = [1, 0, 0]^T$, $v_2 = \frac{1}{\sqrt{2}}[0, 1, -1]^T$ and $v_3 = \frac{1}{\sqrt{2}}[0, 1, 1]^T$. I invite the reader to check the validity of the following equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

It's really neat that to find the inverse of a matrix of orthonormal e-vectors we need only take the

transpose; note $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

XXX— remove comments about e-vectors and e-value before this section and put them here as motivating examples for the proposition that follows.

Proposition 6.2.15.

If Q is a quadratic form on \mathbb{R}^n with matrix A and e-values $\lambda_1, \lambda_2, \dots, \lambda_n$ with orthonormal e-vectors v_1, v_2, \dots, v_n then

$$Q(v_i) = \lambda_i^2$$

for $i = 1, 2, \dots, n$. Moreover, if $P = [v_1 | v_2 | \dots | v_n]$ then

$$Q(\vec{x}) = (P^T \vec{x})^T P^T A P P^T \vec{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

where we defined $\vec{y} = P^T \vec{x}$.

Let me restate the proposition above in simple terms: we can transform a given quadratic form to a diagonal form by finding orthonormalized e-vectors and performing the appropriate coordinate transformation. Since P is formed from orthonormal e-vectors we know that P will be either a rotation or reflection. This proposition says we can remove "cross-terms" by transforming the quadratic forms with an appropriate rotation.

Example 6.2.16. Consider the quadric form $Q(x, y) = 2x^2 + 2xy + 2y^2$. It's not immediately obvious (to me) what the level curves $Q(x, y) = k$ look like. We'll make use of the preceding

⁴think about it, there is a 1-1 correspondance between symmetric matrices and quadratic forms

proposition to understand those graphs. Notice $Q(x, y) = [x, y] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e -values/vectors:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix} = (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$$

Therefore, the e -values are $\lambda_1 = 1$ and $\lambda_2 = 3$.

$$(A - I)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A - 3I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

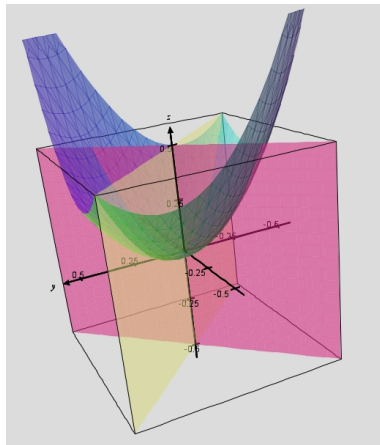
I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1 | \vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}]^T$ defined by $\vec{y} = P^T \vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{array}{l} x = \frac{1}{2}(\bar{x} + \bar{y}) \\ y = \frac{1}{2}(-\bar{x} + \bar{y}) \end{array} \quad \text{or} \quad \begin{array}{l} \bar{x} = \frac{1}{2}(x - y) \\ \bar{y} = \frac{1}{2}(x + y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield⁵:

$$\tilde{Q}(\bar{x}, \bar{y}) = \bar{x}^2 + 3\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x, y) = k$ is an ellipse. If we draw the barred coordinate system superposed over the xy -coordinate system then you'll see that the graph of $Q(x, y) = 2x^2 + 2xy + 2y^2 = k$ is an ellipse rotated by 45 degrees. Or, if you like, we can plot $z = Q(x, y)$:



⁵technically $\tilde{Q}(\bar{x}, \bar{y})$ is $Q(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))$

Example 6.2.17. Consider the quadric form $Q(x, y) = x^2 + 2xy + y^2$. It's not immediately obvious (to me) what the level curves $Q(x, y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x, y) = [x, y] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e -values/vectors:

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{bmatrix} = (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$$

Therefore, the e -values are $\lambda_1 = 0$ and $\lambda_2 = 2$.

$$(A - 0I)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

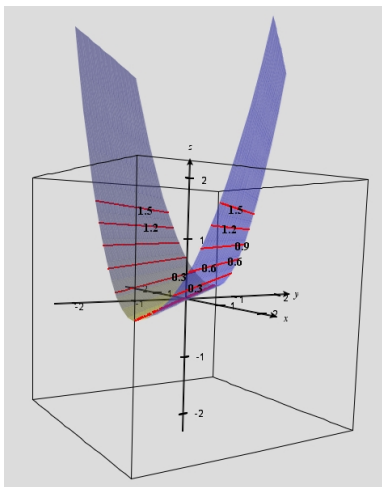
I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1 | \vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}]^T$ defined by $\vec{y} = P^T \vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{array}{l} x = \frac{1}{2}(\bar{x} + \bar{y}) \\ y = \frac{1}{2}(-\bar{x} + \bar{y}) \end{array} \quad \text{or} \quad \begin{array}{l} \bar{x} = \frac{1}{2}(x - y) \\ \bar{y} = \frac{1}{2}(x + y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x}, \bar{y}) = 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x, y) = k$ is a pair of parallel lines. If we draw the barred coordinate system superposed over the xy -coordinate system then you'll see that the graph of $Q(x, y) = x^2 + 2xy + y^2 = k$ is a line with slope -1 . Indeed, with a little algebraic insight we could have anticipated this result since $Q(x, y) = (x + y)^2$ so $Q(x, y) = k$ implies $x + y = \sqrt{k}$ thus $y = \sqrt{k} - x$. Here's a plot which again verifies what we've already found:



Example 6.2.18. Consider the quadric form $Q(x, y) = 4xy$. It's not immediately obvious (to me) what the level curves $Q(x, y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x, y) = [x, y] \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e -values/vectors:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 2 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 - 4 = (\lambda + 2)(\lambda - 2) = 0$$

Therefore, the e -values are $\lambda_1 = -2$ and $\lambda_2 = 2$.

$$(A + 2I)\vec{u}_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

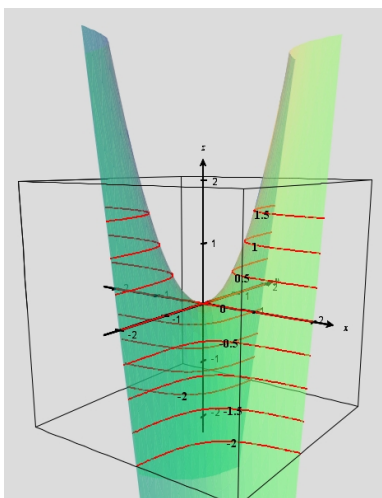
I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1 | \vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}]^T$ defined by $\vec{y} = P^T \vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \begin{aligned} x &= \frac{1}{2}(\bar{x} + \bar{y}) \\ y &= \frac{1}{2}(-\bar{x} + \bar{y}) \end{aligned} \quad \text{or} \quad \begin{aligned} \bar{x} &= \frac{1}{2}(x - y) \\ \bar{y} &= \frac{1}{2}(x + y) \end{aligned}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x}, \bar{y}) = -2\bar{x}^2 + 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x, y) = k$ is a hyperbola. If we draw the barred coordinate system superposed over the xy -coordinate system then you'll see that the graph of $Q(x, y) = 4xy = k$ is a hyperbola rotated by 45 degrees. The graph $z = 4xy$ is thus a hyperbolic paraboloid:



The fascinating thing about the mathematics here is that if you don't want to graph $z = Q(x, y)$, but you do want to know the general shape then you can determine which type of quadric surface you're dealing with by simply calculating the eigenvalues of the form.

Remark 6.2.19.

I made the preceding triple of examples all involved the same rotation. This is purely for my lecturing convenience. In practice the rotation could be by all sorts of angles. In addition, you might notice that a different ordering of the e-values would result in a redefinition of the barred coordinates. ⁶

We ought to do at least one 3-dimensional example.

Example 6.2.20. Consider the quadric form Q defined below:

$$Q(x, y, z) = [x, y, z] \begin{bmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Denote the matrix of the form by A and calculate the e-values/vectors:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 6 - \lambda & -2 & 0 \\ -2 & 6 - \lambda & 0 \\ 0 & 0 & 5 - \lambda \end{bmatrix} \\ &= [(\lambda - 6)^2 - 4](5 - \lambda) \\ &= (5 - \lambda)[\lambda^2 - 12\lambda + 32](5 - \lambda) \\ &= (\lambda - 4)(\lambda - 8)(5 - \lambda) \end{aligned}$$

Therefore, the e-values are $\lambda_1 = 4$, $\lambda_2 = 8$ and $\lambda_3 = 5$. After some calculation we find the following orthonormal e-vectors for A :

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

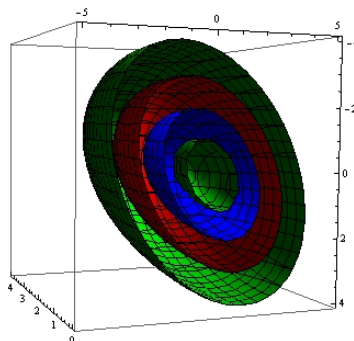
Let $P = [\vec{u}_1 | \vec{u}_2 | \vec{u}_3]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}, \bar{z}]^T$ defined by $\vec{y} = P^T \vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \Rightarrow \begin{array}{lcl} x & = & \frac{1}{2}(\bar{x} + \bar{y}) \\ y & = & \frac{1}{2}(-\bar{x} + \bar{y}) \\ z & = & \bar{z} \end{array} \quad \text{or} \quad \begin{array}{lcl} \bar{x} & = & \frac{1}{2}(x - y) \\ \bar{y} & = & \frac{1}{2}(x + y) \\ \bar{z} & = & z \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x}, \bar{y}, \bar{z}) = 4\bar{x}^2 + 8\bar{y}^2 + 5\bar{z}^2$$

It is clear that in the barred coordinate system the level surface $Q(x, y, z) = k$ is an ellipsoid. If we draw the barred coordinate system superposed over the xyz -coordinate system then you'll see that the graph of $Q(x, y, z) = k$ is an ellipsoid rotated by 45 degrees around the z -axis. Plotted below are a few representative ellipsoids:



In summary, the behaviour of a quadratic form $Q(x) = x^T A x$ is governed by its set of eigenvalues⁷ $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Moreover, the form can be written as $Q(y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_k y_k^2$ by choosing the coordinate system which is built from the orthonormal eigenbasis of $\text{col}(A)$. In this coordinate system the shape of the level-sets of Q becomes manifest from the signs of the e-values.)

Remark 6.2.21.

If you would like to read more about conic sections or quadric surfaces and their connection to e-values/vectors I recommend sections 9.6 and 9.7 of Anton's linear algebra text. I have yet to add examples on how to include translations in the analysis. It's not much more trouble but I decided it would just be an unnecessary complication this semester. Also, section 7.1, 7.2 and 7.3 in Lay's linear algebra text show a bit more about how to use this math to solve concrete applied problems. You might also take a look in Gilbert Strang's linear algebra text, his discussion of tests for positive-definite matrices is much more complete than I will give here.

6.3 second derivative test in many-variables

There is a connection between the shape of level curves $Q(x_1, x_2, \dots, x_n) = k$ and the graph $x_{n+1} = f(x_1, x_2, \dots, x_n)$ of f . I'll discuss $n = 2$ but these comments equally well apply to $w = f(x, y, z)$ or higher dimensional examples. Consider a critical point (a, b) for $f(x, y)$ then the Taylor expansion about (a, b) has the form

$$f(a + h, b + k) = f(a, b) + Q(h, k)$$

where $Q(h, k) = \frac{1}{2}h^2 f_{xx}(a, b) + hk f_{xy}(a, b) + \frac{1}{2}h^2 f_{yy}(a, b) = [h, k][Q](h, k)$. Since $[Q]^T = [Q]$ we can find orthonormal e-vectors \vec{u}_1, \vec{u}_2 for $[Q]$ with e-values λ_1 and λ_2 respective. Using $U = [\vec{u}_1 | \vec{u}_2]$ we

⁷this set is called the spectrum of the matrix

can introduce rotated coordinates $(\bar{h}, \bar{k}) = U(h, k)$. These will give

$$Q(\bar{h}, \bar{k}) = \lambda_1 \bar{h}^2 + \lambda_2 \bar{k}^2$$

Clearly if $\lambda_1 > 0$ and $\lambda_2 > 0$ then $f(a, b)$ yields the local minimum whereas if $\lambda_1 < 0$ and $\lambda_2 < 0$ then $f(a, b)$ yields the local maximum. Edwards discusses these matters on pgs. 148-153. In short, supposing $f \approx f(p) + Q$, if all the e-values of Q are positive then f has a local minimum of $f(p)$ at p whereas if all the e-values of Q are negative then f reaches a local maximum of $f(p)$ at p . Otherwise Q has both positive and negative e-values and we say Q is non-definite and the function has a saddle point. If all the e-values of Q are positive then Q is said to be **positive-definite** whereas if all the e-values of Q are negative then Q is said to be **negative-definite**. Edwards gives a few nice tests for ascertaining if a matrix is positive definite without explicit computation of e-values. Finally, if one of the e-values is zero then the graph will be like a trough.

Example 6.3.1. Suppose $f(x, y) = \exp(-x^2 - y^2 + 2y - 1)$ expand f about the point $(0, 1)$:

$$f(x, y) = \exp(-x^2) \exp(-y^2 + 2y - 1) = \exp(-x^2) \exp(-(y - 1)^2)$$

expanding,

$$f(x, y) = (1 - x^2 + \dots)(1 - (y - 1)^2 + \dots) = 1 - x^2 - (y - 1)^2 + \dots$$

Recenter about the point $(0, 1)$ by setting $x = h$ and $y = 1 + k$ so

$$f(h, 1 + k) = 1 - h^2 - k^2 + \dots$$

If (h, k) is near $(0, 0)$ then the dominant terms are simply those we've written above hence the graph is like that of a quadraic surface with a pair of negative e-values. It follows that $f(0, 1)$ is a local maximum. In fact, it happens to be a global maximum for this function.

Example 6.3.2. Suppose $f(x, y) = 4 - (x - 1)^2 + (y - 2)^2 + A \exp(-(x - 1)^2 - (y - 2)^2) + 2B(x - 1)(y - 2)$ for some constants A, B . Analyze what values for A, B will make $(1, 2)$ a local maximum, minimum or neither. Expanding about $(1, 2)$ we set $x = 1 + h$ and $y = 2 + k$ in order to see clearly the local behaviour of f at $(1, 2)$,

$$\begin{aligned} f(1 + h, 2 + k) &= 4 - h^2 - k^2 + A \exp(-h^2 - k^2) + 2Bhk \\ &= 4 - h^2 - k^2 + A(1 - h^2 - k^2) + 2Bhk \dots \\ &= 4 + A - (A + 1)h^2 + 2Bhk - (A + 1)k^2 + \dots \end{aligned}$$

There is no nonzero linear term in the expansion at $(1, 2)$ which indicates that $f(1, 2) = 4 + A$ may be a local extremum. In this case the quadratic terms are nontrivial which means the graph of this function is well-approximated by a quadraic surface near $(1, 2)$. The quadratic form $Q(h, k) = -(A + 1)h^2 + 2Bhk - (A + 1)k^2$ has matrix

$$[Q] = \begin{bmatrix} -(A + 1) & B \\ B & -(A + 1) \end{bmatrix}.$$

The characteristic equation for Q is

$$\det([Q] - \lambda I) = \det \begin{bmatrix} -(A+1) - \lambda & B \\ B & -(A+1)^2 - \lambda \end{bmatrix} = (\lambda + A + 1)^2 - B^2 = 0$$

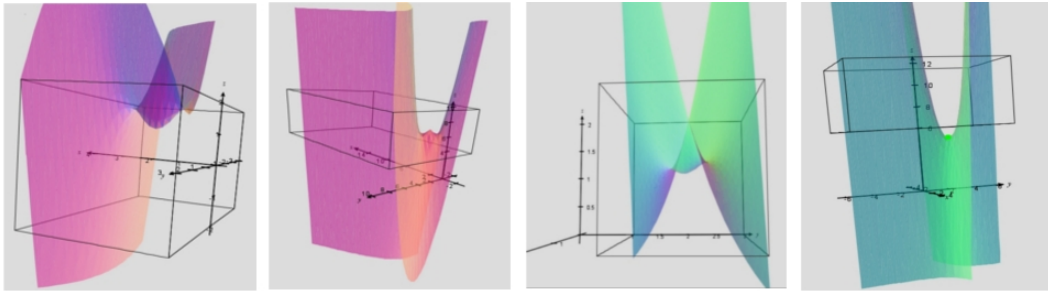
We find solutions $\lambda_1 = -A - 1 + B$ and $\lambda_2 = -A - 1 - B$. The possibilities break down as follows:

1. if $\lambda_1, \lambda_2 > 0$ then $f(1, 2)$ is local minimum.
2. if $\lambda_1, \lambda_2 < 0$ then $f(1, 2)$ is local maximum.
3. if just one of λ_1, λ_2 is zero then f is constant along one direction and min/max along another so technically it is a local extremum.
4. if $\lambda_1 \lambda_2 < 0$ then $f(1, 2)$ is not a local extremum, however it is a saddle point.

In particular, the following choices for A, B will match the choices above

1. Let $A = -3$ and $B = 1$ so $\lambda_1 = 3$ and $\lambda_2 = 1$;
2. Let $A = 3$ and $B = 1$ so $\lambda_1 = -3$ and $\lambda_2 = -5$
3. Let $A = -3$ and $B = -2$ so $\lambda_1 = 0$ and $\lambda_2 = 4$
4. Let $A = 1$ and $B = 3$ so $\lambda_1 = 1$ and $\lambda_2 = -5$

Here are the graphs of the cases above, note the analysis for case 3 is more subtle for Taylor approximations as opposed to simple quadraic surfaces. In this example, case 3 was also a local minimum. In contrast, in Example 6.2.17 the graph was like a trough. The behaviour of f away from the critical point includes higher order terms whose influence turns the trough into a local minimum.



Example 6.3.3. Suppose $f(x, y) = \sin(x) \cos(y)$ to find the Taylor series centered at $(0, 0)$ we can simply multiply the one-dimensional result $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$ and $\cos(y) = 1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \dots$ as follows:

$$\begin{aligned} f(x, y) &= (x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots)(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \dots) \\ &= x - \frac{1}{2}xy^2 + \frac{1}{24}xy^4 - \frac{1}{6}x^3 - \frac{1}{12}x^3y^2 + \dots \\ &= x + \dots \end{aligned}$$

The origin $(0,0)$ is a critical point since $f_x(0,0) = 0$ and $f_y(0,0) = 0$, however, this particular critical point escapes the analysis via the quadratic form term since $Q = 0$ in the Taylor series for this function at $(0,0)$. This is analogous to the inconclusive case of the 2nd derivative test in calculus III.

Example 6.3.4. Suppose $f(x, y, z) = xyz$. Calculate the multivariate Taylor expansion about the point $(1, 2, 3)$. I'll actually calculate this one via differentiation, I have used tricks and/or calculus II results to shortcut any differentiation in the previous examples. Calculate first derivatives

$$f_x = yz \quad f_y = xz \quad f_z = xy,$$

and second derivatives,

$$\begin{array}{lll} f_{xx} = 0 & f_{xy} = z & f_{xz} = y \\ f_{yx} = z & f_{yy} = 0 & f_{yz} = x \\ f_{zx} = y & f_{zy} = x & f_{zz} = 0, \end{array}$$

and the nonzero third derivatives,

$$f_{xyz} = f_{yzx} = f_{zxy} = f_{zyx} = f_{yxz} = f_{xzy} = 1.$$

It follows,

$$\begin{aligned} f(a+h, b+k, c+l) &= \\ &= f(a, b, c) + f_x(a, b, c)h + f_y(a, b, c)k + f_z(a, b, c)l + \\ &\quad \frac{1}{2}(f_{xx}hh + f_{xy}hk + f_{xz}hl + f_{yx}kh + f_{yy}kk + f_{yz}kl + f_{zx}lh + f_{zy}lk + f_{zz}ll) + \dots \end{aligned}$$

Of course certain terms can be combined since $f_{xy} = f_{yx}$ etc... for smooth functions (we assume smooth in this section, moreover the given function here is clearly smooth). In total,

$$f(1+h, 2+k, 3+l) = 6 + 6h + 3k + 2l + \frac{1}{2}(3hk + 2hl + 3kh + kl + 2lh + lk) + \frac{1}{3!}(6)hkl$$

Of course, we could also obtain this from simple algebra:

$$f(1+h, 2+k, 3+l) = (1+h)(2+k)(3+l) = 6 + 6h + 3k + l + 3hk + 2hl + kl + hkl.$$

6.3.1 morse theory and future reading

Chapter 7

multilinear algebra

7.1 dual space

Definition 7.1.1.

Suppose V is a vector space over \mathbb{R} . We define the **dual space** to V to be the set of all linear functions from V to \mathbb{R} . In particular, we denote:

$$V^* = \{f : V \rightarrow \mathbb{R} \mid f(x+y) = f(x) + f(y) \text{ and } f(cx) = cf(x) \quad \forall x, y \in V \text{ and } c \in \mathbb{R}\}$$

If $\alpha \in V^*$ then we say α is a **dual vector**.

I offer several abstract examples to begin, however the majority of this section concerns \mathbb{R}^n .

Example 7.1.2. Suppose \mathcal{F} denotes the set of continuous functions on \mathbb{R} . Define $\alpha(f) = \int_0^1 f(t) dt$. The mapping $\alpha : \mathcal{F} \rightarrow \mathbb{R}$ is linear by properties of definite integrals therefore we identify the definite integral defines a dual-vector to the vector space of continuous functions.

Example 7.1.3. Suppose $V = \mathcal{F}(W, \mathbb{R})$ denotes a set of functions from a vector space W to \mathbb{R} . Note that V is a vector space with respect to point-wise defined addition and scalar multiplication of functions. Let $w_o \in W$ and define $\alpha(f) = f(w_o)$. The mapping $\alpha : V \rightarrow \mathbb{R}$ is linear since $\alpha(cf + g) = (cf + g)(w_o) = cf(w_o) + g(w_o) = c\alpha(f) + \alpha(g)$ for all $f, g \in V$ and $c \in \mathbb{R}$. We find that the **evaluation** map defines a dual-vector $\alpha \in V^*$.

Example 7.1.4. The determinant is a mapping from $\mathbb{R}^{n \times n}$ to \mathbb{R} but it does not define a dual-vector to the vector space of square matrices since $\det(A+B) \neq \det(A) + \det(B)$.

Example 7.1.5. Suppose $\alpha(x) = x \cdot v$ for a particular vector $v \in \mathbb{R}^n$. We argue $\alpha \in V^*$ where we recall $V = \mathbb{R}^n$ is a vector space. Additivity follows from a property of the dot-product on \mathbb{R}^n ,

$$\alpha(x+y) = (x+y) \cdot v = x \cdot v + y \cdot v = \alpha(x) + \alpha(y)$$

for all $x, y \in \mathbb{R}^n$. Likewise, homogeneity follows from another property of the dot-product: observe

$$\alpha(cx) = (cx) \cdot v = c(x \cdot v) = c\alpha(x)$$

for all $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Example 7.1.6. Let $\alpha(x, y) = 2x + 5y$ define a function $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$. Note that

$$\alpha(x, y) = (x, y) \cdot (2, 5)$$

hence by the preceding example we find $\alpha \in (\mathbb{R}^2)^*$.

The preceding example is no accident. It turns out there is a one-one correspondence between row vectors and dual vectors on \mathbb{R}^n . Let $v \in \mathbb{R}^n$ then we define $\alpha_v(x) = x \cdot v$. We proved in Example 7.1.5 that $\alpha_v \in (\mathbb{R}^n)^*$. Suppose $\alpha \in (\mathbb{R}^n)^*$ we see to find $v \in \mathbb{R}^n$ such that $\alpha = \alpha_v$. Recall that a linear function is uniquely defined by its values on a basis; the values of α on the standard basis will show us how to choose v . This is a standard technique. Consider: $v \in \mathbb{R}^n$ with¹ $v = \sum_{j=1}^n v^j e_j$

$$\alpha(x) = \underbrace{\alpha\left(\sum_{j=1}^n x^j e_j\right)}_{\text{additivity}} = \sum_{j=1}^n \underbrace{\alpha(x^j e_j)}_{\text{homogeneity}} = \sum_{j=1}^n x^j \alpha(e_j) = x \cdot v$$

where we define $v = (\alpha(e_1), \alpha(e_2), \dots, \alpha(e_n)) \in \mathbb{R}^n$. The vector which corresponds naturally² to α is simply the vector of the values of α on the standard basis.

The dual space to \mathbb{R}^n is a vector space and the correspondence $v \rightarrow \alpha_v$ gives an isomorphism of \mathbb{R}^n and $(\mathbb{R}^n)^*$. The image of a basis under an isomorphism is once more a basis. Define $\Phi : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^*$ by $\Phi(v) = \alpha_v$ to give the correspondence an explicit label. The image of the standard basis under Φ is called the **standard dual basis** for $(\mathbb{R}^n)^*$. Consider $\Phi(e_j)$, let $x \in \mathbb{R}^n$ and calculate

$$\Phi(e_j)(x) = \alpha_{e_j}(x) = x \cdot e_j$$

In particular, notice that when $x = e_i$ then $\Phi(e_j)(e_i) = e_i \cdot e_j = \delta_{ij}$. Dual vectors are linear transformations therefore we can define the dual basis by its values on the standard basis.

Definition 7.1.7.

The **standard dual basis** of $(\mathbb{R}^n)^*$ is denoted $\{e^1, e^2, \dots, e^n\}$ where we define $e^j : \mathbb{R}^n \rightarrow \mathbb{R}$ to be the linear transformation such that $e^j(e_i) = \delta_{ij}$ for all $i, j \in \mathbb{N}_n$. Generally, given a vector space V with basis $\beta = \{f_1, f_2, \dots, f_m\}$ we say the basis $\beta^* = \{f^1, f^2, \dots, f^n\}$ is dual to β iff $f^j(f_i) = \delta_{ij}$ for all $i, j \in \mathbb{N}_n$.

The term *basis* indicates that $\{e^1, e^2, \dots, e^n\}$ is linearly independent³ and $\text{span}\{e^1, e^2, \dots, e^n\} = (\mathbb{R}^n)^*$. The following calculation is often useful: if $x \in \mathbb{R}^n$ with $x = \sum_{j=1}^n x^j e_j$ then

$$e^i(x) = e^i\left(\sum_{j=1}^n x^j e_j\right) = \sum_{j=1}^n x^j e^i(e_j) = \sum_{j=1}^n x^j \delta_{ij} = x^i \Rightarrow \boxed{e^i(x) = x^i.}$$

¹the super-index is not a power in this context, it is just a notation to emphasize v^j is the component of a vector.

²some authors will say $\mathbb{R}^{n \times 1}$ is dual to $\mathbb{R}^{1 \times n}$ since $\alpha_v(x) = v^T x$ and v^T is a row vector, I will avoid that language in these notes.

³direct proof of LI is left to the reader

The calculation above is a prototype for many that follow in this chapter. Next, suppose $\alpha \in (\mathbb{R}^n)^*$ and suppose $x \in \mathbb{R}^n$ with $x = \sum_{j=1}^n x^j e_j$. Calculate,

$$\alpha(x) = \alpha\left(\sum_{i=1}^n x^i e_i\right) = \sum_{i=1}^n \alpha(e_i) e^i(x) \Rightarrow \boxed{\alpha = \sum_{i=1}^n \alpha(e_i) e^i}$$

this shows every dual vector is in the span of the dual basis $\{e^j\}_{j=1}^n$.

7.2 multilinearity and the tensor product

A multilinear mapping is a function of a Cartesian product of vector spaces which is linear with respect to each "slot". The goal of this section is to explain what that means. It turns out the set of all multilinear mappings on a particular set of vector spaces forms a vector space and we'll show how the tensor product can be used to construct an explicit basis by tensoring a bases which are dual to the bases in the domain. We also examine the concepts of symmetric and antisymmetric multilinear mappings, these form interesting subspaces of the set of all multilinear mappings. Our approach in this section is to treat the case of bilinearity in depth then transition to the case of multilinearity. Naturally this whole discussion demands a familiarity with the preceding section.

7.2.1 bilinear maps

Definition 7.2.1.

Suppose V_1, V_2 are vector spaces then $b : V_1 \times V_2 \rightarrow \mathbb{R}$ is a **bilinear mapping** on $V_1 \times V_2$ iff for all $x, y \in V_1, z, w \in V_2$ and $c \in \mathbb{R}$:

- (1.) $b(cx + y, z) = cb(x, z) + b(y, z)$ (linearity in the first slot)
- (2.) $b(x, cz + w) = cb(x, z) + b(x, w)$ (linearity in the second slot).

bilinear maps on $V \times V$

When $V_1 = V_2 = V$ we simply say that $b : V \times V \rightarrow \mathbb{R}$ is a **bilinear mapping on V** . The set of all bilinear maps of V is denoted $T_0^2 V$. You can show that $T_0^2 V$ forms a vector space under the usual point-wise defined operations of function addition and scalar multiplication⁴. Hopefully you are familiar with the example below.

Example 7.2.2. Define $b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by $b(x, y) = x \cdot y$ for all $x, y \in \mathbb{R}^n$. Linearity in each slot follows easily from properties of dot-products:

$$\begin{aligned} b(cx + y, z) &= (cx + y) \cdot z = cx \cdot z + y \cdot z = cb(x, z) + b(y, z) \\ b(x, cy + z) &= x \cdot (cy + z) = cx \cdot y + x \cdot z = cb(x, y) + b(x, z). \end{aligned}$$

⁴sounds like homework

We can use matrix multiplication to generate a large class of examples with ease.

Example 7.2.3. Suppose $A \in \mathbb{R}^{n \times n}$ and define $b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by $b(x, y) = x^T A y$ for all $x, y \in \mathbb{R}^n$. Observe that, by properties of matrix multiplication,

$$b(cx + y, z) = (cx + y)^T A z = (cx^T + y^T) A z = cx^T A z + y^T A z = cb(x, z) + b(y, z)$$

$$b(x, cy + z) = x^T A(cy + z) = cx^T A y + x^T A z = cb(x, y) + b(x, z)$$

for all $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. It follows that b is bilinear on \mathbb{R}^n .

Suppose $b : V \times V \rightarrow \mathbb{R}$ is bilinear and suppose $\beta = \{e_1, e_2, \dots, e_n\}$ is a basis for V whereas $\beta^* = \{e^1, e^2, \dots, e^n\}$ is a basis of V^* with $e^j(e_i) = \delta_{ij}$

$$\begin{aligned} b(x, y) &= b\left(\sum_{i=1}^n x^i e_i, \sum_{j=1}^n y^j e_j\right) \\ &= \sum_{i,j=1}^n b(x^i e_i, y^j e_j) \\ &= \sum_{i,j=1}^n x^i y^j b(e_i, e_j) \\ &= \sum_{i,j=1}^n b(e_i, e_j) e^i(x) e^j(y) \end{aligned} \tag{7.1}$$

Therefore, if we define $b_{ij} = b(e_i, e_j)$ then we may compute $b(x, y) = \sum_{i,j=1}^n b_{ij} x^i y^j$. The calculation above also indicates that b is a linear combination of certain basic bilinear mappings. In particular, b can be written a linear combination of a tensor product of dual vectors on V .

Definition 7.2.4.

Suppose V is a vector space with dual space V^* . If $\alpha, \beta \in V^*$ then we define $\alpha \otimes \beta : V \times V \rightarrow \mathbb{R}$ by $(\alpha \otimes \beta)(x, y) = \alpha(x)\beta(y)$ for all $x, y \in V$.

Given the notation⁵ preceding this definition, we note $(e^i \otimes e^j)(x, y) = e^i(x)e^j(y)$ hence for all $x, y \in V$ we find:

$$b(x, y) = \sum_{i,j=1}^n b(e_i, e_j) (e^i \otimes e^j)(x, y) \quad \text{therefore,} \quad b = \sum_{i,j=1}^n b(e_i, e_j) e^i \otimes e^j$$

We find⁶ that $T_0^2 V = \text{span}\{e^i \otimes e^j\}_{i,j=1}^n$. Moreover, it can be argued⁷ that $\{e^i \otimes e^j\}_{i,j=1}^n$ is a linearly independent set, therefore $\{e^i \otimes e^j\}_{i,j=1}^n$ forms a basis for $T_0^2 V$. We can count there are n^2 vectors

⁵perhaps you would rather write $(e^i \otimes e^j)(x, y)$ as $e^i \otimes e^j(x, y)$, that is also fine.

⁶with the help of your homework where you will show $\{e^i \otimes e^j\}_{i,j=1}^n \subseteq T_0^2 V$

⁷yes, again, in your homework

in $\{e^i \otimes e^j\}_{i,j=1}^n$ hence $\dim(T_0^2 V) = n^2$.

If $V = \mathbb{R}^n$ and if $\{e^i\}_{i=1}^n$ denotes the standard dual basis, then there is a standard notation for the set of coefficients found in the summation for b . In particular, we denote $B = [b]$ where $B_{ij} = b(e_i, e_j)$ hence, following Equation 7.1,

$$b(x, y) = \sum_{i,j=1}^n x^i y^j b(e_i, e_j) = \sum_{i=1}^n \sum_{j=1}^n x^i B_{ij} y^j = x^T B y$$

Definition 7.2.5.

Suppose $b : V \times V \rightarrow \mathbb{R}$ is a bilinear mapping then we say:

1. b is **symmetric** iff $b(x, y) = b(y, x)$ for all $x, y \in V$
2. b is **antisymmetric** iff $b(x, y) = -b(y, x)$ for all $x, y \in V$

Any bilinear mapping on V can be written as the sum of a symmetric and antisymmetric bilinear mapping, this claim follows easily from the calculation below:

$$b(x, y) = \underbrace{\frac{1}{2} \left(b(x, y) + b(y, x) \right)}_{\text{symmetric}} + \underbrace{\frac{1}{2} \left(b(x, y) - b(y, x) \right)}_{\text{antisymmetric}}.$$

We say S_{ij} is **symmetric** in i, j iff $S_{ij} = S_{ji}$ for all i, j . Likewise, we say A_{ij} is **antisymmetric** in i, j iff $A_{ij} = -A_{ji}$ for all i, j . If S is a symmetric bilinear mapping and A is an antisymmetric bilinear mapping then the components of S are symmetric and the components of A are antisymmetric. Why? Simply note:

$$S(e_i, e_j) = S(e_j, e_i) \Rightarrow S_{ij} = S_{ji}$$

and

$$A(e_i, e_j) = -A(e_j, e_i) \Rightarrow A_{ij} = -A_{ji}.$$

You can prove that the sum or scalar multiple of an (anti)symmetric bilinear mapping is once more (anti)symmetric therefore the set of antisymmetric bilinear maps $\Lambda^2(V)$ and the set of symmetric bilinear maps $ST_2^0 V$ are subspaces of $T_2^0 V$. The notation $\Lambda^2(V)$ is part of a larger discussion on the wedge product, we will return to it in a later section.

Finally, if we consider the special case of $V = \mathbb{R}^n$ once more we find that a bilinear mapping $b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ has a symmetric matrix $[b]^T = [b]$ iff b is symmetric whereas it has an antisymmetric matrix $[b]^T = -[b]$ iff b is antisymmetric.

bilinear maps on $V^* \times V^*$

Suppose $h : V^* \times V^* \rightarrow \mathbb{R}$ is bilinear then we say $h \in T_0^2 V$. In addition, suppose $\beta = \{e_1, e_2, \dots, e_n\}$ is a basis for V whereas $\beta^* = \{e^1, e^2, \dots, e^n\}$ is a basis of V^* with $e^j(e_i) = \delta_{ij}$. Let $\alpha, \beta \in V^*$

$$\begin{aligned}
 h(\alpha, \beta) &= h\left(\sum_{i=1}^n \alpha_i e^i, \sum_{j=1}^n \beta_j e^j\right) \\
 &= \sum_{i,j=1}^n h(\alpha_i e^i, \beta_j e^j) \\
 &= \sum_{i,j=1}^n \alpha_i \beta_j h(e^i, e^j) \\
 &= \sum_{i,j=1}^n h(e^i, e^j) \alpha(e_i) \beta(e_j)
 \end{aligned} \tag{7.2}$$

Therefore, if we define $h^{ij} = h(e^i, e^j)$ then we find the nice formula $h(\alpha, \beta) = \sum_{i,j=1}^n h^{ij} \alpha_i \beta_j$. To further refine the formula above we need a new concept.

The dual of the dual is called the double-dual and it is denoted V^{**} . For a finite dimensional vector space there is a canonical isomorphism of V and V^{**} . In particular, $\Phi : V \rightarrow V^{**}$ is defined by $\Phi(v)(\alpha) = \alpha(v)$ for all $\alpha \in V^*$. It is customary to replace V with V^{**} wherever the context allows. For example, to define the tensor product of two vectors $x, y \in V$ as follows:

Definition 7.2.6.

Suppose V is a vector space with dual space V^* . We define the tensor product of vectors x, y as the mapping $x \otimes y : V^* \times V^* \rightarrow \mathbb{R}$ by $(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y)$ for all $x, y \in V$.

We could just as well have defined $x \otimes y = \Phi(x) \otimes \Phi(y)$ where Φ is once more the canonical isomorphism of V and V^{**} . It's called *canonical* because it has no particular dependence on the coordinates used on V . In contrast, the isomorphism of \mathbb{R}^n and $(\mathbb{R}^n)^*$ was built around the dot-product and the standard basis.

All of this said, note that $\alpha(e_i)\beta(e_j) = e_i \otimes e_j(\alpha, \beta)$ thus,

$$h(\alpha, \beta) = \sum_{i,j=1}^n h(e^i, e^j) e_i \otimes e_j(\alpha, \beta) \Rightarrow h = \sum_{i,j=1}^n h(e^i, e^j) e_i \otimes e_j$$

We argue that $\{e_i \otimes e_j\}_{i,j=1}^n$ is a basis⁸

Definition 7.2.7.

⁸ $T_0^2 V$ is a vector space and we've shown $T_0^2(V) \subseteq \text{span}\{e_i \otimes e_j\}_{i,j=1}^n$ but we should also show $e_i \otimes e_j \in T_0^2$ and check for LI of $\{e_i \otimes e_j\}_{i,j=1}^n$.

Suppose $h : V^* \times V^* \rightarrow \mathbb{R}$ is a bilinear mapping then we say:

1. h is **symmetric** iff $h(\alpha, \beta) = h(\beta, \alpha)$ for all $\alpha, \beta \in V^*$
2. h is **antisymmetric** iff $h(\alpha, \beta) = -h(\beta, \alpha)$ for all $\alpha, \beta \in V^*$

The discussion of the preceding subsection transfers to this context, we simply have to switch some vectors to dual vectors and move some indices up or down. I leave this to the reader.

bilinear maps on $V \times V^*$

Suppose $H : V \times V^* \rightarrow \mathbb{R}$ is bilinear, we say $H \in T_1^1 V$ (or, if the context demands this detail $H \in T_1^{-1} V$). We define $\alpha \otimes x \in T_1^{-1}(V)$ by the natural rule; $(\alpha \otimes x)(y, \beta) = \alpha(x)\beta(y)$ for all $(y, \beta) \in V \times V^*$. We find, by calculations similar to those already given in this section,

$$H(y, \beta) = \sum_{i,j=1}^n H_i^j y^i \beta_j \quad \text{and} \quad H = \sum_{i,j=1}^n H_i^j e^i \otimes e_j$$

where we defined $H_i^j = H(e_i, e^j)$.

bilinear maps on $V^* \times V$

Suppose $G : V^* \times V \rightarrow \mathbb{R}$ is bilinear, we say $G \in T_1^1 V$ (or, if the context demands this detail $G \in T_1^{-1} V$). We define $x \otimes \alpha \in T_1^1 V$ by the natural rule; $(x \otimes \alpha)(\beta, y) = \beta(x)\alpha(y)$ for all $(\beta, y) \in V^* \times V$. We find, by calculations similar to those already given in this section,

$$G(\beta, y) = \sum_{i,j=1}^n G_j^i \beta_i y^j \quad \text{and} \quad G = \sum_{i,j=1}^n G_j^i e_i \otimes e^j$$

where we defined $G_j^i = G(e^i, e_j)$.

7.2.2 trilinear maps

Definition 7.2.8.

Suppose V_1, V_2, V_3 are vector spaces then $T : V_1 \times V_2 \times V_3 \rightarrow \mathbb{R}$ is a **trilinear mapping** on $V_1 \times V_2 \times V_3$ iff for all $u, v \in V_1, w, x \in V_2, y, z \in V_3$ and $c \in \mathbb{R}$:

- (1.) $T(cu + v, w, y) = cT(u, w, y) + T(v, w, y)$ (linearity in the first slot)
- (2.) $T(u, cw + x, y) = cT(u, w, y) + T(u, x, y)$ (linearity in the second slot).
- (3.) $T(u, w, cy + z) = cT(u, w, y) + T(u, w, z)$ (linearity in the third slot).

If $T : V \times V \times V \rightarrow \mathbb{R}$ is trilinear on $V \times V \times V$ then we say T is a **trilinear mapping on V** and we denote the set of all such mappings $T_3^0 V$. The tensor product of three dual vectors is defined much in the same way as it was for two,

$$(\alpha \otimes \beta \otimes \gamma)(x, y, z) = \alpha(x)\beta(y)\gamma(z)$$

Let $\{e_i\}_{i=1}^n$ is a basis for V with dual basis $\{e^i\}_{i=1}^n$ for V^* . If T is trilinear on V it follows

$$T(x, y, z) = \sum_{i,j,k=1}^n T_{ijk} x^i y^j z^k \quad \text{and} \quad T = \sum_{i,j,k=1}^n T_{ijk} e^i \otimes e^j \otimes e^k$$

where we defined $T_{ijk} = T(e_i, e_j, e_k)$ for all $i, j, k \in \mathbb{N}_n$.

Generally suppose that V_1, V_2, V_3 are possibly distinct vector spaces. Moreover, suppose V_1 has basis $\{e_i\}_{i=1}^{n_1}$, V_2 has basis $\{f_j\}_{j=1}^{n_2}$ and V_3 has basis $\{g_k\}_{k=1}^{n_3}$. Denote the dual bases for V_1^*, V_2^*, V_3^* in the usual fashion: $\{e^i\}_{i=1}^{n_1}$, $\{f^j\}_{j=1}^{n_2}$, $\{g^k\}_{k=1}^{n_3}$. With this notation, we can write a trilinear mapping on $V_1 \times V_2 \times V_3$ as follows: (where we define $T_{ijk} = T(e_i, f_j, g_k)$)

$$T(x, y, z) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} T_{ijk} x^i y^j z^k \quad \text{and} \quad T = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} T_{ijk} e^i \otimes f^j \otimes g^k$$

However, if V_1, V_2, V_3 happen to be related by duality then it is customary to use up/down indices. For example, if $T : V \times V \times V^* \rightarrow \mathbb{R}$ is trilinear then we write⁹

$$T = \sum_{i,j,k=1}^n T_{ij}{}^k e^i \otimes e^j \otimes e_k$$

and say $T \in T_2^{-1} V$. On the other hand, if $S : V^* \times V^* \times V$ is trilinear then we'd write

$$T = \sum_{i,j,k=1}^n S^{ij}{}_k e_i \otimes e_j \otimes e^k$$

and say $T \in T_2^2 V$. I'm not sure that I've ever seen this notation elsewhere, but perhaps it could be useful to denote the set of trilinear maps $T : V \times V^* \times V \rightarrow \mathbb{R}$ as $T_1^1{}_1 V$. Hopefully we will not need such silly notation in what we consider this semester.

There was a natural correspondance between bilinear maps on \mathbb{R}^n and square matrices. For a trilinear map we would need a three-dimensional array of components. In some sense you could picture $T : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ as multiplication by a cube of numbers. Don't think too hard about these silly comments, we actually already wrote the useful formulae for dealing with trilinear objects. Let's stop to look at an example.

⁹we identify e_k with its double-dual hence this tensor product is already defined, but to be safe let me write it out in this context $e^i \otimes e^j \otimes e_k(x, y, \alpha) = e^i(x)e^j(y)\alpha(e_k)$.

Example 7.2.9. Define $T : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ by $T(x, y, z) = \det(x|y|z)$. You may not have learned this in your linear algebra course¹⁰ but a nice formula¹¹ for the determinant is given by the Levi-Civita symbol,

$$\det(A) = \sum_{i,j,k=1}^3 \epsilon_{ijk} A_{i1} A_{j2} A_{k3}$$

note that $\text{col}_1(A) = [A_{i1}]$, $\text{col}_2(A) = [A_{i2}]$ and $\text{col}_3(A) = [A_{i3}]$. It follows that

$$T(x, y, z) = \sum_{i,j,k=1}^3 \epsilon_{ijk} x^i y^j z^k$$

Multilinearity follows easily from this formula. For example, linearity in the third slot:

$$T(x, y, cz + w) = \sum_{i,j,k=1}^3 \epsilon_{ijk} x^i y^j (cz + w)^k \quad (7.3)$$

$$= \sum_{i,j,k=1}^3 \epsilon_{ijk} x^i y^j (cz^k + w^k) \quad (7.4)$$

$$= c \sum_{i,j,k=1}^3 \epsilon_{ijk} x^i y^j z^k + \sum_{i,j,k=1}^3 \epsilon_{ijk} x^i y^j w^k \quad (7.5)$$

$$= cT(x, y, z) + T(x, y, w). \quad (7.6)$$

Observe that by properties of determinants, or the Levi-Civita symbol if you prefer, swapping a pair of inputs generates a minus sign, hence:

$$T(x, y, z) = -T(y, x, z) = T(y, z, x) = -T(z, y, x) = T(z, x, y) = -T(x, z, y).$$

If $T : V \times V \times V \rightarrow \mathbb{R}$ is a trilinear mapping such that

$$T(x, y, z) = -T(y, x, z) = T(y, z, x) = -T(z, y, x) = T(z, x, y) = -T(x, z, y)$$

for all $x, y, z \in V$ then we say T is **antisymmetric**. Likewise, if $S : V \times V \times V \rightarrow \mathbb{R}$ is a trilinear mapping such that

$$S(x, y, z) = -S(y, x, z) = S(y, z, x) = -S(z, y, x) = S(z, x, y) = -S(x, z, y).$$

for all $x, y, z \in V$ then we say T is **symmetric**. Clearly the mapping defined by the determinant is antisymmetric. In fact, many authors define the determinant of an $n \times n$ matrix as the antisymmetric n -linear mapping which sends the identity matrix to 1. It turns out these criteria uniquely

¹⁰maybe you haven't even taken linear yet!

¹¹actually, I take this as the definition in linear algebra, it does take considerable effort to recover the expansion by minors formula which I use for concrete examples

define the determinant. That is the motivation behind my Levi-Civita symbol definition. That formula is just the nuts and bolts of complete antisymmetry.

You might wonder, can every trilinear mapping can be written as a the sum of a symmetric and antisymmetric mapping? The answer is no. Take the following trilinear mapping on \mathbb{R}^3 for example:

$$T(x, y, z) = \det[x|y|e_3] + y \cdot z$$

You can verify this is linear in each slot however, it is antisymmetric in the first pair of slots

$$T(x, y, z) = \det[x|y|e_3] + y \cdot z = -\det[y|x|e_3] + y \cdot z = -T(y, x, z)$$

and symmetric in the last pair,

$$T(x, y, z) = \det[x|y|e_3] + y \cdot z = \det[x|y|e_3] + z \cdot y = T(x, z, y).$$

Generally, the decomposition of a multilinear mapping into more basic types is a problem which requires much more thought than we intend here. Representation theory is concerned with precisely this problem: how can we decompose a tensor product into irreducible pieces. Their idea of tensor product is not precisely the same as ours, however algebraically the problems are quite intertwined. I'll leave it at that unless you'd like to do an independent study on representation theory. Ideally you'd already have linear algebra and abstract algebra complete before you attempt that study.

7.2.3 multilinear maps

Definition 7.2.10.

Suppose V_1, V_2, \dots, V_k are vector spaces then $T : V_1 \times V_2 \times \dots \times V_k \rightarrow \mathbb{R}$ is a **k -multilinear mapping** on $V_1 \times V_2 \times \dots \times V_k$ iff for each $c \in \mathbb{R}$ and $x_1, y_1 \in V_1, x_2, y_2 \in V_2, \dots, x_k, y_k \in V_k$

$$T(x_1, \dots, cx_j + y_j, \dots, x_k) = cT(x_1, \dots, x_j, \dots, x_k) + T(x_1, \dots, y_j, \dots, x_k)$$

for $j = 1, 2, \dots, k$. In other words, we assume T is linear in each of its k -slots. If T is multilinear on $V^r \times (V^*)^s$ then we say that $T \in T_r^s V$ and we say T is a **type (r, s) tensor on V** .

The definition above makes a dual vector a type $(1, 0)$ tensor whereas a double dual of a vector a type $(0, 1)$ tensor, a bilinear mapping on V is a type $(2, 0)$ tensor and a bilinear mapping on V^* is a type $(0, 2)$ tensor with respect to V .

We are free to define tensor products in this context in the same manner as we have previously. Suppose $\alpha_1 \in V_1^*, \alpha_2 \in V_2^*, \dots, \alpha_k \in V_k^*$ and $v_1 \in V_1, v_2 \in V_2, \dots, v_k \in V_k$ then

$$\alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_k(v_1, v_2, \dots, v_k) = \alpha_1(v_1)\alpha_2(v_2) \dots \alpha_k(v_k)$$

It is easy to show the tensor produce of k -dual vectors as defined above is indeed a k -multilinear mapping. Moreover, the set of all k -multilinear mappings on $V_1 \times V_2 \times \dots \times V_k$ clearly forms a

vector space of dimension $\dim(V_1)\dim(V_2)\cdots\dim(V_k)$ since it naturally takes the tensor product of the dual bases for $V_1^*, V_2^*, \dots, V_k^*$ as its basis. In particular, suppose for $j = 1, 2, \dots, k$ that V_j has basis $\{E_{ji}\}_{i=1}^{n_j}$ which is dual to $\{E_j^i\}_{i=1}^{n_j}$ the basis for V_j^* . Then we can derive that a k -multilinear mapping can be written as

$$T = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_k=1}^{n_k} T_{i_1 i_2 \dots i_k} E_1^{i_1} \otimes E_2^{i_2} \otimes \cdots \otimes E_k^{i_k}$$

If T is a type (r, s) tensor on V then there is no need for the ugly double indexing on the basis since we need only tensor a basis $\{e_i\}_{i=1}^n$ for V and its dual $\{e^i\}_{i=1}^n$ for V^* in what follows:

$$T = \sum_{i_1, \dots, i_r=1}^n \sum_{j_1, \dots, j_s=1}^n T_{i_1 i_2 \dots i_r}^{j_1 j_2 \dots j_s} e^{i_1} \otimes e^{i_2} \otimes \cdots \otimes e^{i_r} \otimes e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_s}.$$

permutations

Before I define symmetric and antisymmetric for k -linear mappings on V I think it is best to discuss briefly some ideas from the theory of permutations.

Definition 7.2.11.

A permutation on $\{1, 2, \dots, p\}$ is a bijection on $\{1, 2, \dots, p\}$. We define the set of permutations on $\{1, 2, \dots, p\}$ to be \mathcal{S}_p . Further, define the sign of a permutation to be $\text{sgn}(\sigma) = 1$ if σ is the product of an even number of transpositions whereas $\text{sgn}(\sigma) = -1$ if σ is the product of an odd number of transpositions.

Let us consider the set of permutations on $\{1, 2, 3, \dots, n\}$, this is called S_n the symmetric group, its order is $n!$ if you were wondering. Let me remind¹² you how the cycle notation works since it allows us to explicitly present the number of transpositions contained in a permutation,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{pmatrix} \iff \sigma = (12)(356) = (12)(36)(35) \quad (7.7)$$

recall the cycle notation is to be read right to left. If we think about inputting 5 we can read from the matrix notation that we ought to find $5 \mapsto 6$. Clearly that is the case for the first version of σ written in cycle notation; (356) indicates that $5 \mapsto 6$ and nothing else messes with 6 after that. Then consider feeding 5 into the version of σ written with just two-cycles (a.k.a. transpositions), first we note (35) indicates $5 \mapsto 3$, then that 3 hits (36) which means $3 \mapsto 6$, finally the cycle (12) doesn't care about 6 so we again have that $\sigma(5) = 6$. Finally we note that $\text{sgn}(\sigma) = -1$ since it is made of 3 transpositions.

It is always possible to write any permutation as a product of transpositions, such a decomposition is not unique. However, if the number of transpositions is even then it will remain so no matter

¹²or perhaps, more likely, introduce you to this notation

how we rewrite the permutation. Likewise if the permutation is an product of an odd number of transpositions then any other decomposition into transpositions is also comprised of an odd number of transpositions. This is why we can **define** an **even** permutation is a permutation comprised by an even number of transpositions and an **odd** permutation is one comprised of an odd number of transpositions.

Example 7.2.12. Sample cycle calculations: *we rewrite as product of transpositions to determine if the given permutation is even or odd,*

$$\sigma = (12)(134)(152) = (12)(14)(13)(12)(15) \implies \text{sgn}(\sigma) = -1$$

$$\lambda = (1243)(3521) = (13)(14)(12)(31)(32)(35) \implies \text{sgn}(\lambda) = 1$$

$$\gamma = (123)(45678) = (13)(12)(48)(47)(46)(45) \implies \text{sgn}(\gamma) = 1$$

We will not actually write down permutations in the calculations the follow this part of the notes. I merely include this material as to give a logically complete account of antisymmetry. In practice, if you understood the terms as they apply to the bilinear and trilinear case it will usually suffice for concrete examples. Now we are ready to define symmetric and antisymmetric.

Definition 7.2.13.

A k -linear mapping $L : V \times V \times \cdots \times V \rightarrow \mathbb{R}$ is **completely symmetric** if

$$L(x_1, \dots, x, \dots, y, \dots, x_k) = L(x_1, \dots, y, \dots, x, \dots, x_k)$$

for all possible $x, y \in V$. Conversely, if a k -linear mapping on V has

$$L(x_1, \dots, x, \dots, y, \dots, x_p) = -L(x_1, \dots, y, \dots, x, \dots, x_p)$$

for all possible pairs $x, y \in V$ then it is said to be **completely antisymmetric or alternating**. Equivalently a k -linear mapping L is alternating if for all $\pi \in \Sigma_k$

$$L(x_{\pi_1}, x_{\pi_2}, \dots, x_{\pi_k}) = \text{sgn}(\pi)L(x_1, x_2, \dots, x_k)$$

The set of alternating multilinear mappings on V is denoted ΛV , the set of k -linear alternating maps on V is denoted $\Lambda^k V$. Often an alternating k -linear map is called a **k -form**. Moreover, we say the **degree** of a k -form is k .

Similar terminology applies to the components of tensors. We say $T_{i_1 i_2 \dots i_k}$ is completely symmetric in i_1, i_2, \dots, i_k iff $T_{i_1 i_2 \dots i_k} = T_{i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(k)}}$ for all $\sigma \in \Sigma_k$. On the other hand, $T_{i_1 i_2 \dots i_k}$ is completely antisymmetric in i_1, i_2, \dots, i_k iff $T_{i_1 i_2 \dots i_k} = \text{sgn}(\sigma) T_{i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(k)}}$ for all $\sigma \in \Sigma_k$. It is a simple exercise to show that a completely (anti)symmetric tensor¹³ has completely (anti)symmetric components.

¹³in this context a tensor is simply a multilinear mapping, in physics there is more attached to the term

The tensor product is an interesting construction to discuss at length. To summarize, it is associative and distributive across addition. Scalars factor out and it is not generally commutative. For a given vector space V we can in principle generate by tensor products multilinear mappings of arbitrarily high order. This tensor algebra is infinite dimensional. In contrast, the space ΛV of forms on V is a finite-dimensional subspace of the tensor algebra. We discuss this next.

7.3 wedge product

We assume V is a vector space with basis $\{e_i\}_{i=1}^n$ throughout this section. The dual basis is denoted $\{e^i\}_{i=1}^n$ as is our usual custom. Our goal is to find a basis for the alternating maps on V and explore the structure implicit within its construction. This will lead us to call ΛV the **exterior algebra** of V after the discussion below is complete.

7.3.1 wedge product of dual basis generates basis for ΛV

Suppose $b : V \times V \rightarrow \mathbb{R}$ is antisymmetric and $b = \sum_{i,j=1}^n b_{ij} e^i \otimes e^j$, it follows that $b_{ij} = -b_{ji}$ for all $i, j \in \mathbb{N}_n$. Notice this implies that $b_{ii} = 0$ for $i = 1, 2, \dots, n$. For a given pair of indices i, j either $i < j$ or $j < i$ or $i = j$ hence,

$$\begin{aligned}
 b &= \sum_{i < j} b_{ij} e^i \otimes e^j + \sum_{j < i} b_{ij} e^i \otimes e^j + \sum_{i=j} b_{ij} e^i \otimes e^j \\
 &= \sum_{i < j} b_{ij} e^i \otimes e^j + \sum_{j < i} b_{ij} e^i \otimes e^j \\
 &= \sum_{i < j} b_{ij} e^i \otimes e^j - \sum_{j < i} b_{ji} e^i \otimes e^j \\
 &= \sum_{k < l} b_{kl} e^k \otimes e^l - \sum_{k < l} b_{kl} e^l \otimes e^k \\
 &= \sum_{k < l} b_{kl} (e^k \otimes e^l - e^l \otimes e^k). \tag{7.8}
 \end{aligned}$$

Therefore, $\{e^k \otimes e^l - e^l \otimes e^k | l, k \in \mathbb{N}_n \text{ and } l < k\}$ spans the set of antisymmetric bilinear maps on V . Moreover, you can show this set is linearly independent hence it is a basis for $\Lambda^2 V$. We define the wedge product of $e^k \wedge e^l = e^k \otimes e^l - e^l \otimes e^k$. With this notation we find that the alternating bilinear form b can be written as

$$b = \sum_{k < l} b_{kl} e^k \wedge e^l = \sum_{i,j=1}^n \frac{1}{2} b_{ij} e^i \wedge e^j$$

where the summation on the r.h.s. is over all indices¹⁴. Notice that $e^i \wedge e^j$ is an antisymmetric bilinear mapping because $e^i \wedge e^j(x, y) = -e^i \wedge e^j(y, x)$, however, there is more structure here than

¹⁴yes there is something to work out here, probably in your homework

just that. It is also true that $e^i \wedge e^j = -e^j \wedge e^i$. This is a conceptually different antisymmetry, it is the antisymmetry of the wedge product \wedge .

Suppose $b : V \times V \times V \rightarrow \mathbb{R}$ is antisymmetric and $b = \sum_{i,j,k=1}^n b_{ijk} e^i \otimes e^j \otimes e^k$, it follows that $b_{ijk} = b_{jki} = b_{kij}$ and $b_{ijk} = -b_{kji} = -b_{jik} = b_{ikj}$ for all $i, j, k \in \mathbb{N}_n$. Notice this implies that $b_{iii} = 0$ for $i = 1, 2, \dots, n$. A calculation similar to the one just offered for the case of a bilinear map reveals that we can write b as follows:

$$b = \sum_{i < j < k} b_{ijk} \left(e^i \otimes e^j \otimes e^k + e^j \otimes e^k \otimes e^i + e^k \otimes e^i \otimes e^j - e^k \otimes e^j \otimes e^i - e^j \otimes e^i \otimes e^k - e^i \otimes e^k \otimes e^j \right) \quad (7.9)$$

Define $e^i \wedge e^j \wedge e^k = e^i \otimes e^j \otimes e^k + e^j \otimes e^k \otimes e^i + e^k \otimes e^i \otimes e^j - e^k \otimes e^j \otimes e^i - e^j \otimes e^i \otimes e^k - e^i \otimes e^k \otimes e^j$ thus

$$b = \sum_{i < j < k} b_{ijk} e^i \wedge e^j \wedge e^k = \sum_{i,j,k=1}^n \frac{1}{3!} b_{ijk} e^i \wedge e^j \wedge e^k \quad (7.10)$$

and it is clear that $\{e^i \wedge e^j \wedge e^k \mid i, j, k \in \mathbb{N}_n \text{ and } i < j < k\}$ forms a basis for the set of alternating trilinear maps on V .

Following the patterns above, we define the wedge product of p dual basis vectors,

$$e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p} = \sum_{\pi \in \Sigma_p} \text{sgn}(\pi) e^{i_{\pi(1)}} \otimes e^{i_{\pi(2)}} \otimes \dots \otimes e^{i_{\pi(p)}} \quad (7.11)$$

If $x, y \in V$ we would like to show that

$$e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p}(\dots, x, \dots, y, \dots) = -e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p}(\dots, y, \dots, x, \dots) \quad (7.12)$$

follows from the complete antisymmetrization in the definition of the wedge product. Before we give the general argument, let's see how this works in the trilinear case. Consider, $e^i \wedge e^j \wedge e^k =$

$$= e^i \otimes e^j \otimes e^k + e^j \otimes e^k \otimes e^i + e^k \otimes e^i \otimes e^j - e^k \otimes e^j \otimes e^i - e^j \otimes e^i \otimes e^k - e^i \otimes e^k \otimes e^j.$$

Calculate, noting that $e^i \otimes e^j \otimes e^k(x, y, z) = e^i(x)e^j(y)e^k(z) = x^i y^j z^k$ hence

$$e^i \wedge e^j \wedge e^k(x, y, z) = x^i y^j z^k + x^j y^k z^i + x^k y^i z^j - x^k y^j z^i - x^j y^i z^k - x^i y^k z^j$$

Thus,

$$e^i \wedge e^j \wedge e^k(x, z, y) = x^i z^j y^k + x^j z^k y^i + x^k z^i y^j - x^k z^j y^i - x^j z^i y^k - x^i z^k y^j$$

and you can check that $e^i \wedge e^j \wedge e^k(x, y, z) = -e^i \wedge e^j \wedge e^k(x, z, y)$. Similar tedious calculations prove antisymmetry of the interchange of the first and second or the first and third slots. Therefore, $e^i \wedge e^j \wedge e^k$ is an alternating trilinear map as it is clearly trilinear since it is built from the sum of

tensor products which we know are likewise trilinear.

The multilinear case follows essentially the same argument, note

$$e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}(\dots, x_j, \dots, x_k, \dots) = \sum_{\pi \in \Sigma_p} \text{sgn}(\pi) x_1^{i_{\pi(1)}} \cdots x_j^{i_{\pi(j)}} \cdots x_k^{i_{\pi(k)}} \cdots x_p^{i_{\pi(p)}} \quad (7.13)$$

whereas,

$$e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}(\dots, x_k, \dots, x_j, \dots) = \sum_{\sigma \in \Sigma_p} \text{sgn}(\sigma) x_1^{i_{\sigma(1)}} \cdots x_k^{i_{\sigma(k)}} \cdots x_j^{i_{\sigma(j)}} \cdots x_p^{i_{\sigma(p)}}. \quad (7.14)$$

Suppose we take each permutation σ and substitute $\delta \in \Sigma_p$ such that $\sigma(j) = \delta(k)$ and $\sigma(k) = \delta(j)$ and otherwise δ and σ agree. In cycle notation, $\delta(jk) = \sigma$. Substitution δ into Equation 7.14:

$$\begin{aligned} e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}(\dots, x_k, \dots, x_j, \dots) &= \sum_{\delta \in \Sigma_p} \text{sgn}(\delta(jk)) x_1^{i_{\delta(1)}} \cdots x_k^{i_{\delta(j)}} \cdots x_j^{i_{\delta(k)}} \cdots x_p^{i_{\delta(p)}} \\ &= - \sum_{\delta \in \Sigma_p} \text{sgn}(\delta) x_1^{i_{\delta(1)}} \cdots x_j^{i_{\delta(k)}} \cdots x_k^{i_{\delta(j)}} \cdots x_p^{i_{\delta(p)}} \\ &= -e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}(\dots, x_j, \dots, x_k, \dots) \end{aligned} \quad (7.15)$$

Here the sgn of a permutation σ is $(-1)^N$ where N is the number of cycles in σ . We observed that $\delta(jk)$ has one more cycle than δ hence $\text{sgn}(\delta(jk)) = -\text{sgn}(\delta)$. Therefore, we have shown that $e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} \in \Lambda^p V$.

Recall that $e^i \wedge e^j = -e^j \wedge e^i$ in the $p = 2$ case. There is a generalization of that result to the $p > 2$ case. In words, the wedge product is antisymmetric with respect the interchange of any two dual vectors. For $p = 3$ we have the following identities for the wedge product:

$$e^i \wedge e^j \wedge e^k = - \underbrace{e^j \wedge e^i}_{\text{swapped}} \wedge e^k = e^j \wedge \underbrace{e^k \wedge e^i}_{\text{swapped}} = - \underbrace{e^k \wedge e^j}_{\text{swapped}} \wedge e^i = e^k \wedge \underbrace{e^i \wedge e^j}_{\text{swapped}} = - \underbrace{e^i \wedge e^k}_{\text{swapped}} \wedge e^j$$

I've indicated how these signs are consistent with the $p = 2$ antisymmetry. Any permutation of the dual vectors can be thought of as a combination of several transpositions. In any event, it is sometimes useful to just know that the wedge product of three elements is invariant under **cyclic** permutations of the dual vectors,

$$e^i \wedge e^j \wedge e^k = e^j \wedge e^k \wedge e^i = e^k \wedge e^i \wedge e^j$$

and changes by a sign for **anticyclic** permutations of the given object,

$$e^i \wedge e^j \wedge e^k = -e^j \wedge e^i \wedge e^k = -e^k \wedge e^j \wedge e^i = -e^i \wedge e^k \wedge e^j$$

Generally we can argue that, for any permutation $\pi \in \Sigma_p$:

$$\boxed{e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} = \text{sgn}(\pi) e^{i_{\pi(1)}} \wedge e^{i_{\pi(2)}} \wedge \cdots \wedge e^{i_{\pi(p)}}}$$

This is just a slick formula which says the wedge product generates a minus whenever you flip two dual vectors which are wedged.

7.3.2 the exterior algebra

The careful reader will realize we have yet to define wedge products of anything except for the dual basis. But, naturally you must wonder if we can take the wedge product of other dual vectors or morer generally alternating tensors. The answer is yes. Let us define the general wedge product:

Definition 7.3.1. Suppose $\alpha \in \Lambda^p V$ and $\beta \in \Lambda^q V$. We define \mathcal{I}_p to be the set of all increasing lists of p -indices, this set can be empty if $\dim(V)$ is not sufficiently large. Moreover, if $I = (i_1, i_2, \dots, i_p)$ then introduce notation $e^I = e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p}$ hence:

$$\alpha = \sum_{i_1, i_2, \dots, i_p=1}^n \frac{1}{p!} \alpha_{i_1 i_2 \dots i_p} e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p} = \sum_I \frac{1}{p!} \alpha_I e^I = \sum_{I \in \mathcal{I}_p} \alpha_I e^I$$

and

$$\beta = \sum_{j_1, j_2, \dots, j_q=1}^n \frac{1}{q!} \beta_{j_1 j_2 \dots j_q} e^{j_1} \wedge e^{j_2} \wedge \dots \wedge e^{j_q} = \sum_J \frac{1}{q!} \beta_J e^J = \sum_{J \in \mathcal{I}_q} \beta_J e^J$$

Naturally, $e^I \wedge e^J = e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p} \wedge e^{j_1} \wedge e^{j_2} \wedge \dots \wedge e^{j_q}$ and we defined this carefully in the preceding subsection. Define $\alpha \wedge \beta \in \Lambda^{p+q} V$ as follows:

$$\alpha \wedge \beta = \sum_I \sum_J \frac{1}{p!q!} \alpha_I \beta_J e^I \wedge e^J.$$

Again, but with less slick notation:

$$\alpha \wedge \beta = \sum_{i_1, i_2, \dots, i_p=1}^n \sum_{j_1, j_2, \dots, j_q=1}^n \frac{1}{p!q!} \alpha_{i_1 i_2 \dots i_p} \beta_{j_1 j_2 \dots j_q} e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p} \wedge e^{j_1} \wedge e^{j_2} \wedge \dots \wedge e^{j_q}$$

All the definition above really says is that we extend the wedge product on the basis to distribute over the addition of dual vectors. What this means calculationally is that the wedge product obeys the usual laws of addition and scalar multiplication. The one feature that is perhaps foreign is the antisymmetry of the wedge product. We must take care to maintain the order of expressions since the wedge product is not generally commutative.

Proposition 7.3.2.

Let α, β, γ be forms on V and $c \in \mathbb{R}$ then

- | | | |
|-------|---|------------------------------------|
| (i) | $(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma$ | distributes across vector addition |
| (ii) | $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$ | distributes across vector addition |
| (iii) | $(c\alpha) \wedge \beta = \alpha \wedge (c\beta) = c(\alpha \wedge \beta)$ | scalars factor out |
| (iv) | $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$ | associativity |

I leave the proof of this proposition to the reader.

Proposition 7.3.3. *graded commutivity of homogeneous forms.*

Let α, β be forms on V of degree p and q respectively then

$$\alpha \wedge \beta = -(-1)^{pq} \beta \wedge \alpha$$

Proof: suppose $\alpha = \sum_I \frac{1}{p!} e^I$ is a p -form on V and $\beta = \sum_J \frac{1}{q!} e^J$ is a q -form on V . Calculate:

$$\begin{aligned} \alpha \wedge \beta &= \sum_I \sum_J \frac{1}{p!q!} \alpha_I \beta_J e^I \wedge e^J && \text{by defn. of } \wedge, \\ &= \sum_I \sum_J \frac{1}{p!q!} \beta_J \alpha_I e^I \wedge e^J && \text{coefficients are scalars,} \\ &= (-1)^{pq} \sum_I \sum_J \frac{1}{p!q!} \beta_J \alpha_I e^J \wedge e^I && \text{(details on sign given below)} \\ &= (-1)^{pq} \beta \wedge \alpha \end{aligned}$$

Let's expand in detail why $e^J \wedge e^I = (-1)^{pq} e^I \wedge e^J$. Suppose $I = (i_1, i_2, \dots, i_p)$ and $J = (j_1, j_2, \dots, j_q)$, the key is that every swap of dual vectors generates a sign:

$$\begin{aligned} e^I \wedge e^J &= e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p} \wedge e^{j_1} \wedge e^{j_2} \wedge \dots \wedge e^{j_q} \\ &= (-1)^q e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_{p-1}} \wedge e^{j_1} \wedge e^{j_2} \wedge \dots \wedge e^{j_q} \wedge e^{i_p} \\ &= (-1)^q (-1)^q e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_{p-2}} \wedge e^{j_1} \wedge e^{j_2} \wedge \dots \wedge e^{j_q} \wedge e^{i_{p-1}} \wedge e^{i_p} \\ &\quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\ &= \underbrace{(-1)^q (-1)^q \dots (-1)^q}_{p\text{-factors}} e^{j_1} \wedge e^{j_2} \wedge \dots \wedge e^{j_q} \wedge e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p} \\ &= (-1)^{pq} e^J \wedge e^I. \end{aligned} \quad \square$$

Example 7.3.4. Let α be a 2-form defined by

$$\alpha = ae^1 \wedge e^2 + be^2 \wedge e^3$$

And let β be a 1-form defined by

$$\beta = 3e^1$$

Consider then,

$$\begin{aligned} \alpha \wedge \beta &= (ae^1 \wedge e^2 + be^2 \wedge e^3) \wedge (3e^1) \\ &= (3ae^1 \wedge e^2 \wedge e^1 + 3be^2 \wedge e^3 \wedge e^1) \\ &= 3be^1 \wedge e^2 \wedge e^3. \end{aligned} \tag{7.16}$$

whereas,

$$\begin{aligned} \beta \wedge \alpha &= 3e^1 \wedge (ae^1 \wedge e^2 + be^2 \wedge e^3) \\ &= (3ae^1 \wedge e^1 \wedge e^2 + 3be^1 \wedge e^2 \wedge e^3) \\ &= 3be^1 \wedge e^2 \wedge e^3. \end{aligned} \tag{7.17}$$

so this agrees with the proposition, $(-1)^{pq} = (-1)^2 = 1$ so we should have found that $\alpha \wedge \beta = \beta \wedge \alpha$. This illustrates that although the wedge product is antisymmetric on the basis, it is not always antisymmetric, in particular it is commutative for even forms.

The graded commutivity rule $\alpha \wedge \beta = -(-1)^{pq}\beta \wedge \alpha$ has some suprising implications. This rule is ultimately the reason ΛV is finite dimensional. Let's see how that happens.

Proposition 7.3.5. *linear dependent one-forms wedge to zero:*

If $\alpha, \beta \in V^*$ and $\alpha = c\beta$ for some $c \in \mathbb{R}$ then $\alpha \wedge \beta = 0$.

Proof: to begin, note that $\beta \wedge \beta = -\beta \wedge \beta$ hence $2\beta \wedge \beta = 0$ and it follows that $\beta \wedge \beta = 0$. Note:

$$\alpha \wedge \beta = c\beta \wedge \beta = c(0) = 0$$

therefore the proposition is true. \square

Proposition 7.3.6.

Suppose that $\alpha_1, \alpha_2, \dots, \alpha_p$ are linearly dependent 1-forms then

$$\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_p = 0.$$

Proof: by assumption of linear dependence there exist constants c_1, c_2, \dots, c_p not all zero such that

$$c_1\alpha_1 + c_2\alpha_2 + \dots + c_p\alpha_p = 0.$$

Suppose that c_k is a nonzero constant in the sum above, then we may divide by it and consequently we can write α_k in terms of all the other 1-forms,

$$\alpha_k = \frac{-1}{c_k} \left(c_1\alpha_1 + \dots + c_{k-1}\alpha_{k-1} + c_{k+1}\alpha_{k+1} + \dots + c_p\alpha_p \right)$$

Insert this sum into the wedge product in question,

$$\begin{aligned} \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_p &= \alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_k \wedge \dots \wedge \alpha_p \\ &= (-c_1/c_k)\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_1 \wedge \dots \wedge \alpha_p \\ &\quad + (-c_2/c_k)\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_2 \wedge \dots \wedge \alpha_p + \dots \\ &\quad + (-c_{k-1}/c_k)\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_{k-1} \wedge \dots \wedge \alpha_p \\ &\quad + (-c_{k+1}/c_k)\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_{k+1} \wedge \dots \wedge \alpha_p + \dots \\ &\quad + (-c_p/c_k)\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_p \wedge \dots \wedge \alpha_p \\ &= 0. \end{aligned} \tag{7.18}$$

We know all the wedge products are zero in the above because in each there is at least one 1-form repeated, we simply permute the wedge products till they are adjacent and by the previous proposition the term vanishes. The proposition follows. \square

Let us pause to reflect on the meaning of the proposition above for a n -dimensional vector space V . The dual space V^* is likewise n -dimensional, this is a general result which applies to all finite-dimensional vector spaces¹⁵. Thus, any set of more than n dual vectors is necessarily linearly dependent. Consequently, using the proposition above, we find the wedge product of more than n one-forms is trivial. Therefore, while it is possible to construct $\Lambda^k V$ for $k > n$ we should understand that this space only contains zero. The highest degree of a nontrivial form over a vector space of dimension n is an n -form.

Moreover, we can use the proposition to deduce the dimension of a basis for $\Lambda^p V$, it must consist of the wedge product of distinct linearly independent one-forms. The number of ways to choose p distinct objects from a list of n distinct objects is precisely "n choose p",

$$\binom{n}{p} = \frac{n!}{(n-p)!p!} \quad \text{for } 0 \leq p \leq n. \quad (7.19)$$

Proposition 7.3.7.

If V is an n -dimensional vector space then $\Lambda^k V$ is an $\binom{n}{k}$ -dimensional vector space of k -forms. Moreover, the direct sum of all forms over V has the structure

$$\Lambda V = \mathbb{R} \oplus \Lambda^1 V \oplus \cdots \oplus \Lambda^{n-1} V \oplus \Lambda^n V$$

and is a vector space of dimension 2^n

Proof: define $\Lambda^0 V = \mathbb{R}$ then it is clear $\Lambda^k V$ forms a vector space for $k = 0, 1, \dots, n$. Moreover, $\Lambda^j V \cap \Lambda^k V = \{0\}$ for $j \neq k$ hence the term "direct sum" is appropriate. It remains to show $\dim(\Lambda V) = 2^n$ where $\dim(V) = n$. A natural basis β for ΛV is found from taking the union of the bases for each subspace of k -forms,

$$\beta = \{1, e^{i_1}, e^{i_1} \wedge e^{i_2}, \dots, e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_n} \mid 1 \leq i_1 < i_2 < \cdots < i_n \leq n\}$$

But, we can count the number of vectors N in the set above as follows:

$$N = 1 + n + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

Recall the binomial theorem states

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k = a^n + na^{n-1}b + \cdots + nab^{n-1} + b^n.$$

Recognize that $N = (1+1)^n$ and the proposition follows. \square

¹⁵however, in infinite dimensions, the story is not so simple

We should note that in the basis above the space of n -forms is one-dimensional because there is only one way to choose a strictly increasing list of n integers in \mathbb{N}_n . In particular, it is useful to note $\Lambda^n V = \text{span}\{e^1 \wedge e^2 \wedge \cdots \wedge e^n\}$. The form $e^1 \wedge e^2 \wedge \cdots \wedge e^n$ is sometimes called the *top-form*¹⁶.

Example 7.3.8. exterior algebra of \mathbb{R}^2 *Let us begin with the standard dual basis $\{e^1, e^2\}$. By definition we take the $p = 0$ case to be the field itself; $\Lambda^0 V \equiv \mathbb{R}$, it has basis 1. Next, $\Lambda^1 V = \text{span}(e^1, e^2) = V^*$ and $\Lambda^2 V = \text{span}(e^1 \wedge e^2)$ is all we can do here. This makes ΛV a $2^2 = 4$ -dimensional vector space with basis*

$$\{1, e^1, e^2, e^1 \wedge e^2\}.$$

Example 7.3.9. exterior algebra of \mathbb{R}^3 *Let us begin with the standard dual basis $\{e^1, e^2, e^3\}$. By definition we take the $p = 0$ case to be the field itself; $\Lambda^0 V \equiv \mathbb{R}$, it has basis 1. Next, $\Lambda^1 V = \text{span}(e^1, e^2, e^3) = V^*$. Now for something a little more interesting,*

$$\Lambda^2 V = \text{span}(e^1 \wedge e^2, e^1 \wedge e^3, e^2 \wedge e^3)$$

and finally,

$$\Lambda^3 V = \text{span}(e^1 \wedge e^2 \wedge e^3).$$

This makes ΛV a $2^3 = 8$ -dimensional vector space with basis

$$\{1, e^1, e^2, e^3, e^1 \wedge e^2, e^1 \wedge e^3, e^2 \wedge e^3, e^1 \wedge e^2 \wedge e^3\}$$

it is curious that the number of independent one-forms and 2-forms are equal.

Example 7.3.10. exterior algebra of \mathbb{R}^4 *Let us begin with the standard dual basis $\{e^1, e^2, e^3, e^4\}$. By definition we take the $p = 0$ case to be the field itself; $\Lambda^0 V \equiv \mathbb{R}$, it has basis 1. Next, $\Lambda^1 V = \text{span}(e^1, e^2, e^3, e^4) = V^*$. Now for something a little more interesting,*

$$\Lambda^2 V = \text{span}(e^1 \wedge e^2, e^1 \wedge e^3, e^1 \wedge e^4, e^2 \wedge e^3, e^2 \wedge e^4, e^3 \wedge e^4)$$

and three forms,

$$\Lambda^3 V = \text{span}(e^1 \wedge e^2 \wedge e^3, e^1 \wedge e^2 \wedge e^4, e^1 \wedge e^3 \wedge e^4, e^2 \wedge e^3 \wedge e^4).$$

and $\Lambda^4 V = \text{span}(e^1 \wedge e^2 \wedge e^3 \wedge e^4)$. Thus ΛV a $2^4 = 16$ -dimensional vector space. Note that, in contrast to \mathbb{R}^3 , we do not have the same number of independent one-forms and two-forms over \mathbb{R}^4 .

Let's explore how this algebra fits with calculations we already know about determinants.

Example 7.3.11. *Suppose $A = [A_1 | A_2]$. I propose the determinant of A is given by the top-form on \mathbb{R}^2 via the formula $\det(A) = (e^1 \wedge e^2)(A_1, A_2)$. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A_1 = (a, c)$ and*

¹⁶or *volume form* for reasons we will explain later, other authors begin the discussion of forms from the consideration of volume, see Chapter 4 in Bernard Schutz' *Geometrical methods of mathematical physics*

$A_2 = (b, d)$. Thus,

$$\begin{aligned} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= (e^1 \wedge e^2)(A_1, A_2) \\ &= (e^1 \otimes e^2 - e^2 \otimes e^1)((a, c), (b, d)) \\ &= e^1(a, c)e^2(b, d) - e^2(a, c)e^1(b, d) \\ &= ad - bc. \end{aligned}$$

I hope this is not surprising!

Example 7.3.12. Suppose $A = [A_1|A_2|A_3]$. I propose the determinant of A is given by the top-form on \mathbb{R}^3 via the formula $\det(A) = (e^1 \wedge e^2 \wedge e^3)(A_1, A_2, A_3)$. Let's see if we can find the expansion by cofactors. By the definition we have $e^1 \wedge e^2 \wedge e^3 =$

$$\begin{aligned} &= e^1 \otimes e^2 \otimes e^3 + e^2 \otimes e^3 \otimes e^1 + e^3 \otimes e^1 \otimes e^2 - e^3 \otimes e^2 \otimes e^1 - e^2 \otimes e^1 \otimes e^3 - e^1 \otimes e^3 \otimes e^2 \\ &= e^1 \otimes (e^2 \otimes e^3 - e^3 \otimes e^2) - e^2 \otimes (e^1 \otimes e^3 - e^3 \otimes e^1) + e^3 \otimes (e^1 \otimes e^2 - e^2 \otimes e^1) \\ &= e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) + e^3 \otimes (e^1 \wedge e^2). \end{aligned}$$

I submit to the reader that this is precisely the cofactor expansion formula with respect to the first column of A . Suppose $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $A_1 = (a, d, g)$, $A_2 = (b, e, h)$ and $A_3 = (c, f, i)$. Calculate,

$$\begin{aligned} \det(A) &= e^1(A_1)(e^2 \wedge e^3)(A_2, A_3) - e^2(A_1)(e^1 \wedge e^3)(A_2, A_3) + e^3(A_1)(e^1 \wedge e^2)(A_2, A_3) \\ &= a(e^2 \wedge e^3)(A_2, A_3) - d(e^1 \wedge e^3)(A_2, A_3) + g(e^1 \wedge e^2)(A_2, A_3) \\ &= a(ei - fh) - d(bi - ch) + g(bf - ce) \end{aligned}$$

which is precisely my claim.

7.3.3 connecting vectors and forms in \mathbb{R}^3

There are a couple ways to connect vectors and forms in \mathbb{R}^3 . Mainly we need the following maps:

Definition 7.3.13.

Given $v = \langle a, b, c \rangle \in \mathbb{R}^3$ we can construct a corresponding one-form $\omega_v = ae^1 + be^2 + ce^3$ or we can construct a corresponding two-form $\Phi_v = ae^2 \wedge e^3 + be^3 \wedge e^1 + ce^1 \wedge e^2$

Recall that $\dim(\Lambda^1 \mathbb{R}^3) = \dim(\Lambda^2 \mathbb{R}^3) = 3$ hence the space of vectors, one-forms, and also two-forms are isomorphic as vector spaces. It is not difficult to show that $\omega_{v_1 + cv_2} = \omega_{v_1} + c\omega_{v_2}$ and $\Phi_{v_1 + cv_2} = \Phi_{v_1} + c\Phi_{v_2}$ for all $v_1, v_2 \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Moreover, $\omega_v = 0$ iff $v = 0$ and $\Phi_v = 0$ iff $v = 0$ hence $\ker(\omega) = \{0\}$ and $\ker(\Phi) = \{0\}$ but this means that ω and Φ are injective and since

the dimensions of the domain and codomain are 3 and these are linear transformations¹⁷ it follows ω and Φ are isomorphisms.

It appears we have two ways to represent vectors with forms in \mathbb{R}^3 . We'll see why this is important as we study integration of forms. It turns out the two-forms go with surfaces whereas the one-forms attach to curves. This corresponds to the fact in calculus III we have two ways to integrate a vector-field, we can either calculate flux or work. Partly for this reason the mapping ω is called the **work-form correspondence** and Φ is called the **flux-form correspondence**. Integration has to wait a bit, for now we focus on algebra.

Example 7.3.14. Suppose $v = \langle 2, 0, 3 \rangle$ and $w = \langle 0, 1, 2 \rangle$ then $\omega_v = 2e^1 + 3e^3$ and $\omega_w = e^2 + 2e^3$. Calculate the wedge product,

$$\begin{aligned}
 \omega_v \wedge \omega_w &= (2e^1 + 3e^3) \wedge (e^2 + 2e^3) \\
 &= 2e^1 \wedge (e^2 + 2e^3) + 3e^3 \wedge (e^2 + 2e^3) \\
 &= 2e^1 \wedge e^2 + 4e^1 \wedge e^3 + 3e^3 \wedge e^2 + 6e^3 \wedge e^3 \\
 &= -3e^2 \wedge e^3 - 4e^3 \wedge e^1 + 2e^1 \wedge e^2 \\
 &= \Phi_{\langle -3, -4, 2 \rangle} \\
 &= \Phi_{v \times w}
 \end{aligned} \tag{7.20}$$

Coincidence? Nope.

Proposition 7.3.15.

Suppose $v, w \in \mathbb{R}^3$ then $\omega_v \wedge \omega_w = \Phi_{v \times w}$ where $v \times w$ denotes the cross-product which is defined by $v \times w = \sum_{i,j,k=1}^3 \epsilon_{ijk} v_i w_j e_k$.

Proof: Suppose $v = \sum_{i=1}^3 v_i e_i$ and $w = \sum_{j=1}^3 w_j e_j$ then $\omega_v = \sum_{i=1}^3 v_i e^i$ and $\omega_w = \sum_{j=1}^3 w_j e^j$. Calculate,

$$\omega_v \wedge \omega_w = \left(\sum_{i=1}^3 v_i e^i \right) \wedge \left(\sum_{j=1}^3 w_j e^j \right) = \sum_{i=1}^3 \sum_{j=1}^3 v_i w_j e^i \wedge e^j$$

In invite the reader to show $e^i \wedge e^j = \Phi(\sum_{k=1}^3 \epsilon_{ijk} e_k)$ where I'm using $\Phi_v = \Phi(v)$ to make the argument of the flux-form mapping easier to read, hence,

$$\omega_v \wedge \omega_w = \underbrace{\sum_{i=1}^3 \sum_{j=1}^3 v_i w_j \Phi\left(\sum_{k=1}^3 \epsilon_{ijk} e_k\right)}_{\text{linearity of } \Phi} = \Phi\left(\sum_{i,j,k=1}^3 v_i w_j \epsilon_{ijk} e_k\right) = \Phi(v \times w) = \Phi_{v \times w}$$

Of course, if you don't like my proof you could just work it out like the example that precedes this proposition. I gave the proof to show off the mappings a bit more. \square

¹⁷this is not generally true, note $f(x) = x^2$ has $f(x) = 0$ iff $x = 0$ and yet f is not injective. The linearity is key.

Is the wedge product just the cross-product generalized? Well, not really. I think they're quite different animals. The wedge product is an associative product which makes sense in any vector space. The cross-product only matches the wedge product after we interpret it through a pair of isomorphisms (ω and ϕ) which are special to \mathbb{R}^3 . However, there is debate, largely the question comes down to what you think makes the cross-product the cross-product. If you think it must pick a unique perpendicular direction to a pair of given directions then that is only going to work in \mathbb{R}^3 since even in \mathbb{R}^4 there is a whole plane of perpendicular vectors to a given pair. On the other hand, if you think the cross-product in \mathbb{R}^4 should be pick the unique perpendicular to a given triple of vectors then you could set something up. You could define $v \times w \times x = \omega^{-1}(\psi(\omega_v \wedge \omega_w \wedge \omega_x))$ where $\psi : \Lambda^3 \mathbb{R}^4 \rightarrow \Lambda^1 \mathbb{R}^4$ is an isomorphism we'll describe in a upcoming section. But, you see it's no longer a product of two vectors, it's not a binary operation, it's a tertiary operation. In any event, you can read a lot more on this if you wish. We have all the tools we need for this course. The wedge product provides the natural antisymmetric algebra for n -dimensiona and the work and flux-form maps naturally connect us to the special world of three-dimensional mathematics.

There is more algebra for forms on \mathbb{R}^3 however we defer it to a later section where we have a few more tools. Chief among those is the Hodge dual. But, before we can discuss Hodge duality we need to generalize our idea of a dot-product just a little.

7.4 bilinear forms and geometry; metric duality

The concept of a metric goes beyond the familar case of the dot-product. If you want a more strict generalization of the dot-product then you should think about an *inner-product*. In contrast to the definition below, the inner-product replaces non-degeneracy with the stricter condition of positive-definite which would read $g(x, x) > 0$ for $x \neq 0$. I included a discussion of inner products at the end of this section for the interested reader although we are probably not going to need all of that material.

7.4.1 metric geometry

A **geometry** is a vector space paired with a metric. For example, if we pair \mathbb{R}^n with the dot-product you get Euclidean space. However, if we pair \mathbb{R}^4 with the Minkowski metric then we obtain Minkowski space.

Definition 7.4.1.

If V is a vector space and $g : V \times V \rightarrow \mathbb{R}$ is

1. **bilinear:** $g \in T_2^0 V$,
2. **symmetric:** $g(x, y) = g(y, x)$ for all $x, y \in V$,
3. **nondegenerate:** $g(x, y) = 0$ for all $x \in V$ implies $y = 0$.

the we call g a **metric** on V .

If $V = \mathbb{R}^n$ then we can write $g(x, y) = x^T G y$ where $[g] = G$. Moreover, $g(x, y) = g(y, x)$ implies $G^T = G$. Nondegenerate means that $g(x, y) = 0$ for all $y \in \mathbb{R}^n$ iff $x = 0$. It follows that $G y = 0$ has no non-trivial solutions hence G^{-1} exists.

Example 7.4.2. Suppose $g(x, y) = x^T y$ for all $x, y \in \mathbb{R}^n$. This defines a metric for \mathbb{R}^n , it is just the dot-product. Note that $g(x, y) = x^T y = x^T I y$ hence we see $[g] = I$ where I denotes the identity matrix in $\mathbb{R}^{n \times n}$.

Example 7.4.3. Suppose $v = (v^0, v^1, v^2, v^3), w = (w^0, w^1, w^2, w^3) \in \mathbb{R}^4$ then define the **Minkowski product** of v and w as follows:

$$g(v, w) = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$

It is useful to write the Minkowski product in terms of a matrix multiplication. Observe that for $x, y \in \mathbb{R}^4$,

$$g(x, y) = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3 = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \end{pmatrix} \equiv x^t \eta y$$

where we have introduced η the matrix of the Minkowski product. Notice that $\eta^T = \eta$ and $\det(\eta) = -1 \neq 0$ hence $g(x, y) = x^t \eta y$ makes g a symmetric, nondegenerate bilinear form on \mathbb{R}^4 . The formula is clearly related to the dot-product. Suppose $\bar{v} = (v^0, \vec{v})$ and $\bar{w} = (w^0, \vec{w})$ then note

$$g(v, w) = -v^0 w^0 + \vec{v} \cdot \vec{w}$$

For vectors with zero in the zeroth slot this Minkowski product reduces to the dot-product. However, for vectors which have nonzero entries in both the zeroth and later slots much differs. Recall that any vector's dot-product with itself gives the square of the vectors length. Of course this means that $\vec{x} \cdot \vec{x} = 0$ iff $\vec{x} = 0$. Contrast that with the following: if $v = (1, 1, 0, 0)$ then

$$g(v, v) = -1 + 1 = 0$$

Yet $v \neq 0$. Why study such a strange generalization of length? The answer lies in physics. I'll give you a brief account by defining a few terms: Let $v = (v^0, v^1, v^2, v^3) \in \mathbb{R}^4$ then we say

1. v is a timelike vector if $\langle v, v \rangle < 0$
2. v is a lightlike vector if $\langle v, v \rangle = 0$
3. v is a spacelike vector if $\langle v, v \rangle > 0$

If we consider the trajectory of a massive particle in \mathbb{R}^4 that begins at the origin then at any later time the trajectory will be located at a timelike vector. If we consider a light beam emitted from the origin then at any future time it will be located at the tip of a lightlike vector. Finally, spacelike vectors point to points in \mathbb{R}^4 which cannot be reached by the motion of physical particles that pass through the origin. We say that massive particles are confined within their light cones, this means that they are always located at timelike vectors relative to their current position in space time. If you'd like to know more I can recommend a few books.

At this point you might wonder if there are other types of metrics beyond these two examples. Surprisingly, in a certain sense, no. A rather old theorem of linear algebra due to Sylvester states that we can change coordinates so that the metric more or less resembles either the dot-product or something like it with some sign-flips. We'll return to this in a later section.

7.4.2 metric duality for tensors

Throughout this section we consider a vector space V paired with a metric $g : V \times V \rightarrow \mathbb{R}$. Moreover, the vector space V has basis $\{e_i\}_{i=1}^n$ which has a g -dual basis $\{e^i\}_{i=1}^n$. Up to this point we always have used a g -dual basis where the duality was offered by the dot-product. In the context of Minkowski geometry that sort of duality is no longer natural. Instead we must follow the definition below:

Definition 7.4.4.

If V is a vector space with metric g and basis $\{e_i\}_{i=1}^n$ then we say the basis $\{e^i\}_{i=1}^n$ is g -dual iff

Suppose $e^i(e_j) = \delta_{ij}$ and consider $g = \sum_{i,j=1}^n g_{ij} e^i \otimes e^j$. Furthermore, suppose g^{ij} are the components of the inverse matrix to (g_{ij}) this means that $\sum_{k=1}^n g_{ik} g^{kj} = \delta_{ij}$. We use the components of the metric and its inverse to raise and lower indices on tensors. Here are the basic conventions: given an object A^j which has the *contravariant* index j we can lower it to be covariant by contracting against the metric components as follows:

$$A_i = \sum_j g_{ij} A^j$$

On the other hand, given an object B_j which has a *covariant* index j we can raise it to be contravariant by contracting against the inverse components of the metric:

$$B^i = \sum_j g^{ij} B_j$$

I like to think of this as some sort of conservation of indices. Strict adherence to the notation drives us to write things such as $\sum_{k=1}^n g_{ik} g^{kj} = \delta_i^j$ just to keep up the up/down index pattern. I should mention that these formulas are much more beautiful in the physics literature, you can

look at my old Math 430 notes from NCSU if you'd like a healthy dose of that notation¹⁸. I use Einstein's implicit summation notation throughout those notes and I discuss this index calculation more in the way a physicist typically approaches it. Here I am trying to be careful enough that these equations are useful to mathematicians. Let me show you some examples:

Example 7.4.5. *Specialize for this example to $V = \mathbb{R}^4$ with $g(x, y) = x^T \eta y$. Suppose $x = \sum_{\mu=0}^4 x^\mu e_\mu$ the components x^μ are called **contravariant components**. The metric allows us to define **covariant components** by*

$$x_\nu = \sum_{\mu=0}^4 \eta_{\nu\mu} x^\mu.$$

For the Minkowski metric this just adjoins a minus to the zeroth component: if $(x^\mu) = (a, b, c, d)$ then $x_\mu = (-a, b, c, d)$.

Example 7.4.6. *Suppose we are working on \mathbb{R}^n with the Euclidean metric $g_{ij} = \delta_{ij}$ and it follows that $g^{ij} = \delta_{ij}$ or to be a purist for a moment $\sum_k g_{ik} g^{kj} = \delta_i^j$. In this case $v^i = \sum_j g^{ij} v_j = \sum_j \delta_{ij} v_j = v_i$. The covariant and contravariant components are the same. This is why it was ok to ignore up/down indices when we work with a dot-product exclusively.*

What if we raise an index and then lower it back down once more? Do we really get back where we started? Given x^μ we lower the index by $x_\nu = \sum_\mu g_{\nu\mu} x^\mu$ then we raise it once more by

$$x^\alpha = \sum_\nu g^{\alpha\nu} x_\nu = \sum_\nu g^{\alpha\nu} \sum_\mu g_{\nu\mu} x^\mu = \sum_{\mu,\nu} g^{\alpha\nu} g_{\nu\mu} x^\mu = \sum_\mu \delta_\mu^\alpha x^\mu$$

and the last summation squishes down to x^α once more. It would seem this procedure of raising and lowering indices is at least consistent.

Example 7.4.7. *Suppose we raise the index on the basis $\{e_i\}$ and formally obtain $\{e^j = \sum_k g^{jk} e_k\}$ on the other hand suppose we lower the index on the dual basis $\{e^l\}$ to formally obtain $\{e_m = \sum_l g_{ml} e^l\}$. I'm curious, are these consistent? We should get $e^j(e_m) = \delta_m^j$, I'll be nice and look at $e_m(e^j)$ in the following sense:*

$$\sum_l g_{ml} e^l \left(\sum_k g^{jk} e_k \right) = \sum_{l,k} g_{ml} g^{jk} e^l(e_k) = \sum_{l,k} g_{ml} g^{jk} \delta_k^l = \sum_k g_{mk} g^{jk} = \delta_m^j$$

Interesting, but what does it mean?

I used the term *formal* in the preceding example to mean that the example makes sense in as much as you accept the equations which are written. If you think harder about it then you'll find it was rather meaningless. That said, this index notation is rather forgiving.

¹⁸just a taste: $v_\mu = \eta_{\mu\nu} v^\nu$ or $v^\mu = \eta^{\mu\nu} v_\nu$ or $v^\mu v_\mu = \eta^{\mu\nu} v_\nu v_\mu = \eta_{\mu\nu} v^\mu v^\nu$

Ok, but what are we doing? Recall that I insisted on using lower indices for forms and upper indices for vectors? The index conventions I'm toying with above are the reason for this strange notation. When we lower an index we might be changing a vector to a dual vector, or vice-versa when we raise an index we might be changing a dual vector into a vector. Let me be explicit.

1. given $v \in V$ we create $\alpha_v \in V^*$ by the rule $\alpha_v(x) = g(x, v)$.
2. given $\alpha \in V^*$ we create $v_\alpha \in V^{**}$ by the rule $v_\alpha(\beta) = g^{-1}(\alpha, \beta)$ where $g^{-1}(\alpha, \beta) = \sum_{ij} \alpha_i \beta_j g^{ij}$.

Recall we at times identify V and V^{**} . Let's work out the component structure of α_v and see how it relates to v ,

$$\alpha_v(e_i) = g(v, e_i) = g\left(\sum_j v^j e_j, e_i\right) = \sum_j v^j g(e_j, e_i) = \sum_j v^j g_{ji}$$

Thus, $\alpha_v = \sum_i v_i e^i$ where $v_i = \sum_j v^j g_{ji}$. When we lower the index we're actually using an isomorphism which is provided by the metric to map vectors to forms. The process of raising the index is just the inverse of this isomorphism.

$$v_\alpha(e^i) = g^{-1}(\alpha, e^i) = g^{-1}\left(\sum_j \alpha_j e^j, e^i\right) = \sum_j \alpha_j g^{ji}$$

thus $v_\alpha = \sum_i \alpha^i e_i$ where $\alpha^i = \sum_j \alpha_j g^{ji}$.

Suppose we want to change a type $(0, 2)$ tensor to a type $(2, 0)$ tensor. We're given $T : V^* \times V^*$ where $T = \sum_{ij} T^{ij} e_i \otimes e_j$. Define $\tilde{T} : V \times V \rightarrow \mathbb{R}$ as follows:

$$\tilde{T}(v, w) = T(\alpha_v, \alpha_w)$$

What does this look like in components? Note $\alpha_{e_i}(e_j) = g(e_i, e_j) = g_{ij}$ hence $\alpha_{e_i} = \sum_j g_{ij} e^j$ and

$$\tilde{T}(e_i, e_j) = T(\alpha_{e_i}, \alpha_{e_j}) = T\left(\sum_k g_{ik} e^k, \sum_l g_{jl} e^l\right) = \sum_{k,l} g_{ki} g_{lj} T(e^k, e^l) = \sum_{k,l} g_{ki} g_{lj} T^{kl}$$

Or, as is often customary, we could write $T_{ij} = \sum_{k,l} g_{ik} g_{jl} T^{kl}$. However, this is an abuse of notation since T_{ij} are not technically components for T . If we have a metric we can recover either T from \tilde{T} or vice-versa. Generally, if we are given two tensors, say T_1 of rank (r, s) and the T_2 of rank (r', s') , then these might be equivalent if $r + s = r' + s'$. It may be that through raising and lowering indices (a.k.a. appropriately composing with the vector \leftrightarrow dual vector isomorphisms) we can convert T_1 to T_2 . If you read *Gravitation* by Misner, Thorne and Wheeler you'll find many more thoughts on this equivalence. Challenge: can you find the explicit formulas like $\tilde{T}(v, w) = T(\alpha_v, \alpha_w)$ which back up the index calculations below?

$$T_{ij}{}^k = \sum_{a,b} g_{ia} g_{jb} T^{abk} \quad \text{or} \quad S^{ij} = \sum_{a,b} g^{ia} g^{jb} S_{ab}$$

I hope I've given you enough to chew on in this section to put these together.

7.4.3 inner products and induced norm

There are generalized dot-products on many abstract vector spaces, we call them **inner-products**.

Definition 7.4.8.

Suppose V is a vector space. If $\langle, \rangle: V \times V \rightarrow \mathbb{R}$ is a function such that for all $x, y, z \in V$ and $c \in \mathbb{R}$:

1. $\langle x, y \rangle = \langle y, x \rangle$ (symmetric)
2. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ (additive in the first slot)
3. $\langle cx, y \rangle = c \langle x, y \rangle$ (homogeneity in the first slot)
4. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$

then we say (V, \langle, \rangle) is an **inner-product space** with inner product \langle, \rangle .

Given an inner-product space (V, \langle, \rangle) we can easily induce a norm for V by the formula $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in V$. Properties (1.), (3.) and (4.) in the definition of the norm are fairly obvious for the induced norm. Let's think through the triangle inequality for the induced norm:

$$\begin{aligned}
 \|x + y\|^2 &= \langle x + y, x + y \rangle && \text{def. of induced norm} \\
 &= \langle x, x + y \rangle + \langle y, x + y \rangle && \text{additive prop. of inner prod.} \\
 &= \langle x + y, x \rangle + \langle x + y, y \rangle && \text{symmetric prop. of inner prod.} \\
 &= \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle && \text{additive prop. of inner prod.} \\
 &= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2
 \end{aligned}$$

At this point we're stuck. A nontrivial identity¹⁹ called the **Cauchy-Schwarz** identity helps us proceed; $\langle x, y \rangle \leq \|x\| \|y\|$. It follows that $\|x + y\|^2 \leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$. However, the induced norm is clearly positive²⁰ so we find $\|x + y\| \leq \|x\| + \|y\|$.

Most linear algebra texts have a whole chapter on inner-products and their applications, you can look at my notes for a start if you're curious. That said, this is a bit of a digression for this course.

¹⁹I prove this for the dot-product in my linear notes, however, the proof is written in such a way it equally well applies to a general inner-product

²⁰note: if you have $(-5)^2 < (-7)^2$ it does not follow that $-5 < -7$, in order to take the squareroot of the inequality we need positive terms squared

7.5 hodge duality

We can prove that $\binom{n}{p} = \binom{n}{n-p}$. This follows from explicit computation of the formula for $\binom{n}{p}$ or from the symmetry of Pascal's triangle if you prefer. In any event, this equality suggests there is some isomorphism between p and $(n-p)$ -forms. When we are given a metric g on a vector space V (and the notation of the preceding section) it is fairly simple to construct the isomorphism. Suppose we are given $\alpha \in \Lambda^p V$ and following our usual notation:

$$\alpha = \sum_{i_1, i_2, \dots, i_p=1}^n \frac{1}{p!} \alpha_{i_1 i_2 \dots i_p} e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_p}$$

Then, define $*\alpha$ the **hodge dual** to be the $(n-p)$ -form given below:

$$*\alpha = \sum_{i_1, i_2, \dots, i_n=1}^n \frac{1}{p!(n-p)!} \alpha_{i_1 i_2 \dots i_p} \epsilon_{i_1 i_2 \dots i_p j_1 j_2 \dots j_{n-p}} e^{j_1} \wedge e^{j_2} \wedge \dots \wedge e^{j_{n-p}}$$

I should admit, to prove this is a reasonable definition we'd need to do some work. It's clearly a linear transformation, but bijectivity and coordinate invariance of this definition might take a little work. I intend to omit those details and instead focus on how this works for \mathbb{R}^3 or \mathbb{R}^4 . My advisor taught a course on fiber bundles and there is a much more general and elegant presentation of the hodge dual over a manifold. Ask if interested, I think I have a pdf.

7.5.1 hodge duality in euclidean space \mathbb{R}^3

To begin, consider a scalar 1, this is a 0-form so we expect the hodge dual to give a 3-form:

$$*1 = \sum_{i,j,k} \frac{1}{0!3!} \epsilon_{ijk} e^i \wedge e^j \wedge e^k = e^1 \wedge e^2 \wedge e^3$$

Interesting, the hodge dual of 1 is the top-form on \mathbb{R}^3 . Conversely, calculate the dual of the top-form, note $e^1 \wedge e^2 \wedge e^3 = \sum_{ijk} \frac{1}{6} \epsilon_{ijk} e^i \wedge e^j \wedge e^k$ reveals the components of the top-form are precisely ϵ_{ijk} thus:

$$*(e^1 \wedge e^2 \wedge e^3) = \sum_{i,j,k=1}^3 \frac{1}{3!(3-3)!} \epsilon^{ijk} \epsilon_{ijk} = \frac{1}{6} (1 + 1 + 1 + (-1)^2 + (-1)^2 + (-1)^2) = 1.$$

Next, consider e^1 , note that $e^1 = \sum_k \delta_{k1} e^k$ hence the components are δ_{k1} . Thus,

$$*e^1 = \sum_{i,j,k} \frac{1}{1!2!} \epsilon_{ijk} \delta^{k1} e^i \wedge e^j = \frac{1}{2} \sum_{i,j} \epsilon_{ij1} e^i \wedge e^j = \frac{1}{2} (\epsilon_{231} e^2 \wedge e^3 + \epsilon_{321} e^3 \wedge e^2) = e^2 \wedge e^3$$

Similar calculations reveal $*e^2 = e^3 \wedge e^1$ and $*e^3 = e^1 \wedge e^2$. What about the duals of the two-forms? Begin with $\alpha = e^1 \wedge e^2$ note that $e^1 \wedge e^2 = e^1 \otimes e^2 - e^2 \otimes e^1$ thus we can see the components are

$\alpha_{ij} = \delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}$. Thus,

$$*(e^1 \wedge e^2) = \sum_{i,j,k} \frac{1}{2!1!} \epsilon_{ijk} (\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}) e^k = \frac{1}{2} \left(\sum_k \epsilon_{12k} e^k - \sum_k \epsilon_{21k} e^k \right) = \frac{1}{2} (e^3 - (-e^3)) = e^3.$$

Similar calculations show that $*(e^2 \wedge e^3) = e^1$ and $*(e^3 \wedge e^1) = e^2$. Put all of this together and we find that

$$*(ae^1 + be^2 + ce^3) = ae^2 \wedge e^3 + be^3 \wedge e^1 + ce^1 \wedge e^2$$

and

$$*(ae^2 \wedge e^3 + be^3 \wedge e^1 + ce^1 \wedge e^2) = ae^1 + be^2 + ce^3$$

Which means that $\boxed{*\omega_v = \Phi_v}$ and $\boxed{*\Phi_v = \omega_v}$. Hodge duality links the two different form-representations of vectors in a natural manner. Moreover, for \mathbb{R}^3 we should also note that $**\alpha = \alpha$ for all $\alpha \in \Lambda\mathbb{R}^3$. In general, for other metrics, we can have a change of signs which depends on the degree of α .

We can summarize hodge duality for three-dimensional Euclidean space as follows:

$*1 = e^1 \wedge e^2 \wedge e^3$	$*(e^1 \wedge e^2 \wedge e^3) = 1$
$*e^1 = e^2 \wedge e^3$	$*(e^2 \wedge e^3) = e^1$
$*e^2 = e^3 \wedge e^1$	$*(e^3 \wedge e^1) = e^2$
$*e^3 = e^1 \wedge e^2$	$*(e^1 \wedge e^2) = e^3$

A simple rule to calculate the hodge dual of a basis form is as follows

1. begin with the top-form $e^1 \wedge e^2 \wedge e^3$
2. permute the forms until the basis form you wish to hodge dual is to the left of the expression, whatever remains to the right is the hodge dual.

For example, to calculate the dual of $e^2 \wedge e^3$ note

$$e^1 \wedge e^2 \wedge e^3 = \underbrace{e^2 \wedge e^3}_{\text{to be dual}} \wedge \underbrace{e^1}_{\text{the dual}} \Rightarrow *(e^2 \wedge e^3) = e^1.$$

Consider what happens if we calculate $**\alpha$, since the dual is a linear operation it suffices to think about the basis forms. Let me sketch the process of $**e^I$ where I is a multi-index:

1. begin with $e^1 \wedge e^2 \wedge e^3$
2. write $e^1 \wedge e^2 \wedge e^3 = (-1)^N e^I \wedge e^J$ and identify $*e^I = (-1)^N e^J$.
3. then to calculate the second dual once more begin with $e^1 \wedge e^2 \wedge e^3$ and note

$$e^1 \wedge e^2 \wedge e^3 = (-1)^N e^J \wedge e^I$$

since the same N transpositions are required to push e^I to the left or e^J to the right.

4. It follows that $**e^I = e^I$ for any multi-index hence $**\alpha = \alpha$ for all $\alpha \in \Lambda\mathbb{R}^3$.

I hope that once you get past the index calculation you can see the hodge dual is not a terribly complicated construction. Some of the index calculation in this section was probably gratuitous, but I would like you to be aware of such techniques. Brute-force calculation has it's place, but a well-thought index notation can bring far more insight with much less effort.

7.5.2 hodge duality in minkowski space \mathbb{R}^4

The logic here follows fairly close to the last section, however the wrinkle is that the metric here demands more attention. We must take care to raise the indices on the forms when we Hodge dual them. First let's list the basis forms, we have to add time to the mix (again $c = 1$ so $x^0 = ct = t$ if you worried about it) Remember that the Greek indices are defined to range over 0, 1, 2, 3. Here

Name	Degree	Typical Element	Basis for $\Lambda^p\mathbb{R}^4$
function	$p = 0$	f	1
one-form	$p = 1$	$\alpha = \sum_{\mu} \alpha_{\mu} e^{\mu}$	e^0, e^1, e^2, e^3
two-form	$p = 2$	$\beta = \sum_{\mu, \nu} \frac{1}{2} \beta_{\mu\nu} e^{\mu} \wedge e^{\nu}$	$e^2 \wedge e^3, e^3 \wedge e^1, e^1 \wedge e^2$ $e^0 \wedge e^1, e^0 \wedge e^2, e^0 \wedge e^3$
three-form	$p = 3$	$\gamma = \sum_{\mu, \nu, \alpha} \frac{1}{3!} \gamma_{\mu\nu\alpha} e^{\mu} \wedge e^{\nu} \wedge e^{\alpha}$	$e^1 \wedge e^2 \wedge e^3, e^0 \wedge e^2 \wedge e^3$ $e^0 \wedge e^1 \wedge e^3, e^0 \wedge e^1 \wedge e^2$
four-form	$p = 4$	$ge^0 \wedge e^1 \wedge e^2 \wedge e^3$	$e^0 \wedge e^1 \wedge e^2 \wedge e^3$

the top form is degree four since in four dimensions we can have at most four dual-basis vectors without a repeat. Wedge products work the same as they have before, just now we have e^0 to play with. Hodge duality may offer some surprises though.

Definition 7.5.1. The antisymmetric symbol in flat \mathbb{R}^4 is denoted $\epsilon_{\mu\nu\alpha\beta}$ and it is defined by the value

$$\epsilon_{0123} = 1$$

plus the demand that it be completely antisymmetric.

We must not assume that this symbol is invariant under a cyclic exchange of indices. Consider,

$$\begin{aligned}
 \epsilon_{0123} &= -\epsilon_{1023} && \text{flipped (01)} \\
 &= +\epsilon_{1203} && \text{flipped (02)} \\
 &= -\epsilon_{1230} && \text{flipped (03)}.
 \end{aligned} \tag{7.21}$$

In four dimensions we'll use antisymmetry directly and forego the cyclicity shortcut. Its not a big deal if you notice it before it confuses you.

Example 7.5.2. Find the Hodge dual of $\gamma = e^1$ with respect to the Minkowski metric $\eta_{\mu\nu}$, to begin notice that dx has components $\gamma_{\mu} = \delta_{\mu}^1$ as is readily verified by the equation $e^1 = \sum_{\mu} \delta_{\mu}^1 e^{\mu}$. Lets

raise the index using η as we learned previously,

$$\gamma^\mu = \sum_\nu \eta^{\mu\nu} \gamma_\nu = \sum_\nu \eta^{\mu\nu} \delta_\nu^1 = \eta^{1\mu} = \delta^{1\mu}$$

Starting with the definition of Hodge duality we calculate

$$\begin{aligned} *(e^1) &= \sum_{\alpha,\beta,\mu,\nu} \frac{1}{p!} \frac{1}{(n-p)!} \gamma^\mu \epsilon_{\mu\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\ &= \sum_{\alpha,\beta,\mu,\nu} (1/6) \delta^{1\mu} \epsilon_{\mu\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\ &= \sum_{\alpha,\beta,\nu} (1/6) \epsilon_{1\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\ &= (1/6) [\epsilon_{1023} e^0 \wedge e^2 \wedge e^3 + \epsilon_{1230} e^2 \wedge e^3 \wedge e^0 + \epsilon_{1302} e^3 \wedge e^0 \wedge e^2 \\ &\quad + \epsilon_{1320} e^3 \wedge e^2 \wedge e^0 + \epsilon_{1203} e^2 \wedge e^0 \wedge e^3 + \epsilon_{1032} e^0 \wedge e^3 \wedge e^2] \\ &= (1/6) [-e^0 \wedge e^2 \wedge e^3 - e^2 \wedge e^3 \wedge e^0 - e^3 \wedge e^0 \wedge e^2 \\ &\quad + e^3 \wedge e^2 \wedge e^0 + e^2 \wedge e^0 \wedge e^3 + e^0 \wedge e^3 \wedge e^2] \\ &= -e^2 \wedge e^3 \wedge e^0 = -e^0 \wedge e^2 \wedge e^3. \end{aligned} \tag{7.22}$$

the difference between the three and four dimensional Hodge dual arises from two sources, for one we are using the Minkowski metric so indices up or down makes a difference, and second the antisymmetric symbol has more possibilities than before because the Greek indices take four values.

I suspect we can calculate the hodge dual by the following pattern: suppose we wish to find the dual of α where α is a basis form for $\Lambda\mathbb{R}^4$ with the Minkowski metric

1. begin with the top-form $e^0 \wedge e^1 \wedge e^2 \wedge e^3$
2. permute factors as needed to place α to the left,
3. the form which remains to the right will be the hodge dual of α if no e^0 is in α otherwise the form to the right multiplied by -1 is $*\alpha$.

Note this works for the previous example as follows:

1. begin with $e^0 \wedge e^1 \wedge e^2 \wedge e^3$
2. note $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = -e^1 \wedge e^0 \wedge e^2 \wedge e^3 = e^1 \wedge (-e^0 \wedge e^2 \wedge e^3)$
3. identify $*e^1 = -e^0 \wedge e^2 \wedge e^3$ (no extra sign since no e^0 appears in e^1)

Follow the algorithm for finding the dual of e^0 ,

1. begin with $e^0 \wedge e^1 \wedge e^2 \wedge e^3$
2. note $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = e^0 \wedge (e^1 \wedge e^2 \wedge e^3)$

3. identify $*e^0 = -e^1 \wedge e^2 \wedge e^3$ (added sign since e^0 appears in form being hodge dual)

Let's check from the definition if my algorithm worked out right.

Example 7.5.3. Find the Hodge dual of $\gamma = e^0$ with respect to the Minkowski metric $\eta_{\mu\nu}$, to begin notice that e^0 has components $\gamma_\mu = \delta_\mu^0$ as is readily verified by the equation $e^0 = \sum_\mu \delta_\mu^0 e^\mu$. Lets raise the index using η as we learned previously,

$$\gamma^\mu = \sum_\nu \eta^{\mu\nu} \gamma_\nu = \sum_\nu \eta^{\mu\nu} \delta_\nu^0 = \eta^{\mu 0} = -\delta^{0\mu}$$

the minus sign is due to the Minkowski metric. Starting with the definition of Hodge duality we calculate

$$\begin{aligned} *(e^0) &= \sum_{\alpha,\beta,\mu,\nu} \frac{1}{p!} \frac{1}{(n-p)!} \gamma^\mu \epsilon_{\mu\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\ &= \sum_{\alpha,\beta,\mu,\nu} -(1/6) \delta^{0\mu} \epsilon_{\mu\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\ &= \sum_{\alpha,\beta,\nu} -(1/6) \epsilon_{0\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\ &= \sum_{i,j,k} -(1/6) \epsilon_{0ijk} e^i \wedge e^j \wedge e^k \\ &= \sum_{i,j,k} -(1/6) \epsilon_{ijk} \epsilon_{ijk} e^1 \wedge e^2 \wedge e^3 \quad \leftarrow \text{sneaky step} \\ &= -e^1 \wedge e^2 \wedge e^3. \end{aligned} \tag{7.23}$$

Notice I am using the convention that Greek indices sum over 0,1,2,3 whereas Latin indices sum over 1,2,3.

Example 7.5.4. Find the Hodge dual of $\gamma = e^0 \wedge e^1$ with respect to the Minkowski metric $\eta_{\mu\nu}$, to begin notice the following identity, it will help us find the components of γ

$$e^0 \wedge e^1 = \sum_{\mu,\nu} \frac{1}{2} 2 \delta_\mu^0 \delta_\nu^1 e^\mu \wedge e^\nu$$

now we antisymmetrize to get the components of the form,

$$e^0 \wedge e^1 = \sum_{\mu,\nu} \frac{1}{2} \delta_{[\mu}^0 \delta_{\nu]}^1 dx^\mu \wedge dx^\nu$$

where $\delta_{[\mu}^0 \delta_{\nu]}^1 = \delta_\mu^0 \delta_\nu^1 - \delta_\nu^0 \delta_\mu^1$ and the factor of two is used up in the antisymmetrization. Lets raise the index using η as we learned previously,

$$\gamma^{\alpha\beta} = \sum_{\mu,\nu} \eta^{\alpha\mu} \eta^{\beta\nu} \gamma_{\mu\nu} = \sum_{\mu,\nu} \eta^{\alpha\mu} \eta^{\beta\nu} \delta_{[\mu}^0 \delta_{\nu]}^1 = -\eta^{\alpha 0} \eta^{\beta 1} + \eta^{\beta 0} \eta^{\alpha 1} = -\delta^{[\alpha 0} \delta^{\beta] 1}$$

the minus sign is due to the Minkowski metric. Starting with the definition of Hodge duality we

calculate

$$\begin{aligned}
*(e^0 \wedge e^1) &= \frac{1}{p!} \frac{1}{(n-p)!} \gamma^{\alpha\beta} \epsilon_{\alpha\beta\mu\nu} e^\mu \wedge e^\nu \\
&= (1/4)(-\delta^{[\alpha 0} \delta^{\beta] 1}) \epsilon_{\alpha\beta\mu\nu} e^\mu \wedge e^\nu \\
&= -(1/4)(\epsilon_{01\mu\nu} e^\mu \wedge e^\nu - \epsilon_{10\mu\nu} e^\mu \wedge e^\nu) \\
&= -(1/2) \epsilon_{01\mu\nu} e^\mu \wedge e^\nu \\
&= -(1/2)[\epsilon_{0123} e^2 \wedge e^3 + \epsilon_{0132} e^3 \wedge e^2] \\
&= -e^2 \wedge e^3
\end{aligned} \tag{7.24}$$

Note, the algorithm works out the same,

$$e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \underbrace{e^0 \wedge e^1}_{\text{has } e^0} \wedge (e^2 \wedge e^3) \Rightarrow *(e^0 \wedge e^1) = -e^2 \wedge e^3$$

The other Hodge duals of the basic two-forms calculate by almost the same calculation. Let us make a table of all the basic Hodge dualities in Minkowski space, I have grouped the terms to emphasize

$*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$	$*(e^0 \wedge e^1 \wedge e^2 \wedge e^3) = -1$
$*(e^1 \wedge e^2 \wedge e^3) = -e^0$	$*e^0 = -e^1 \wedge e^2 \wedge e^3$
$*(e^0 \wedge e^2 \wedge e^3) = -e^1$	$*e^1 = -e^2 \wedge e^3 \wedge e^0$
$*(e^0 \wedge e^3 \wedge e^1) = -e^2$	$*e^2 = -e^3 \wedge e^1 \wedge e^0$
$*(e^0 \wedge e^1 \wedge e^2) = -e^3$	$*e^3 = -e^1 \wedge e^2 \wedge e^0$
$*(e^3 \wedge e^0) = e^1 \wedge e^2$	$*(e^1 \wedge e^2) = -e^3 \wedge e^0$
$*(e^1 \wedge e^0) = e^2 \wedge e^3$	$*(e^2 \wedge e^3) = -e^1 \wedge e^0$
$*(e^2 \wedge e^0) = e^3 \wedge e^1$	$*(e^3 \wedge e^1) = -e^2 \wedge e^0$

the isomorphisms between the one-dimensional $\Lambda^0 \mathbb{R}^4$ and $\Lambda^4 \mathbb{R}^4$, the four-dimensional $\Lambda^1 \mathbb{R}^4$ and $\Lambda^3 \mathbb{R}^4$, the six-dimensional $\Lambda^2 \mathbb{R}^4$ and itself. Notice that the dimension of $\Lambda \mathbb{R}^4$ is 16 which we have explained in depth in the previous section. Finally, it is useful to point out the three-dimensional work and flux form mappings to provide some useful identities in this 1 + 3-dimensional setting.

$$*\omega_{\vec{v}} = -e^0 \wedge \Phi_{\vec{v}} \quad * \Phi_{\vec{v}} = e^0 \wedge \omega_{\vec{v}} \quad *(e^0 \wedge \Phi_{\vec{v}}) = \omega_{\vec{v}}$$

I leave verification of these formulas to the reader (use the table). Finally let us analyze the process of taking two hodge duals in succession. In the context of \mathbb{R}^3 we found that $**\alpha = \alpha$, we seek to discern if a similar formula is available in the context of \mathbb{R}^4 with the minkowski metric. We can calculate one type of example with the identities above:

$$*\omega_{\vec{v}} = -e^0 \wedge \Phi_{\vec{v}} \Rightarrow **\omega_{\vec{v}} = -*(e^0 \wedge \Phi_{\vec{v}}) = -\omega_{\vec{v}} \Rightarrow **\omega_{\vec{v}} = -\omega_{\vec{v}}$$

Perhaps this is true in general?

If we accept my algorithm then it's not too hard to sort through using multi-index notation: since hodge duality is linear it suffices to consider a basis element e^I where I is a multi-index,

1. transpose dual vectors so that $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = (-1)^N e^I \wedge e^J$
2. if $0 \notin I$ then $*e^I = (-1)^N e^J$ and $0 \in J$ since $I \cup J = \{0, 1, 2, 3\}$. Take a second dual by writing $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = (-1)^N e^J \wedge e^I$ but note $*((-1)^N e^J) = -e^I$ since $0 \in J$. We find $**e^I = -e^I$ for all I not containing the 0-index.
3. if $0 \in I$ then $*e^I = -(-1)^N e^J$ and $0 \notin J$ since $I \cup J = \{0, 1, 2, 3\}$. Take a second dual by writing $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = -(-1)^N e^J \wedge (-e^I)$ and hence $*(-(-1)^N e^J) = -e^I$ since $0 \notin J$. We find $**e^I = -e^I$ for all I containing the 0-index.
4. it follows that $\boxed{**\alpha = -\alpha}$ for all $\alpha \in \Lambda\mathbb{R}^4$ with the minkowski metric.

To conclude, I would warn the reader that the results in this section pertain to our choice of notation for \mathbb{R}^4 . Some other texts use a metric which is $-\eta$ relative to our notation. This modifies many signs in this section. See Misner, Thorne and Wheeler's *Gravitation* or Bertlmann's *Anomalies in Field Theory* for future reading on Hodge duality and a more systematic explanation of how and when these signs arise from the metric.

7.6 coordinate change

Suppose V has two bases $\bar{\beta} = \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n\}$ and $\beta = \{f_1, f_2, \dots, f_n\}$. If $v \in V$ then we can write v in as a linear combination of the $\bar{\beta}$ basis or the β basis:

$$v = x^1 f_1 + x^2 f_2 + \dots + x^n f_n \quad \text{and} \quad v = \bar{x}^1 \bar{f}_1 + \bar{x}^2 \bar{f}_2 + \dots + \bar{x}^n \bar{f}_n$$

given the notation above, we define **coordinate maps** as follows:

$$\Phi_\beta(v) = (x^1, x^2, \dots, x^n) = x \quad \text{and} \quad \Phi_{\bar{\beta}}(v) = (\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n) = \bar{x}$$

We sometimes use the notation $\Phi_\beta(v) = [v]_\beta = x$ whereas $\Phi_{\bar{\beta}}(v) = [v]_{\bar{\beta}} = \bar{x}$. A coordinate map takes an abstract vector v and maps it to a particular representative in \mathbb{R}^n . A natural question to ask is how do different representatives compare? How do x and \bar{x} compare in our current notation? Because the coordinate maps are isomorphisms it follows that $\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isomorphism and given the domain and codomain we can write its formula via matrix multiplication:

$$\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}(u) = Pu \quad \Rightarrow \quad \Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}(\bar{x}) = P\bar{x}$$

However, $\Phi_{\bar{\beta}}^{-1}(\bar{x}) = v$ hence $\Phi_\beta(v) = P\bar{x}$ and consequently, $\boxed{x = P\bar{x}}$. Conversely, to switch to barred coordinates we multiply the coordinate vectors by P^{-1} ; $\boxed{\bar{x} = P^{-1}x}$.

Continuing this discussion we turn to the dual space. Suppose $\bar{\beta}^* = \{\bar{f}^j\}_{j=1}^n$ is dual to $\bar{\beta} = \{\bar{f}_j\}_{j=1}^n$ and $\beta^* = \{f^j\}_{j=1}^n$ is dual to $\beta = \{f_j\}_{j=1}^n$. By definition we are given that $f^j(f_i) = \delta_{ij}$ and $\bar{f}^j(\bar{f}_i) = \delta_{ij}$ for all $i, j \in \mathbb{N}_n$. Suppose $\alpha \in V^*$ is a dual vector with components α_j with respect to the β^* basis and components $\bar{\alpha}_j$ with respect to the $\bar{\beta}^*$ basis. In particular this means we can either write $\alpha = \sum_{j=1}^n \alpha_j f^j$ or $\alpha = \sum_{j=1}^n \bar{\alpha}_j \bar{f}^j$. Likewise, given a vector $v \in V$ we can either write $v = \sum_{i=1}^n x^i f_i$ or $v = \sum_{i=1}^n \bar{x}^i \bar{f}_i$. With these notations in mind calculate:

$$\alpha(v) = \left(\sum_{j=1}^n \alpha_j f^j \right) \left(\sum_{i=1}^n x^i f_i \right) = \sum_{i,j=1}^n \alpha_j x^i f^j(f_i) = \sum_{i=1}^n \sum_{j=1}^n \alpha_j x^i \delta_{ij} = \sum_{i=1}^n \alpha_i x^i$$

and by the same calculation in the barred coordinates we find, $\alpha(v) = \sum_{i=1}^n \bar{\alpha}_i \bar{x}^i$. Therefore,

$$\sum_{i=1}^n \alpha_i x^i = \sum_{i=1}^n \bar{\alpha}_i \bar{x}^i.$$

Recall, $x = P\bar{x}$. In components, $x^i = \sum_{k=1}^n P_k^i \bar{x}^k$. Substituting,

$$\sum_{i=1}^n \sum_{k=1}^n \alpha_i P_k^i \bar{x}^k = \sum_{i=1}^n \bar{\alpha}_i \bar{x}^i.$$

But, this formula holds for all possible vectors v and hence all possible coordinate vectors \bar{x} . If we consider $v = \bar{f}_j$ then $\bar{x}^i = \delta_{ij}$ hence $\sum_{i=1}^n \bar{\alpha}_i \bar{x}^i = \sum_{i=1}^n \bar{\alpha}_i \delta_{ij} = \bar{\alpha}_j$. Moreover, $\sum_{i=1}^n \sum_{k=1}^n \alpha_i P_k^i \bar{x}^k = \sum_{i=1}^n \sum_{k=1}^n \alpha_i P_k^i \delta_{kj} = \sum_{i=1}^n \alpha_i P_j^i$. Thus, $\bar{\alpha}_j = \sum_{i=1}^n P_j^i \alpha_i$. Compare how vectors and dual vectors transform:

$$\boxed{\bar{\alpha}_j = \sum_{i=1}^n P_j^i \alpha_i \quad \text{verses} \quad \bar{x}^j = \sum_{i=1}^n (P^{-1})_i^j x^i.}$$

It is customary to use lower-indices on the components of dual-vectors and upper-indices on the components of vectors: we say $x = \sum_{i=1}^n x^i e_i \in \mathbb{R}^n$ has **contravariant components** whereas $\alpha = \sum_{j=1}^n \alpha_j e^j \in (\mathbb{R}^n)^*$ has **covariant components**. These terms arise from the coordinate change properties we derived in this section. The convenience of the up/down index notation will be more apparent as we continue our study to more complicated objects. It is interesting to note the basis elements transform inversely:

$$\boxed{\bar{f}^j = \sum_{i=1}^n (P^{-1})_i^j f^i \quad \text{verses} \quad \bar{f}_j = \sum_{i=1}^n P_j^i f_i.}$$

The formulas above can be derived by arguments similar to those we already gave in this section,

however I think it may be more instructive to see how these rules work in concert:

$$\begin{aligned}
 x &= \sum_{i=1}^n \bar{x}^i \bar{f}_i = \sum_{i=1}^n \sum_{j=1}^n (P^{-1})_j^i x^j \bar{f}_i \\
 &= \sum_{i=1}^n \sum_{j=1}^n (P^{-1})_j^i x^j \sum_{k=1}^n P_i^k f_k \\
 &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n (P^{-1})_j^i P_i^k x^j f_k \\
 &= \sum_{j=1}^n \sum_{k=1}^n \delta_j^k x^j f_k \\
 &= \sum_{k=1}^n x^k f_k.
 \end{aligned} \tag{7.25}$$

7.6.1 coordinate change for $T_2^0(V)$

For an abstract vector space, or for \mathbb{R}^n with a nonstandard basis, we have to replace v, w with their coordinate vectors. If V has basis $\bar{\beta} = \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n\}$ with dual basis $\beta^* = \{\bar{f}^1, \bar{f}^2, \dots, \bar{f}^n\}$ and v, w have coordinate vectors \bar{x}, \bar{y} (which means $v = \sum_{i=1}^n \bar{x}^i \bar{f}_i$ and $w = \sum_{i=1}^n \bar{y}^i \bar{f}_i$) then,

$$b(v, w) = \sum_{i,j=1}^n \bar{x}^i \bar{y}^j \bar{B}_{ij} = \bar{x}^T \bar{B} \bar{y}$$

where $\bar{B}_{ij} = b(\bar{f}_i, \bar{f}_j)$. If $\beta = \{f_1, f_2, \dots, f_n\}$ is another basis on V with dual basis β^* then we define $B_{ij} = b(f_i, f_j)$ and we have

$$b(v, w) = \sum_{i,j=1}^n x^i y^j B_{ij} = x^T B y.$$

Recall that $\bar{f}_i = \sum_{k=1}^n P_i^k f_k$. With this in mind calculate:

$$\bar{B}_{ij} = b(\bar{f}_i, \bar{f}_j) = b\left(\sum_{k=1}^n P_i^k f_k, \sum_{l=1}^n P_j^l f_l\right) = \sum_{k,l=1}^n P_i^k P_j^l b(f_k, f_l) = \sum_{k,l=1}^n P_i^k P_j^l B_{kl}$$

We find the components of a bilinear map transform as follows:

$$\boxed{\bar{B}_{ij} = \sum_{k,l=1}^n P_i^k P_j^l B_{kl}}$$

XXX- include general coordinate change and metrics with sylvester's theorem.

Chapter 8

manifold theory

In this chapter I intend to give you a fairly accurate account of the modern definition of a manifold¹. In a nutshell, a manifold is simply a set which allows for calculus locally. Alternatively, many people say that a manifold is simply a set which is locally "flat", or it locally "looks like \mathbb{R}^n ". This covers most of the objects you've seen in calculus III. However, the technical details most closely resemble the parametric view-point.

¹the definitions we follow are primarily taken from Burns and Gidea's *Differential Geometry and Topology With a View to Dynamical Systems*, I like their notation, but you should understand this definition is known to many authors

8.1 manifolds

Definition 8.1.1.

We define a **smooth manifold of dimension m** as follows: suppose we are given a set M , a collection of open subsets U_i of \mathbb{R}^m , and a collection of mappings $\phi_i : U_i \subseteq \mathbb{R}^m \rightarrow V_i \subseteq M$ which satisfies the following three criteria:

1. each map $\phi_i : U_i \rightarrow V_i$ is injective
2. if $V_i \cap V_j \neq \emptyset$ then there exists a smooth mapping

$$\theta_{ij} : \phi_j^{-1}(V_i \cap V_j) \rightarrow \phi_i^{-1}(V_i \cap V_j)$$

such that $\phi_j = \phi_i \circ \theta_{ij}$

3. $M = \cup_i \phi_i(U_i)$

Moreover, we call the mappings ϕ_i the **local parametrizations** or **patches** of M and the space U_i is called the **parameter space**. The range V_i together with the inverse ϕ_i^{-1} is called a **coordinate chart** on M . The component functions of a chart (V, ϕ^{-1}) are usually denoted $\phi_i^{-1} = (x^1, x^2, \dots, x^m)$ where $x^j : V \rightarrow \mathbb{R}$ for each $j = 1, 2, \dots, m$.

We could add to this definition that i is taken from an *index set* \mathcal{I} (which could be an infinite set). The union given in criteria (3.) is called a **covering of M** . Most often, we deal with finitely covered manifolds. You may recall that there are infinitely many ways to parametrize the lines or surfaces we dealt with in calculus III. The story here is no different. It follows that when we consider classification of manifolds the definition we just offered is a bit lacking. We would also like to lump in all other possible compatible parametrizations. In short, the definition we gave says a manifold is a set together with an **atlas of compatible charts**. If we take that atlas and adjoin to it all possible compatible charts then we obtain the so-called **maximal atlas** which defines a **differentiable structure** on the set M . Many other authors define a manifold as a set together with a differentiable structure. That said, our less ambitious definition will do.

We should also note that $\theta_{ij} = \phi_i^{-1} \circ \phi_j$ hence $\theta_{ij}^{-1} = (\phi_i^{-1} \circ \phi_j)^{-1} = \phi_j^{-1} \circ \phi_i = \theta_{ji}$. The functions θ_{ij} are called the **transition functions** of \mathcal{M} . These explain how we change coordinates locally.

I now offer a few examples so you can appreciate how general this definition is, in contrast to the level-set definition we explored previously. We will recover those as examples of this more general definition later in this chapter.

Example 8.1.2. Let $M = \mathbb{R}^m$ and suppose $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the identity mapping ($\phi(u) = u$ for all $u \in \mathbb{R}^m$) defines the collection of parametrizations on M . In this case the collection is just one mapping and $U = V = \mathbb{R}^m$, clearly ϕ is injective and V covers \mathbb{R}^m . The remaining overlap criteria is trivially satisfied since there is only one patch to consider.

Example 8.1.3. Let $U = \mathbb{R}^m$ and suppose $p_o \in \mathbb{R}^p$ then $\phi : U \rightarrow \mathbb{R}^p \times \mathbb{R}^m$ defined by $\phi(u) = p_o \times u$ makes $\phi(U) = M$ an m -dimensional manifold. Again we have no overlap and the covering criteria is clearly satisfied so that leaves injectivity of ϕ . Note $\phi(u) = \phi(u')$ implies $p_o \times u = p_o \times u'$ hence $u = u'$.

Example 8.1.4. Suppose V is an m -dimensional vector space over \mathbb{R} with basis $\beta = \{e_i\}_{i=1}^n$. Define $\phi : \mathbb{R}^m \rightarrow V$ as follows, for each $u = (u^1, u^2, \dots, u^m) \in \mathbb{R}^m$

$$\phi(u) = u^1 e_1 + u^2 e_2 + \dots + u^m e_m.$$

Injectivity of the map follows from the linear independence of β . The overlap criteria is trivially satisfied. Moreover, $\text{span}(\beta) = V$ thus we know that $\phi(\mathbb{R}^m) = V$ which means the vector space is covered. All together we find V is an m -dimensional manifold. Notice that the inverse of ϕ of the coordinate mapping Φ_β from our earlier work and so we find the coordinate chart is a coordinate mapping in the context of a vector space. Of course, this is a very special case since most manifolds are not spanned by a basis.

You might notice that there seems to be little contact with criteria two in the examples above. These are rather special cases in truth. When we deal with curved manifolds we cannot avoid it any longer. I should mention we can (and often do) consider other coordinate systems on \mathbb{R}^m . Moreover, in the context of a vector space we also have infinitely many coordinate systems to use. We will have to analyze compatibility of those new coordinates as we adjoin them. For the vector space it's simple to see the transition maps are smooth since they'll just be invertible linear mappings. On the other hand, it is more work to show new curvilinear coordinates on \mathbb{R}^m are compatible with Cartesian coordinates. The inverse function theorem would likely be needed.

Example 8.1.5. Let $M = \{(\cos(\theta), \sin(\theta)) \mid \theta \in [0, 2\pi)\}$. Define $\phi_1(u) = (\cos(u), \sin(u))$ for all $u \in (0, 3\pi/2) = U_1$. Also, define $\phi_2(v) = (\cos(v), \sin(v))$ for all $v \in (\pi, 2\pi) = U_2$. Injectivity follows from the basic properties of sine and cosine and covering follows from the obvious geometry of these mappings. However, overlap we should check. Let $V_1 = \phi_1(U_1)$ and $V_2 = \phi_2(U_2)$. Note $V_1 \cap V_2 = \{(\cos(\theta), \sin(\theta)) \mid \pi < \theta < 3\pi/2\}$. We need to find the formula for

$$\theta_{12} : \phi_2^{-1}(V_1 \cap V_2) \rightarrow \phi_1^{-1}(V_1 \cap V_2)$$

In this example, this means

$$\theta_{12} : (\pi, 3\pi/2) \rightarrow (\pi, 3\pi/2)$$

Example 8.1.6. Let's return to the vector space example. This time we want to allow for all possible coordinate systems. Once more suppose V is an m -dimensional vector space over \mathbb{R} . Note that for each basis $\beta = \{e_i\}_{i=1}^n$. Define $\phi_\beta : \mathbb{R}^m \rightarrow V$ as follows, for each $u = (u^1, u^2, \dots, u^m) \in \mathbb{R}^m$

$$\phi_\beta(u) = u^1 e_1 + u^2 e_2 + \dots + u^m e_m.$$

Suppose β, β' are bases for V which define local parametrizations $\phi_\beta, \phi_{\beta'}$ respectively. The transition functions $\theta : \mathbb{R}^m \rightarrow \mathbb{R}^m$ are given by

$$\theta = \phi_\beta^{-1} \circ \phi_{\beta'}$$

Note θ is the composition of linear mappings and is therefore a linear mapping on \mathbb{R}^m . It follows that $\theta(x) = Px$ for some $M \in GL(m) = \{X \in \mathbb{R}^{m \times m} \mid \det(X) \neq 0\}$. It follows that θ is a smooth mapping since each component function of θ is simply a linear combination of the variables in \mathbb{R}^m . Let's take a moment to connect with linear algebra notation. If $\theta = \phi_\beta^{-1} \circ \phi_{\beta'}$ then $\theta \circ \phi_{\beta'}^{-1} = \phi_\beta^{-1}$ hence $\theta \circ \Phi_{\beta'} = \Phi_\beta$ as we used $\Phi_\beta : V \rightarrow \mathbb{R}^m$ as the coordinate chart in linear algebra and $\phi_\beta^{-1} = \Phi_\beta$. Thus, $\theta \circ \Phi_{\beta'}(v) = \Phi_\beta(v)$ implies $P[v]_{\beta'} = [v]_\beta$. This matrix P is the coordinate change matrix from linear algebra.

The contrast of Examples 8.1.3 and 8.1.6 stems in the allowed coordinate systems. In Example 8.1.3 we had just one coordinate system whereas in Example 8.1.6 we allowed infinitely many. We could construct other manifolds over the set V . We could take all coordinate systems that are of a particular type. If $V = \mathbb{R}^m$ then it is often interesting to consider only those coordinate systems for which the Pythagorean theorem holds true, such coordinates have transition functions in the group of orthogonal transformations. Or, if $V = \mathbb{R}^4$ then we might want to consider only inertially related coordinates. Inertially related coordinates on \mathbb{R}^4 preserve the *interval* defined by the Minkowski product and the transition functions form a group of Lorentz transformations. Orthogonal matrices and Lorentz matrices are simply the matrices of the aptly named transformations. In my opinion this is one nice feature of saving the maximal atlas concept for the differentiable structure. Manifolds as we have defined them give us a natural mathematical context to restrict the choice of coordinates. From the viewpoint of physics, the maximal atlas contains many coordinate systems which are unnatural for physics. Of course, it is possible to take a given theory of physics and translate physically natural equations into less natural equations in non-standard coordinates. For example, look up how Newton's simple equation $\vec{F} = m\vec{a}$ is translated into rotating coordinate systems.

8.1.1 embedded manifolds

The manifolds in Examples 8.1.2, 8.1.3 and 8.1.5 were all defined as subsets of euclidean space. Generally, if a manifold is a subset of Euclidean space \mathbb{R}^n then we say the manifold is **embedded in \mathbb{R}^n** . In contrast, Examples 8.1.4 and 8.1.6 are called **abstract manifolds** since the points in the manifold were not found in Euclidean space². If you are only interested in embedded manifolds³ then the definition is less abstract:

Definition 8.1.7. *embedded manifold.*

We say \mathcal{M} is a **smooth embedded manifold of dimension m** iff we are given a set $\mathcal{M} \subseteq \mathbb{R}^n$ such that at each $p \in \mathcal{M}$ there is a set an open subsets U_i of \mathbb{R}^m and open subsets V_i of \mathcal{M} containing p such that the mapping $\phi_i : U_i \subseteq \mathbb{R}^m \rightarrow V_i \subseteq \mathcal{M}$ satisfies the following criteria:

1. each map $\phi_i : U_i \rightarrow V_i$ is injective
2. each map $\phi_i : U_i \rightarrow V_i$ is smooth
3. each map $\phi_i^{-1} : V_i \rightarrow U_i$ is continuous
4. the differential $d\phi_x$ has rank m for each $x \in U_i$.

You may identify that this definition more closely resembles the parametrized objects from your multivariate calculus course. There are two key differences with this definition:

1. the set V_i is assumed to be "open in \mathcal{M} " where $\mathcal{M} \subseteq \mathbb{R}^n$. This means that for each point $p \in V_i$ there exists an open n -ball $B \subset \mathbb{R}^n$ such that $B \cap \mathcal{M}$ contains p . This is called the **subspace topology** for \mathcal{M} induced from the euclidean topology of \mathbb{R}^n . No topological assumptions were given for V_i in the abstract definition. In practice, for the abstract case, we use the charts to lift open sets to \mathcal{M} , we need not assume any topology on \mathcal{M} since the machinery of the manifold allows us to build our own. However, this can lead to some pathological cases so those cases are usually ruled out by stating that our manifold is *Hausdorff* and the covering has a countable basis of open sets⁴. I will leave it at that since this is not a topology course.
2. the condition that the inverse of the local parametrization be continuous and ϕ_i be smooth were not present in the abstract definition. Instead, we assumed smoothness of the transition functions.

One can prove that the embedded manifold of Definition 8.1.7 is simply a subcase of the abstract manifold given by Definition 8.1.1. See Munkres Theorem 24.1 where he shows the transition

²a vector space could be euclidean space, but it could also be a set of polynomials, operators or a lot of other rather abstract objects.

³The definition I gave for embedded manifold here is mostly borrowed from Munkres' excellent text *Analysis on Manifolds* where he primarily analyzes embedded manifolds

⁴see Burns and Gidea page 11 in *Differential Geometry and Topology With a View to Dynamical Systems*

functions of an embedded manifold are smooth. In fact, his theorem is given for the case of a manifold with boundary which adds a few complications to the discussion. We'll discuss manifolds with boundary at the conclusion of this chapter.

Example 8.1.8. *A line is a one dimensional manifold with a global coordinate patch:*

$$\phi(t) = \vec{r}_o + t\vec{v}$$

for all $t \in \mathbb{R}$. We can think of this as the mapping which takes the real line and glues it in \mathbb{R}^n along some line which points in the direction \vec{v} and the new origin is at \vec{r}_o . In this case $\phi: \mathbb{R} \rightarrow \mathbb{R}^n$ and $d\phi_t$ has matrix \vec{v} which has rank one iff $\vec{v} \neq 0$.

Example 8.1.9. *A plane is a two dimensional manifold with a global coordinate patch: suppose \vec{A}, \vec{B} are any two linearly independent vectors in the plane, and \vec{r}_o is a particular point in the plane,*

$$\phi(u, v) = \vec{r}_o + u\vec{A} + v\vec{B}$$

for all $(u, v) \in \mathbb{R}^2$. This amounts to pasting a copy of the xy -plane in \mathbb{R}^n where we moved the origin to \vec{r}_o . If we just wanted a little parallelogram then we could restrict $(u, v) \in [0, 1] \times [0, 1]$, then we would envision that the unit-square has been pasted on to a parallelogram. Lengths and angles need not be maintained in this process of gluing. Note that the rank two condition for $d\phi$ says the derivative $\phi'(u, v) = [\frac{\partial \phi}{\partial u} | \frac{\partial \phi}{\partial v}] = [\vec{A} | \vec{B}]$ must have rank two. But, this amounts to insisting the vectors \vec{A}, \vec{B} are linearly independent. In the case of \mathbb{R}^3 this is conveniently tested by computation of $\vec{A} \times \vec{B}$ which happens to be the normal to the plane.

Example 8.1.10. *A cone is almost a manifold, define*

$$\phi(t, z) = (z \cos(t), z \sin(t), z)$$

for $t \in [0, 2\pi]$ and $z \geq 0$. What two problems does this potential coordinate patch $\phi: U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ suffer from? Can you find a modification of U which makes $\phi(U)$ a manifold (it could be a subset of what we call a cone)

The cone is not a manifold because of its point. Generally a space which is mostly like a manifold except at a finite, or discrete, number of *singular* points is called an **orbifold**. Recently, in the past decade or two, the orbifold has been used in string theory. The singularities can be used to fit various *charge* to fields through a mathematical process called the *blow-up*.

Example 8.1.11. *Let $\phi(\theta, \gamma) = (\cos(\theta) \cosh(\gamma), \sin(\theta) \cosh(\gamma), \sinh(\gamma))$ for $\theta \in (0, 2\pi)$ and $\gamma \in \mathbb{R}$. This gives us a patch on the hyperboloid $x^2 + y^2 - z^2 = 1$*

Example 8.1.12. *Let $\phi(x, y, z, t) = (x, y, z, R \cos(t), R \sin(t))$ for $t \in (0, 2\pi)$ and $(x, y, z) \in \mathbb{R}^3$. This gives a copy of \mathbb{R}^3 inside \mathbb{R}^5 where a circle has been attached at each point of space in the two transverse directions of \mathbb{R}^5 . You could imagine that R is nearly zero so we cannot traverse these extra dimensions.*

Example 8.1.13. *The following patch describes the **Möbius band** which is obtained by gluing a line segment to each point along a circle. However, these segments twist as you go around the circle and the structure of this manifold is less trivial than those we have thus far considered. The Möbius band is an example of a manifold which is not oriented. This means that there is not a well-defined normal vectorfield over the surface. The patch is:*

$$\phi(t, \lambda) = \left(\left[1 + \frac{1}{2}\lambda \cos\left(\frac{t}{2}\right)\right] \cos(t), \left[1 + \frac{1}{2}\lambda \sin\left(\frac{t}{2}\right)\right] \sin(t), \frac{1}{2}\lambda \sin\left(\frac{t}{2}\right) \right)$$

for $0 \leq t \leq 2\pi$ and $-1 \leq \lambda \leq 1$. To understand this mapping better try studying the map evaluated at various values of t ;

$$\phi(0, \lambda) = (1 + \lambda/2, 0, 0), \quad \phi(\pi, \lambda) = (-1, 0, \lambda/2), \quad \phi(2\pi, \lambda) = (1 - \lambda/2, 0, 0)$$

Notice the line segment parametrized by $\phi(0, \lambda)$ and $\phi(2\pi, \lambda)$ is the same set of points, however the orientation is reversed.

Example 8.1.14. *A **regular surface** is a two-dimensional manifold embedded in \mathbb{R}^3 . We need $\phi_i : U_i \subseteq \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ such that, for each i , $d\phi_{i,u,v}$ has rank two for all $(u, v) \in U_i$. Moreover, in this case we can define a normal vector field $N(u, v) = \partial_u \phi \times \partial_v \phi$ and if we visualize these vectors as attached to the surface they will point in or out of the surface and provide the normal to the tangent plane at the point considered. The surface S is called **orientable** iff the normal vector field is non-vanishing on S . XXX, clean up normal idea*

Example 8.1.15. *Graphs*

Example 8.1.16. *$(n - p)$ -dimensional level set.*

8.1.2 manifolds defined by charts

Given patches which define a manifold you can derive charts, conversely, given properly constructed charts you can just as well find patches and hence the manifold structure. I include these examples to illustrate why charts are a nice starting point for certain examples. Recall that if $\phi : U \subseteq \mathbb{R}^m \rightarrow V \subseteq M$ is a local parametrization then $\phi^{-1} : V \rightarrow U$ is called a local coordinate chart. We seek to translate the definition which we gave in patch-notation into chart notation for this subsection. Throughout this subsection we denote $\phi_i^{-1} = \chi_i$ then we must insist, for all i , $\chi_i : V_i \rightarrow U_i$ where U_i is open in \mathbb{R}^m satisfies

1. each map $\chi_i : V_i \rightarrow U_i$ is bijective
2. if $V_i \cap V_j \neq \emptyset$ then there exists a smooth mapping

$$\theta_{ij} : \chi_j(V_i \cap V_j) \rightarrow \chi_i(V_i \cap V_j)$$

such that $\theta_{ij} = \chi_i \circ \chi_j^{-1}$.

3. $\mathcal{M} = \cup_i V_i$

For convenience of discussion we suppose the local parametrizations are also bijective. There is no loss of generality since we can always make an injective map bijective by simply shrinking the codomain. The original definition only assumed injectivity since that was sufficient. Now that we talk about inverse mappings it is convenient to add the supposition of surjectivity.

Example 8.1.17. Let $\mathcal{M} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

1. Let $V_+ = \{(x, y) \in \mathcal{M} \mid y > 0\} = \text{dom}(\chi_+)$ and define $\chi_+(x, y) = x$
2. Let $V_- = \{(x, y) \in \mathcal{M} \mid y < 0\} = \text{dom}(\chi_-)$ and define $\chi_-(x, y) = x$
3. Let $V_R = \{(x, y) \in \mathcal{M} \mid x > 0\} = \text{dom}(\chi_R)$ and define $\chi_R(x, y) = y$
4. Let $V_L = \{(x, y) \in \mathcal{M} \mid x < 0\} = \text{dom}(\chi_L)$ and define $\chi_L(x, y) = y$

The set of charts $\mathcal{A} = \{(V_+, \chi_+), (V_-, \chi_-), (V_R, \chi_R), (V_L, \chi_L)\}$ forms an atlas on \mathcal{M} which gives the circle a differentiable structure⁵. It is not hard to show the transition functions are smooth on the image of the intersection of their respect domains. For example, $V_+ \cap V_R = W_{+R} = \{(x, y) \in \mathcal{M} \mid x, y > 0\}$, it's easy to calculate that $\chi_+^{-1}(x) = (x, \sqrt{1-x^2})$ hence

$$(\chi_R \circ \chi_+^{-1})(x) = \chi_R(x, \sqrt{1-x^2}) = \sqrt{1-x^2}$$

for each $x \in \chi_R(W_{+R})$. Note $x \in \chi_R(W_{+R})$ implies $0 < x < 1$ hence it is clear the transition function is smooth. Similar calculations hold for all the other overlapping charts. This manifold is usually denoted $\mathcal{M} = S_1$.

A cylinder is the Cartesian product of a line and a circle. In other words, we can create a cylinder by gluing a copy of a circle at each point along a line. If all these copies line up and don't twist around then we get a cylinder. The example that follows here illustrates a more general pattern, we can take a given manifold and paste a copy at each point along another manifold by using a Cartesian product.

Example 8.1.18. Let $\mathcal{P} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

1. Let $V_+ = \{(x, y, z) \in \mathcal{P} \mid y > 0\} = \text{dom}(\chi_+)$ and define $\chi_+(x, y, z) = (x, z)$
2. Let $V_- = \{(x, y, z) \in \mathcal{P} \mid y < 0\} = \text{dom}(\chi_-)$ and define $\chi_-(x, y, z) = (x, z)$
3. Let $V_R = \{(x, y, z) \in \mathcal{P} \mid x > 0\} = \text{dom}(\chi_R)$ and define $\chi_R(x, y, z) = (y, z)$
4. Let $V_L = \{(x, y, z) \in \mathcal{P} \mid x < 0\} = \text{dom}(\chi_L)$ and define $\chi_L(x, y, z) = (y, z)$

The set of charts $\mathcal{A} = \{(V_+, \chi_+), (V_-, \chi_-), (V_R, \chi_R), (V_L, \chi_L)\}$ forms an atlas on \mathcal{P} which gives the cylinder a differentiable structure. It is not hard to show the transition functions are smooth on the

⁵meaning that if we adjoin the infinity of likewise compatible charts that defines a differentiable structure on \mathcal{M}

image of the intersection of their respective domains. For example, $V_+ \cap V_R = W_{+R} = \{(x, y, z) \in \mathcal{P} \mid x, y > 0\}$, it's easy to calculate that $\chi_+^{-1}(x, z) = (x, \sqrt{1-x^2}, z)$ hence

$$(\chi_R \circ \chi_+^{-1})(x, z) = \chi_R(x, \sqrt{1-x^2}, z) = (\sqrt{1-x^2}, z)$$

for each $(x, z) \in \chi_R(W_{+R})$. Note $(x, z) \in \chi_R(W_{+R})$ implies $0 < x < 1$ hence it is clear the transition function is smooth. Similar calculations hold for all the other overlapping charts.

Generally, given two manifolds \mathcal{M} and \mathcal{N} we can construct $\mathcal{M} \times \mathcal{N}$ by taking the Cartesian product of the charts. Suppose $\chi_{\mathcal{M}} : V \subseteq \mathcal{M} \rightarrow U \subseteq \mathbb{R}^m$ and $\chi_{\mathcal{N}} : V' \subseteq \mathcal{N} \rightarrow U' \subseteq \mathbb{R}^n$ then you can define the product chart $\chi : V \times V' \rightarrow U \times U'$ as $\chi = \chi_{\mathcal{M}} \times \chi_{\mathcal{N}}$. The Cartesian product $\mathcal{M} \times \mathcal{N}$ together with all such product charts naturally is given the structure of an $(m+n)$ -dimensional manifold. For example, in the preceding example we took $\mathcal{M} = S_1$ and $\mathcal{N} = \mathbb{R}$ to construct $\mathcal{P} = S_1 \times \mathbb{R}$.

Example 8.1.19. The 2-torus, or donut, is constructed as $T_2 = S_1 \times S_1$. The n -torus is constructed by taking the product of n -circles:

$$T_n = \underbrace{S_1 \times S_1 \times \cdots \times S_1}_{n \text{ copies}}$$

The atlas on this space can be obtained by simply taking the product of the S_1 charts n -times.

One of the surprising discoveries in manifold theory is that a particular set of points may have many different possible differentiable structures. This is why mathematicians often say a manifold is a set together with a maximal atlas. For example, higher-dimensional spheres (S_7, S_8, \dots) have more than one differentiable structure. In contrast, S_n for $n \leq 6$ has just one differentiable structure. XXX-add reference.

The most familar example of a manifold is just \mathbb{R}^2 or \mathbb{R}^3 itself. One may ask which coordinates are in the atlas which contains the standard Cartesian coordinate chart. The most commonly used charts other than Cartesian would probably be the spherical and cylindrical coordinate systems for \mathbb{R}^3 or the polar coordinate system for \mathbb{R}^2 . Technically, certain restrictions must be made on the domain of these non-Cartesian coordinates if we are to correctly label them "coordinate charts". Interestingly, applications are greedier than manifold theorists, we do need to include those points in \mathbb{R}^n which spoil the injectivity of spherical or cylindrical coordinates. On the other hand, those bad points are just the origin and a ray of points which do not contribute noticable in the calculation of a surface or volume integral.

I will not attempt to make explicit the domain of the coordinate charts in the following two examples (you might find them in a homework):

Example 8.1.20. Define $\chi_{\text{spherical}}(x, y, z) = (r, \theta, \phi)$ implicitly by the coordinate transformations

$$x = r \cos(\theta) \sin(\phi), \quad y = r \sin(\theta) \sin(\phi), \quad z = r \cos(\phi)$$

These can be inverted,

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \left[\frac{y}{x} \right], \quad \phi = \cos^{-1} \left[\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right]$$

To show compatibility with the standard Cartesian coordinates we would need to select a subset of \mathbb{R}^3 for which $\chi_{\text{spherical}}$ is 1-1 and the since $\chi_{\text{Cartesian}} = \text{Id}$ the transition functions are just $\chi_{\text{spherical}}^{-1}$.

Example 8.1.21. Define $\chi_{\text{cylindrical}}(x, y, z) = (s, \theta, z)$ implicitly by the coordinate transformations

$$x = s \cos(\theta), \quad y = s \sin(\theta), \quad z = z$$

These can be inverted,

$$s = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \left[\frac{y}{x} \right], \quad z = z$$

You can take $\text{dom}(\chi_{\text{cylindrical}}) = \{(x, y, z) \mid 0 < \theta < 2\pi, \} - \{(0, 0, 0)\}$

Remark 8.1.22.

I would encourage you to read Burns and Gidea and/or Munkres for future study. You'll find much more material, motivation and depth in those texts and if you were interested in an independent study after you've completed real analysis feel free to ask.

8.1.3 diffeomorphism

At the outset of this study I emphasized that the purpose of a manifold was to give a natural language for calculus on curved spaces. This definition begins to expose how this is accomplished.

Definition 8.1.23. *smoothness on manifolds.*

Suppose \mathcal{M} and \mathcal{N} are smooth manifolds and $f : \mathcal{M} \rightarrow \mathcal{N}$ is a function then we say f is **smooth** iff for each $p \in \mathcal{M}$ there exists local parametrizations $\phi_M : U_M \subseteq \mathbb{R}^m \rightarrow V_M \subseteq \mathcal{M}$ and $\phi_N : U_N \subseteq \mathbb{R}^n \rightarrow V_N \subseteq \mathcal{N}$ such that $p \in U_M$ and $\phi_N^{-1} \circ f \circ \phi_M$ is a smooth mapping from \mathbb{R}^m to \mathbb{R}^n . If $f : \mathcal{M} \rightarrow \mathcal{N}$ is a smooth bijection then we say f is a **diffeomorphism**. Moreover, if f is a diffeomorphism then we say \mathcal{M} and \mathcal{N} are **diffeomorphic**.

In other words, f is smooth iff its local coordinate representative is smooth. It suffices to check one representative since any other will be related by transition functions which are smooth: suppose we have patches $\bar{\phi}_M : \bar{U}_M \subseteq \mathbb{R}^m \rightarrow \bar{V}_M \subseteq \mathcal{M}$ and $\bar{\phi}_N : \bar{U}_N \subseteq \mathbb{R}^n \rightarrow \bar{V}_N \subseteq \mathcal{N}$ such that $p \in \bar{U}_M$,

$$\underbrace{\phi_N^{-1} \circ f \circ \phi_M}_{\text{local rep. of } f} = \underbrace{\phi_N^{-1} \circ \bar{\phi}_N}_{\text{trans. fnct.}} \circ \underbrace{\bar{\phi}_N^{-1} \circ f \circ \bar{\phi}_M}_{\text{local rep. of } f} \circ \underbrace{\bar{\phi}_M^{-1} \circ \phi_M}_{\text{trans. fnct.}}$$

follows from the chain rule for mappings. This formula shows that if f is smooth with respect to a particular pair of coordinates then its representative will likewise be smooth for any other pair of compatible patches.

Example 8.1.24. Recall in Example 8.1.3 we studied $\mathcal{M} = \{p_o\} \times \mathbb{R}^m$. Recall we have one parametrization $\phi : \mathbb{R}^m \rightarrow \mathcal{M}$ which is defined by $\phi(u) = p_o \times u$. Clearly $\phi^{-1}(p_o, u) = u$ for all $(p_o, u) \in \mathcal{M}$. Let \mathbb{R}^m have Cartesian coordinates so the identity map is the patch for \mathbb{R}^m . Consider the function $f = \phi : \mathbb{R}^m \rightarrow \mathcal{M}$, we have only the local coordinate representative $\phi^{-1} \circ f \circ Id$ to consider. Let $x \in \mathbb{R}^m$,

$$\phi^{-1} \circ f \circ Id = \phi^{-1} \circ \phi \circ Id = Id.$$

Hence, ϕ is a smooth bijection from \mathbb{R}^m to \mathcal{M} and we find \mathcal{M} is diffeomorphic to \mathbb{R}^m

The preceding example naturally generalizes to an arbitrary coordinate chart. Suppose \mathcal{M} is a manifold and $\phi^{-1} : V \rightarrow \mathbb{R}^m$ is a coordinate chart around the point $p \in \mathcal{M}$. We argue that ϕ^{-1} is a diffeomorphism. Once more take the Cartesian coordinate system for \mathbb{R}^m and suppose $\bar{\phi} : \bar{U} \rightarrow \bar{V}$ is a local parametrization with $p \in \bar{V}$. The local coordinate representative of ϕ^{-1} is simply the transition function since:

$$\bar{\phi} \circ \phi^{-1} \circ Id = \bar{\phi} \circ \phi^{-1}.$$

We find ϕ^{-1} is smooth on $V \cap \bar{V}$. It follows that ϕ^{-1} is a diffeomorphism since we know transition functions are smooth on a manifold. We arrive at the following characterization of a manifold: *a manifold is a space which is locally diffeomorphic to \mathbb{R}^m .*

However, just because a manifold is locally diffeomorphic to \mathbb{R}^m that does not mean it is actually diffeomorphic to \mathbb{R}^n . For example, it is a well-known fact that there does not exist a smooth bijection between the 2-sphere and \mathbb{R}^2 . The curvature of a manifold gives an obstruction to making such a mapping.

8.2 tangent space

Since a manifold is generally an abstract object we would like to give a definition for the tangent space which is not directly based on the traditional geometric meaning. On the other hand, we should expect that the definition which is given in the abstract reduces to the usual geometric meaning for the context of an embedded manifold. It turns out there are three common viewpoints.

1. a tangent vector is an *equivalence class* of curves.
2. a tangent vector is a *contravariant vector*.
3. a tangent vector is a *derivation*.

I will explain each case and we will find explicit isomorphisms between each language. We assume that \mathcal{M} is an m -dimensional smooth manifold throughout this section.

8.2.1 equivalence classes of curves

I essentially used case (1.) as the definition for the tangent space of a level-set. Suppose $\gamma : I \subseteq \mathbb{R} \rightarrow \mathcal{M}$ is a smooth curve with $\gamma(0) = p$. In this context, this means that all the local coordinate representatives of γ are smooth curves on \mathbb{R}^m ; that is, for every parametrization $\phi : U \subseteq \mathbb{R}^m \rightarrow V \subseteq \mathcal{M}$ containing $p \in V$ the mapping⁶ $\phi^{-1} \circ \gamma$ is a smooth mapping from \mathbb{R} to \mathbb{R}^m . Given the coordinate system defined by ϕ^{-1} , we define two smooth curves γ_1, γ_2 on \mathcal{M} with $\gamma_1(0) = \gamma_2(0) = p \in \mathcal{M}$ to be **similar at p** iff $(\phi^{-1} \circ \gamma_1)'(0) = (\phi^{-1} \circ \gamma_2)'(0)$. If smooth curves γ_1, γ_2 are similar at $p \in \mathcal{M}$ then we denote this by writing $\gamma_1 \sim_p \gamma_2$. We insist the curves be parametrized such that they reach the point of interest at the parameter $t = 0$, this is not a severe restriction since we can always reparametrize a given curve which reaches p at $t = t_o$ by replacing the parameter with $t - t_o$. Observe that \sim_p defines an **equivalence relation** on the set of smooth curves through p which reach p at $t = 0$ in their domain.

- (i) reflexive: $\gamma \sim_p \gamma$ iff $\gamma(0) = p$ and $(\phi^{-1} \circ \gamma)'(0) = (\phi^{-1} \circ \gamma)'(0)$. If γ is a smooth curve on \mathcal{M} with $\gamma(0) = p$ then clearly the reflexive property holds true.
- (ii) symmetric: Suppose $\gamma_1 \sim_p \gamma_2$ then $(\phi^{-1} \circ \gamma_1)'(0) = (\phi^{-1} \circ \gamma_2)'(0)$ hence $(\phi^{-1} \circ \gamma_2)'(0) = (\phi^{-1} \circ \gamma_1)'(0)$ and we find $\gamma_2 \sim_p \gamma_1$ thus \sim_p is a symmetric relation.
- (iii) transitive: if $\gamma_1 \sim_p \gamma_2$ and $\gamma_2 \sim_p \gamma_3$ then $(\phi^{-1} \circ \gamma_1)'(0) = (\phi^{-1} \circ \gamma_2)'(0)$ and $(\phi^{-1} \circ \gamma_2)'(0) = (\phi^{-1} \circ \gamma_3)'(0)$ thus $(\phi^{-1} \circ \gamma_1)'(0) = (\phi^{-1} \circ \gamma_3)'(0)$ which shows $\gamma_1 \sim_p \gamma_3$.

The equivalence classes of \sim_p partition the set of smooth curves with $\gamma(0) = p$. Each equivalence class $\tilde{\gamma} = \{\beta : I \subseteq \mathbb{R} \rightarrow \mathcal{M} \mid \beta \sim_p \gamma\}$ corresponds uniquely to a particular vector $(\phi^{-1} \circ \gamma)'(0)$ in

⁶Note, we may have to restrict the domain of $\phi^{-1} \circ \gamma$ such that the image of γ falls inside V , keep in mind this poses no threat to the construction since we only consider the derivative of the curve at zero in the final construction. That said, keep in mind as we construct composites in this section we always suppose the domain of a curve includes some nbhd. of zero. We need this assumption in order that the derivative at zero exist.

\mathbb{R}^m . Conversely, given $v = (v^1, v^2, \dots, v^m) \in \mathbb{R}^m$ we can write the equation for a line in \mathbb{R}^m with direction v and base-point $\phi^{-1}(p)$:

$$\vec{r}(t) = \phi^{-1}(p) + tv.$$

We compose \vec{r} with ϕ to obtain a smooth curve through $p \in \mathcal{M}$ which corresponds to the vector v . In invite the reader to verify that $\gamma = \phi \circ \vec{r}$ has

$$(1.) \quad \gamma(0) = p \quad (2.) \quad (\phi^{-1} \circ \gamma)'(0) = v.$$

Notice that the correspondence is made between a vector in \mathbb{R}^m and a whole family of curves. There are naturally many curves that share the same tangent vector to a given point.

Moreover, we show these equivalence classes are not coordinate dependent. Suppose $\gamma \sim_p \beta$ relative to the chart $\phi^{-1} : V \rightarrow U$, with $p \in V$. In particular, we suppose $\gamma(0) = \beta(0) = p$ and $(\phi^{-1} \circ \gamma)'(0) = (\phi^{-1} \circ \beta)'(0)$. Let $\bar{\phi}^{-1} : \bar{V} \rightarrow \bar{U}$, with $p \in \bar{V}$, we seek to show $\gamma \sim_p \beta$ relative to the chart $\bar{\phi}^{-1}$. Note that $\bar{\phi}^{-1} \circ \gamma = \bar{\phi}^{-1} \circ \phi \circ \phi^{-1} \circ \gamma$ hence, by the chain rule,

$$(\bar{\phi}^{-1} \circ \gamma)'(0) = (\bar{\phi}^{-1} \circ \phi)'(\phi^{-1}(p))(\phi^{-1} \circ \gamma)'(0)$$

Likewise, $(\bar{\phi}^{-1} \circ \beta)'(0) = (\bar{\phi}^{-1} \circ \phi)'(\phi^{-1}(p))(\phi^{-1} \circ \beta)'(0)$. Recognize that $(\bar{\phi}^{-1} \circ \phi)'(\phi^{-1}(p))$ is an invertible matrix since it is the derivative of the invertible transition functions, label $(\bar{\phi}^{-1} \circ \phi)'(\phi^{-1}(p)) = P$ to obtain:

$$(\bar{\phi}^{-1} \circ \gamma)'(0) = P(\phi^{-1} \circ \gamma)'(0) \quad \text{and} \quad (\bar{\phi}^{-1} \circ \beta)'(0) = P(\phi^{-1} \circ \beta)'(0)$$

the equality $(\bar{\phi}^{-1} \circ \gamma)'(0) = (\bar{\phi}^{-1} \circ \beta)'(0)$ follows and this shows that $\gamma \sim_p \beta$ relative to the $\bar{\phi}$ coordinate chart. We find that the equivalence classes of curves are independent of the coordinate system.

With the analysis above in mind we define addition and scalar multiplication of equivalence classes of curves as follows: given a coordinate chart $\phi^{-1} : V \rightarrow U$ with $p \in V$, equivalence classes $\tilde{\gamma}_1, \tilde{\gamma}_2$ at p and $c \in \mathbb{R}^m$, if $\tilde{\gamma}_1$ has $(\phi^{-1} \circ \gamma_1)'(0) = v_1$ in \mathbb{R}^m and $\tilde{\gamma}_2$ has $(\phi^{-1} \circ \gamma_2)'(0) = v_2$ in \mathbb{R}^m then we define

$$(i) \quad \tilde{\gamma}_1 + \tilde{\gamma}_2 = \tilde{\alpha} \text{ where } (\phi^{-1} \circ \alpha)'(0) = v_1 + v_2$$

$$(ii) \quad c\tilde{\gamma}_1 = \tilde{\beta} \text{ where } (\phi^{-1} \circ \beta)'(0) = cv_1.$$

We know α and β exist because we can simply push the lines in \mathbb{R}^m based at $\phi^{-1}(p)$ with directions $v_1 + v_2$ and cv_1 up to \mathcal{M} to obtain the desired curve and hence the required equivalence class. Moreover, we know this construction is coordinate independent since the equivalence classes are independent of coordinates.

Definition 8.2.1.

Suppose \mathcal{M} is an m -dimensional smooth manifold. We define the **tangent space at $p \in \mathcal{M}$** to be the set of \sim_p -equivalence classes of curves. In particular, denote:

$$\text{curve}T_p\mathcal{M} = \{\tilde{\gamma} \mid \gamma \text{ smooth and } \gamma(0) = p\}$$

Keep in mind this is just one of three equivalent definitions which are commonly implemented.

8.2.2 contravariant vectors

If $(\bar{\phi}^{-1} \circ \gamma)'(0) = \bar{v}$ and $(\phi^{-1} \circ \gamma)'(0) = v$ then $\bar{v} = Pv$ where $P = (\bar{\phi}^{-1} \circ \phi)'(\phi^{-1}(p))$. With this in mind we could use the pair (p, v) or (p, \bar{v}) to describe a tangent vector at p . The cost of using (p, v) is it brings in questions of coordinate dependence.

The equivalence class viewpoint is at times quite useful, but the definition of vector offered here is a bit easier in certain respects. In particular, relative to a particular coordinate chart $\phi^{-1} : V \rightarrow U$, with $p \in V$, we define (temporary notation)

$$\text{vect}T_p\mathcal{M} = \{(p, v) \mid v \in \mathbb{R}^m\}$$

Vectors are added and scalar multiplied in the obvious way:

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2) \quad \text{and} \quad c(p, v_1) = (p, cv_1)$$

for all $(p, v_1), (p, v_2) \in \text{vect}T_p\mathcal{M}$ and $c \in \mathbb{R}$. Moreover, if we change from the ϕ^{-1} chart to the $\bar{\phi}^{-1}$ coordinate chart then the vector changes form as indicated in the previous subsection; $(p, v) \rightarrow (p, \bar{v})$ where $\bar{v} = Pv$ and $P = (\bar{\phi}^{-1} \circ \phi)'(\phi^{-1}(p))$. The components of (p, v) are said to transform **contravariantly**.

Technically, this is also an equivalence class construction. A more honest notation would be to replace (p, v) with (p, ϕ, v) and then we could state that $(p, \phi, v) \sim (p, \bar{\phi}, \bar{v})$ iff $\bar{v} = Pv$ and $P = (\bar{\phi}^{-1} \circ \phi)'(\phi^{-1}(p))$. However, this notation is tiresome so we do not pursue it further. I prefer the notation of the next viewpoint.

8.2.3 derivations

To begin, let us define the set of locally smooth functions at $p \in \mathcal{M}$:

$$C^\infty(p) = \{f : \mathcal{M} \rightarrow \mathbb{R} \mid f \text{ is smooth on an open set containing } p\}$$

In particular, we suppose $f \in C^\infty(p)$ to mean there exists a patch $\phi : U \rightarrow V \subseteq \mathcal{M}$ such that f is smooth on V . Since we use Cartesian coordinates on \mathbb{R} by convention it follows that $f : V \rightarrow \mathbb{R}$ smooth indicates the local coordinate representative $f \circ \phi : U \rightarrow \mathbb{R}$ is smooth (it has continuous partial derivatives of all orders).

Definition 8.2.2.

Suppose $X_p : C^\infty(p) \rightarrow \mathbb{R}$ is a linear transformation which satisfies the Leibniz rule then we say X_p is a **derivation** on $C^\infty(p)$. Moreover, we denote $X_p \in \mathcal{D}_p\mathcal{M}$ iff $X_p(f + cg) = X_p(f) + cX_p(g)$ and $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$ for all $f, g \in C^\infty(p)$ and $c \in \mathbb{R}$.

Example 8.2.3. Let $\mathcal{M} = \mathbb{R}$ and consider $X_{t_o} = d/dt|_{t_o}$. Clearly X is a derivation on smooth functions near t_o .

Example 8.2.4. Consider $\mathcal{M} = \mathbb{R}^2$. Pick $p = (x_o, y_o)$ and define $X_p = \frac{\partial}{\partial x}|_p$ and $Y_p = \frac{\partial}{\partial y}|_p$. Once more it is clear that $X_p, Y_p \in \mathcal{D}(p)\mathbb{R}^2$. These derivations action is accomplished by partial differentiation followed by evaluation at p .

Example 8.2.5. Suppose $\mathcal{M} = \mathbb{R}^m$. Pick $p \in \mathbb{R}^m$ and define $X = \frac{\partial}{\partial x^j}|_p$. Clearly this is a derivation for any $j \in \mathbb{N}_m$.

Are the other types of derivations? Is the only thing a derivation is is a partial derivative operator? Before we can explore this question we need to define partial differentiation on a manifold. We should hope the definition is consistent with the language we already used in multivariate calculus (and the preceding pair of examples) and yet is also general enough to be stated on any abstract smooth manifold.

Definition 8.2.6.

Let \mathcal{M} be a smooth m -dimensional manifold and let $\phi : U \rightarrow V$ be a local parametrization with $p \in V$. The j -th **coordinate function** $x^j : V \rightarrow \mathbb{R}$ is the j -component function of $\phi^{-1} : V \rightarrow U$. In other words:

$$\phi^{-1}(p) = x(p) = (x^1(p), x^2(p), \dots, x^m(p))$$

These x^j are **manifold coordinates**. In contrast, we will denote the standard Cartesian coordinates in $U \subseteq \mathbb{R}^m$ via u^j so a typical point has the form (u^1, u^2, \dots, u^m) and viewed as functions $u^j : \mathbb{R}^m \rightarrow \mathbb{R}$ where $u^j(v) = e^j(v) = v^j$. We define the **partial derivative with respect to x^j** at p for $f \in C^\infty(p)$ as follows:

$$\frac{\partial f}{\partial x^j}(p) = \frac{\partial}{\partial u^j} \left[(f \circ \phi)(u) \right] \Big|_{u=\phi^{-1}(p)} = \frac{\partial}{\partial u^j} \left[f \circ x^{-1} \right] \Big|_{x(p)}.$$

The idea of the definition is simply to take the function f with domain in \mathcal{M} then pull it back to a function $f \circ x^{-1} : U \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ on \mathbb{R}^m . Then we can take partial derivatives of $f \circ x^{-1}$ in the same way we did in multivariate calculus. In particular, the partial derivative w.r.t. u^j is calculated by:

$$\frac{\partial f}{\partial x^j}(p) = \frac{d}{dt} \left[(f \circ \phi)(x(p) + te_j) \right]_{t=0}$$

which is precisely the directional derivative of $f \circ x^{-1}$ in the j -direction at $x(p)$. In fact, Note

$$(f \circ \phi)(x(p) + te_j) = f(x^{-1}(x(p) + te_j)).$$

The curve $t \rightarrow x^{-1}(x(p) + te_j)$ is the curve on \mathcal{M} through p where all coordinates are fixed except the j -coordinate. It is a *coordinate curve* on \mathcal{M} .

XXX-need a picture of coordinate curves on manifold!

Notice in the case that $\mathcal{M} = \mathbb{R}^m$ is given Cartesian coordinate $\phi = Id$ then $x^{-1} = Id$ as well and the $t \rightarrow x^{-1}(x(p) + te_j)$ reduces to $t \rightarrow p + te_j$ which is just the j -th coordinate curve through p on \mathbb{R}^m . It follows that the partial derivative defined for manifolds naturally reduces to the ordinary partial derivative in the context of $\mathcal{M} = \mathbb{R}^m$ with Cartesian coordinates. The beautiful thing is that almost everything we know for ordinary partial derivatives equally well transfers to $\frac{\partial}{\partial x^j}|_p$.

Theorem 8.2.7. *Partial differentiation on manifolds*

Let \mathcal{M} be a smooth m -dimensional manifold with coordinates x^1, x^2, \dots, x^m near p . Furthermore, suppose coordinates y^1, y^2, \dots, y^m are also defined near p . Suppose $f, g \in C^\infty(p)$ and $c \in \mathbb{R}$ then:

1. $\frac{\partial}{\partial x^j}|_p[f + g] = \frac{\partial f}{\partial x^j}|_p + \frac{\partial g}{\partial x^j}|_p$
2. $\frac{\partial}{\partial x^j}|_p[cf] = c \frac{\partial f}{\partial x^j}|_p$
3. $\frac{\partial}{\partial x^j}|_p[fg] = f(p) \frac{\partial g}{\partial x^j}|_p + \frac{\partial f}{\partial x^j}|_p g(p)$
4. $\frac{\partial x^i}{\partial x^j}|_p = \delta_{ij}$
5. $\sum_{k=1}^m \frac{\partial x^k}{\partial y^j}|_p \frac{\partial y^i}{\partial x^k}|_p = \delta_{ij}$
6. $\frac{\partial f}{\partial y^j}|_p = \sum_{k=1}^m \frac{\partial x^k}{\partial y^j}|_p \frac{\partial f}{\partial x^k}|_p$

Proof: The proof of (1.) and (2.) follows from the calculation below:

$$\begin{aligned}
 \frac{\partial(f + cg)}{\partial x^j}(p) &= \frac{\partial}{\partial u^j} \left[(f + cg) \circ x^{-1} \right] \Big|_{x(p)} \\
 &= \frac{\partial}{\partial u^j} \left[f \circ x^{-1} + cg \circ x^{-1} \right] \Big|_{x(p)} \\
 &= \frac{\partial}{\partial u^j} \left[f \circ x^{-1} \right] \Big|_{x(p)} + c \frac{\partial}{\partial u^j} \left[g \circ x^{-1} \right] \Big|_{x(p)} \\
 &= \frac{\partial f}{\partial x^j}(p) + c \frac{\partial g}{\partial x^j}(p)
 \end{aligned} \tag{8.1}$$

The key in this argument is that composition $(f + cg) \circ x^{-1} = f \circ x^{-1} + cg \circ x^{-1}$ along side the linearity of the partial derivative. Item (3.) follows from the identity $(fg) \circ x^{-1} = (f \circ x^{-1})(g \circ x^{-1})$ in tandem with the product rule for a partial derivative on \mathbb{R}^m . The reader may be asked to complete the argument for (3.) in the homework. Continuing to (4.) we calculate from the definition:

$$\left. \frac{\partial x^i}{\partial x^j} \right|_p = \frac{\partial}{\partial u^j} \left[(x^i \circ x^{-1})(u) \right] \Big|_{x(p)} = \frac{\partial u^i}{\partial u^j} \Big|_{x(p)} = \delta_{ij}.$$

where the last equality is known from multivariate calculus. In invite the reader to prove it from the definition if unaware of this fact. Before we prove (5.) it helps to have a picture and a bit more notation in mind. Near the point p we have two coordinate charts $x : V \rightarrow U \subseteq \mathbb{R}^m$ and $y : V \rightarrow W \subseteq \mathbb{R}^m$, we take the chart domain V to be small enough so that both charts are defined. Denote Cartesian coordinates on U by u^1, u^2, \dots, u^m and for W we likewise use Cartesian coordinates w^1, w^2, \dots, w^m . Let us denote patches ϕ, ψ as the inverses of these charts; $\phi^{-1} = x$ and $\psi^{-1} = y$. Transition functions $\psi^{-1} \circ \phi = y \circ x^{-1}$ are mappings from $U \subseteq \mathbb{R}^m$ to $W \subseteq \mathbb{R}^m$ and we note

$$\frac{\partial}{\partial w^j} \left[(y^i \circ x^{-1})(u) \right] = \frac{\partial y^i}{\partial x^j}$$

Likewise, the inverse transition functions $\phi^{-1} \circ \psi = x \circ y^{-1}$ are mappings from $W \subseteq \mathbb{R}^m$ to $U \subseteq \mathbb{R}^m$

$$\frac{\partial}{\partial w^j} \left[(x^i \circ y^{-1})(w) \right] = \frac{\partial x^i}{\partial y^j}$$

Recall that if $F, G : \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $F \circ G = Id$ then $F'G' = I$ by the chainrule, hence $(F')^{-1} = G'$. Apply this general fact to the transition functions, we find their derivative matrices are inverses. Item (5.) follows. In matrix notation we item (5.) reads $\frac{\partial x}{\partial y} \frac{\partial y}{\partial x} = I$. Item (6.) follows from:

$$\begin{aligned} \left. \frac{\partial f}{\partial y^j} \right|_p &= \frac{\partial}{\partial w^j} \left[(f \circ y^{-1})(w) \right] \Big|_{y(p)} \\ &= \frac{\partial}{\partial w^j} \left[(f \circ x^{-1} \circ x \circ y^{-1})(w) \right] \Big|_{y(p)} \\ &= \frac{\partial}{\partial w^j} \left[(f \circ x^{-1})(u^1(w), \dots, u^m(w)) \right] \Big|_{y(p)} \quad : \text{ where } u^k(w) = (x \circ y^{-1})^k(w) \\ &= \sum_{k=1}^m \frac{\partial (x \circ y^{-1})^k}{\partial w^j} \Big|_{y(p)} \frac{\partial (f \circ x^{-1})}{\partial u^k} \Big|_{(x \circ y^{-1})(y(p))} \quad : \text{ chain rule} \\ &= \sum_{k=1}^m \frac{\partial (x^k \circ y^{-1})}{\partial w^j} \Big|_{y(p)} \frac{\partial (f \circ x^{-1})}{\partial u^k} \Big|_{x(p)} \\ &= \sum_{k=1}^m \frac{\partial x^k}{\partial y^j} \Big|_p \frac{\partial f}{\partial x^k} \Big|_p \end{aligned}$$

The key step was the multivariate chain rule. \square

This theorem proves we can lift calculus on \mathbb{R}^m to \mathcal{M} in a natural manner. Moreover, we should note that items (1.), (2.) and (3.) together show $\frac{\partial}{\partial x^i}\big|_p$ is a derivation at p . Item (6.) should remind the reader of the contravariant vector discussion. Removing the f from the equation reveals that

$$\frac{\partial}{\partial y^j}\bigg|_p = \sum_{k=1}^m \frac{\partial x^k}{\partial y^j}\bigg|_p \frac{\partial}{\partial x^k}\bigg|_p$$

A notation convenient to the current discussion is that a contravariant transformation is $(p, v_x) \rightarrow (p, v_y)$ where $v_y = Pv_x$ and $P = (y \circ x^{-1})'(x(p)) = \frac{\partial y}{\partial x}\big|_{x(p)}$. Notice this is the inverse of what we see in (6.). This suggests that the partial derivatives change coordinates like as a basis for the tangent space. To complete this thought we need a few well-known propositions for derivations.

Proposition 8.2.8. *derivations on constant function gives zero.*

If $f \in C^\infty(p)$ is a constant function and $X_p \in \mathcal{D}_p\mathcal{M}$ then $X_p(f) = 0$.

Proof: Suppose $f(x) = c$ for all $x \in V$, define $g(x) = 1$ for all $x \in V$ and note $f = fg$ on V . Since X_p is a derivation it satisfies the Leibniz rule hence

$$X_p(f) = X_p(fg) = f(p)X_p(g) + X_p(f)g(p) = cX_p(g) + X_p(f) \Rightarrow cX_p(g) = 0.$$

Moreover, by homogeneity of X_p , note $cX_p(g) = X_p(cg) = X_p(f)$. Thus, $X_p(f) = 0$. \square

Proposition 8.2.9.

If $f, g \in C^\infty(p)$ and $f(x) = g(x)$ for all $x \in V$ and $X_p \in \mathcal{D}_p\mathcal{M}$ then $X_p(f) = X_p(g)$.

Proof: Note that $f(x) = g(x)$ implies $h(x) = f(x) - g(x) = 0$ for all $x \in V$. Thus, the previous proposition yields $X_p(h) = 0$. Thus, $X_p(f - g) = 0$ and by linearity $X_p(f) - X_p(g) = 0$. The proposition follows. \square

Proposition 8.2.10.

Suppose $X_p \in \mathcal{D}_p\mathcal{M}$ and x is a chart defined near p ,

$$X_p = \sum_{j=1}^m X_p(x^j) \frac{\partial}{\partial x^j}\bigg|_p$$

Proof: this is a less trivial proposition. We need a standard lemma before we begin.

Lemma 8.2.11.

Let p be a point in smooth manifold \mathcal{M} and let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth function. If $x : V \rightarrow U$ is a chart with $p \in V$ and $x(p) = 0$ then there exist smooth functions $g_j : \mathcal{M} \rightarrow \mathbb{R}$ whose values at p satisfy $g_j(p) = \frac{\partial f}{\partial x^j}(p)$. In addition, for all q near enough to p we have $f(q) = f(p) + \sum_{k=1}^m x^k(q)g_k(q)$

Proof: follows from proving a similar identity on \mathbb{R}^m then lifting to the manifold. I leave this as a nontrivial exercise for the reader. This can be found in many texts, see Burns and Gidea page 29 for one source. ∇

Consider $f \in C^\infty(p)$, and use the lemma, we assume $x(p) = 0$ and $g_j(p) = \frac{\partial f}{\partial x^j}(p)$:

$$\begin{aligned} X_p(f) &= X_p\left(f(p) + \sum_{k=1}^m x^k(q)g_k(q)\right) \\ &= X_p(f(p)) + \sum_{k=1}^m X_p(x^k(q)g_k(q)) \\ &= \sum_{k=1}^m \left[X_p(x^k)g_k(p) + x^k(p)X_p(g_k(q)) \right] \\ &= \sum_{k=1}^m X_p(x^k) \frac{\partial f}{\partial x^k}(p). \end{aligned}$$

The calculation above holds for arbitrary $f \in C^\infty(p)$ hence the proposition follows. \square

We've answered the question posed earlier in this section. It is true that every derivation of a manifold is simply a linear combination of partial derivatives. We can say more. The set of derivations at p naturally forms a vector space under the usual addition and scalar multiplication of operators: if $X_p, Y_p \in \mathcal{D}_p\mathcal{M}$ then we define $X_p + Y_p$ by $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$ and cX_p by $(cX_p)(f) = cX_p(f)$ for all $f, g \in C^\infty(p)$ and $c \in \mathbb{R}$. It is easy to show $\mathcal{D}_p\mathcal{M}$ is a vector space under these operations. Moreover, the preceding proposition shows that $\mathcal{D}_p\mathcal{M} = \text{span}\left\{\frac{\partial f}{\partial x^j}\Big|_p\right\}_{j=1}^m$ hence $\mathcal{D}_p\mathcal{M}$ is an m -dimensional vector space⁷.

Finally, let's examine coordinate change for derivations. Given two coordinate charts x, y at $p \in \mathcal{M}$ we have two ways to write the derivation X_p :

$$X_p = \sum_{j=1}^m X_p(x^j) \frac{\partial}{\partial x^j} \Big|_p \quad \text{or} \quad X_p = \sum_{k=1}^m X_p(y^k) \frac{\partial}{\partial y^k} \Big|_p$$

⁷technically, we should show the coordinate derivations $\frac{\partial}{\partial x^j}\Big|_p$ are linearly independent to make this conclusion. I don't suppose we've done that directly at this juncture. You might find this as a homework

It is simple to connect these formulas. Whereas, for y -coordinates,

$$X_p(y^k) = \sum_{j=1}^m X_p(x^j) \frac{\partial y^k}{\partial x^j} \Big|_p \quad (8.2)$$

This is the contravariant transformation rule. In contrast, recall $\frac{\partial}{\partial y^j} \Big|_p = \sum_{k=1}^m \frac{\partial x^k}{\partial y^j} \Big|_p \frac{\partial}{\partial x^k} \Big|_p$. We should have anticipated this pattern since from the outset it is clear there is no coordinate dependence in the definition of a derivation.

8.2.4 dictionary between formalisms

We have three competing views of how to characterize a tangent vector.

1. $curveT_p\mathcal{M} = \{\tilde{\gamma} \mid \gamma \text{ smooth and } \gamma(0) = p\}$
2. $vectT_p\mathcal{M} = \{(p, v) \mid v \in \mathbb{R}^m\}$
3. $derT_p\mathcal{M} = \mathcal{D}_p\mathcal{M}$

Perhaps it is not terribly obvious how to get a derivation from an equivalence class of curves. Suppose $\tilde{\gamma}$ is a tangent vector to \mathcal{M} at p and let $f, g \in C^\infty(p)$. Define an operator V_p associated to $\tilde{\gamma}$ via $V_p(f) = (f \circ \gamma)'(0)$. Consider, $(f + cg) \circ \gamma(t) = (f + cg)(\gamma(t)) = f(\gamma(t)) + cg(\gamma(t))$ differentiate at set $t = 0$ to verify that $V_p(f + cg)(p) = (f + cg) \circ \gamma'(0) = V_p(f)(p) + cV_p(g)$. Furthermore, observe that $((fg) \circ \gamma)(t) = f(\gamma(t))g(\gamma(t))$ therefore by the product rule from calculus I,

$$((fg) \circ \gamma)'(t) = (f \circ \gamma)'(t)g(\gamma(t)) + f(\gamma(t))(g \circ \gamma)'(t)$$

hence, noting $\gamma(0) = p$ we verify the Leibniz rule for V_p ,

$$V_p(fg) = ((fg) \circ \gamma)'(0) = (f \circ \gamma)'(0)g(p) + f(p)(g \circ \gamma)'(0) = V_p(f)g(p) + f(p)V_p(g)$$

In view of these calculations we find that $\Xi : curveT_p\mathcal{M} \rightarrow derT_p\mathcal{M}$ defined by $\Xi(\tilde{\gamma}) = V_p$ is well-defined. Moreover, we can show Ξ is an isomorphism. To be clear, we define:

$$\Xi(\tilde{\gamma})(f) = V_p(f) = (f \circ \gamma)'(0).$$

I'll begin with injectivity. Suppose $\Xi(\tilde{\gamma}) = \Xi(\tilde{\beta})$ then for all $f \in C^\infty(p)$ we have $\Xi(\tilde{\gamma})(f) = \Xi(\tilde{\beta})(f)$ hence $(f \circ \gamma)'(0) = (f \circ \beta)'(0)$ for all smooth functions f at p . Take $f = x : V \rightarrow U$ and it follows that $\gamma \sim_p \beta$ hence $\tilde{\gamma} = \tilde{\beta}$ and we have shown Ξ is injective. Linearity of Ξ must be judged on the basis of our definition for the addition of equivalence classes of curves. I leave linearity and surjectivity to the reader. Once those are established it follows that Ξ is an isomorphism and $curveT_p\mathcal{M} \approx derT_p\mathcal{M}$.

The isomorphism between $vectT_p\mathcal{M}$ and $derT_p\mathcal{M}$ was nearly given in the previous subsection. Essentially we can just paste the components from $vectT_p\mathcal{M}$ onto the partial derivative basis for

derivations. Define $\Upsilon : \text{vect}T_p\mathcal{M} \rightarrow \text{der}T_p\mathcal{M}$ for each $(p, v_x) \in \text{vect}T_p\mathcal{M}$, relative to coordinates x at $p \in \mathcal{M}$,

$$\Upsilon\left(p, \sum_{k=1}^m v_x^k e_k\right) = \sum_{k=1}^m v_x^k \frac{\partial}{\partial x^k} \Big|_p$$

Note that if we used a different chart y then $(p, v_x) \rightarrow (p, v_y)$ and consequently

$$\Upsilon\left(p, \sum_{k=1}^m v_y^k e_k\right) = \sum_{k=1}^m v_y^k \frac{\partial}{\partial y^k} \Big|_p = \sum_{k=1}^m v_x^k \frac{\partial}{\partial x^k} \Big|_p.$$

Thus Υ is single-valued on each equivalence class of vectors. Furthermore, the inverse mapping is simple to write: for a chart x at p ,

$$\Upsilon^{-1}(X_p) = (p, \sum_{k=1}^m X_p(x^k) e_k)$$

and the value of the mapping above is related contravariantly if we were to use a different chart y

$$\Upsilon^{-1}(X_p) = (p, \sum_{k=1}^m X_p(y^k) e_k).$$

See Equation 8.2 and the surrounding discussion if you forgot. It is not hard to verify that Υ is bijective and linear thus Υ is an isomorphism. We have shown $\text{vect}T_p\mathcal{M} \approx \text{der}T_p\mathcal{M}$. Let us summarize:

$$\boxed{\text{vect}T_p\mathcal{M} \approx \text{der}T_p\mathcal{M} \approx \text{curve}T_p\mathcal{M}}$$

Sorry to be anticlimatic here, but we choose the following for future use:

Definition 8.2.12. *tangent space*

We denote $T_p\mathcal{M} = \text{der}T_p\mathcal{M}$.

8.3 the differential

In this section we generalize the concept of the differential to the context of manifolds. Recall that for $F : U \subseteq \mathbb{R}^m \rightarrow V \subseteq \mathbb{R}^n$ the differential $d_p F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ was a linear transformation which best approximated the change in F near p . Notice that while the domain of F could be a mere subset of \mathbb{R}^m the differential always took all of \mathbb{R}^m as its domain. This suggests we should really think of the differential as a mapping which transports tangent vectors to U to tangent vectors at V . Often $d_p f$ is called the **push-forward** by f at p because it pushes tangent vectors along side the mapping.

Definition 8.3.1. *differential for manifolds.*

Suppose \mathcal{M} and \mathcal{N} are smooth manifolds of dimension m and n respectively. Furthermore, suppose $f : \mathcal{M} \rightarrow \mathcal{N}$ is a smooth mapping. We define $d_p f : T_p \mathcal{M} \rightarrow T_{f(p)} \mathcal{N}$ as follows: for each $X_p \in T_p \mathcal{M}$ and $g \in C^\infty(f(p))$

$$d_p f(X_p)(g) = X_p(g \circ f).$$

Notice that $g : \text{dom}(g) \subseteq \mathcal{N} \rightarrow \mathbb{R}$ and consequently $g \circ f : \mathcal{M} \rightarrow \mathbb{R}$ and it follows $g \circ f \in C^\infty(p)$ and it is natural to find $g \circ f$ in the domain of X_p . In addition, it is not hard to show $d_p f(X_p) \in T_{f(p)} \mathcal{N}$. Observe:

1. $d_p f(X_p)(g + h) = X_p((g + h) \circ f) = X_p(g \circ f + h \circ f) = X_p(g \circ f) + X_p(h \circ f)$
2. $d_p f(X_p)(cg) = X_p((cg) \circ f) = X_p(cg \circ f) = cX_p(g \circ f) = cd_p f(X_p)(g)$

The proof of the Leibniz rule is similar. Now that we have justified the definition let's look at an interesting application to the study of surfaces in \mathbb{R}^3 .

Suppose $S \subset \mathbb{R}^3$ is an embedded two-dimensional manifold. In particular suppose S is a **regular surface** which means that for each parametrization $\phi : U \rightarrow V$ the normal vector field $N(u, v) = (\phi_u \times \phi_v)(u, v)$ is a smooth non-vanishing vector field on S . Recall that the unit-sphere $S_2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ is also manifold, perhaps you showed this in a homework. In any event, the mapping $U : S \rightarrow S_2$ defined by

$$G(u, v) = \frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|}$$

provides a smooth mapping from the surface to the unit sphere. The change in G measures how the normal deflects as we move about the surface S . One natural scalar we can use to quantify that curving of the normal is called the **Gaussian curvature** which is defined by $K = \det(dG)$. Likewise, we define $H = \text{trace}(dG)$ which is the **mean curvature** of S . If k_1, k_2 are the eigenvalues the operator $d_p G$ then it is a well-known result of linear algebra that $\det(d_p G) = k_1 k_2$ and $\text{trace}(d_p G) = k_1 + k_2$. The eigenvalues are called the **principal curvatures**. Moreover, it can be shown that the matrix of $d_p G$ is symmetric and a theorem of linear algebra says that the eigenvalues are real and we can select an orthogonal basis of eigenvectors for $T_p S$.

Example 8.3.2. Consider the plane S with base point r_o and containing the vectors \vec{A}, \vec{B} , write

$$\phi(u, v) = r_o + u\vec{A} + v\vec{B}$$

to place coordinates u, v on the plane S . Calculate the Gauss map,

$$G(u, v) = \frac{\phi_u \times \phi_v}{\|\phi_u \times \phi_v\|} = \frac{\vec{A} \times \vec{B}}{\|\vec{A} \times \vec{B}\|}$$

This is constant on S hence $d_p G = 0$ for each $p \in S$. The curvatures (mean, Gaussian and principles) are all zero for this case. Makes sense, a plane isn't curved!

Let me outline how to calculate the curvature directly when G is not trivial. Calculate,

$$d_p G \left(\frac{\partial}{\partial x^k} \right) (y^j) = \frac{\partial}{\partial x^k} (y^j \circ G) = \frac{\partial (y^j \circ G)}{\partial x^k}$$

Thus, using the discussion of the preceding section,

$$d_p G \left(\frac{\partial}{\partial x^k} \right) = \sum_{j=1}^2 \frac{\partial (y^j \circ G)}{\partial x^k} \frac{\partial}{\partial y^j}$$

Therefore, the matrix of $d_p G$ is the 2×2 matrix $\left[\frac{\partial (y^j \circ G)}{\partial x^k} \right]$ with respect to the choice of coordinates x^1, x^2 on S and y^1, y^2 on the sphere.

Example 8.3.3. Suppose $\phi(u, v) = (u, v, \sqrt{R^2 - u^2 - v^2})$ parameterizes part of a sphere S_R of radius $R > 0$. You can calculate the Gauss map and the result should be geometrically obvious:

$$G(u, v) = \frac{1}{R} (u, v, \sqrt{R^2 - u^2 - v^2})$$

Then the u and v components of $G(u, v)$ are simply u/R and v/R respective. Calculate,

$$[d_p G] = \begin{bmatrix} \frac{\partial}{\partial u} [\frac{u}{R}] & \frac{\partial}{\partial v} [\frac{u}{R}] \\ \frac{\partial}{\partial u} [\frac{v}{R}] & \frac{\partial}{\partial v} [\frac{v}{R}] \end{bmatrix} = \begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{bmatrix}$$

Thus the Gaussian curvature of the sphere $K = 1/R^2$. The principle curvatures are $k_1 = k_2 = 1/R$ and the mean curvature is simply $H = 2/R$. Notice that as $R \rightarrow \infty$ we find agreement with the curvature of a plane.

Example 8.3.4. Suppose S is a cylinder which is parametrized by $\phi(u, v) = (R \cos u, R \sin u, v)$. The Gauss map yields $G(u, v) = (\cos u, \sin u, 0)$. I leave the explicit details to the reader, but it can be shown that $k_1 = 1/R, k_2 = 0$ and hence $K = 0$ whereas $H = 1/R$.

The differential is actually easier to frame in the equivalence class curve formulation of $T_p\mathcal{M}$. In particular, suppose $\tilde{\gamma} = [\gamma]$ as a more convenient notation for what follows. In addition, suppose $F : \mathcal{M} \rightarrow \mathcal{N}$ is a smooth function and $[\gamma] \in \text{curve}T_p\mathcal{M}$ then we define $d_pF : \text{curve}T_p\mathcal{M} \rightarrow \text{curve}T_{F(p)}\mathcal{N}$ as follows:

$$d_pF([\gamma]) = [F \circ \gamma]$$

There is a chain-rule for differentials. It's the natural rule you'd expect. If $F : \mathcal{M} \rightarrow \mathcal{N}$ and $G : \mathcal{N} \rightarrow \mathcal{P}$ then, denoting $q = F(p)$,

$$\boxed{d_p(G \circ F) = d_qG \circ d_pF.}$$

The proof is simple in the curve notation:

$$(d_qG \circ d_pF)([\gamma]) = d_qG(d_pF([\gamma])) = d_qG([F \circ \gamma]) = [G \circ (F \circ \gamma)] = d_p(G \circ F)[\gamma].$$

You can see why the curve formulation of tangent vectors is useful. It does simply certain questions. That said, we will insist $T_p\mathcal{M} = \mathcal{D}_p\mathcal{M}$ in sequel.

The push-forward need not be an abstract⁸ exercise.

Example 8.3.5. Suppose $F : \mathbb{R}_{r,\theta}^2 \rightarrow \mathbb{R}_{x,y}^2$ is the polar coordinate transformation. In particular,

$$F(r, \theta) = (r \cos \theta, r \sin \theta)$$

Let's examine where F pushes $\frac{\partial}{\partial r}|_p$ and $\frac{\partial}{\partial \theta}|_p$. We use (x, y) as coordinates in the codomain and the problem is to calculate, for $X_p = \partial_r|_p$ or $X_p = \partial_\theta|_p$, the values of $d_pF(X_p)(x)$ and $d_pF(X_p)(y)$ as we know $d_pF(X_p) = d_pF(X_p)(x)\partial_x|_q + d_pF(X_p)(y)\partial_y|_q$ where $q = F(p)$.

$$\begin{aligned} d_pF(\partial_r|_p) &= \partial_r|_p(x \circ F)\partial_x|_q + \partial_r|_p(y \circ F)\partial_y|_q \\ &= \partial_r|_p(r \cos \theta)\partial_x|_q + \partial_r|_p(r \sin \theta)\partial_y|_q \\ &= \cos \theta \partial_x|_q + \sin \theta \partial_y|_q \end{aligned}$$

A similar calculation follows for $d_pF(\partial_\theta|_p)$. However, let me do the calculation in the traditional notation from multivariate calculus. If we denote $F = (x, y)$ and drop the point dependence on the partials we find the formulas for $\partial/\partial\theta$ below:

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} = -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y}.$$

Therefore, the push-forward is a tool which we can use to change coordinates for vectors. Given the coordinate transformation on a manifold we just push the vector of interest presented in one coordinate system to the other through the formulas above. In multivariate calculus we simply thought of this as changing notation on a given problem. I would be good if you came to the same understanding here.

⁸it's not my idea of abstract that is wrong... think about that. \smile

8.4 cotangent space

The tangent space to a smooth manifold \mathcal{M} is a vector space of derivations and we denote it by $T_p\mathcal{M}$. The dual space to this vector space is called the **cotangent space** and the typical elements are called **covectors**.

Definition 8.4.1. *cotangent space $T_p\mathcal{M}^*$*

Suppose \mathcal{M} is a smooth manifold and $T_p\mathcal{M}$ is the tangent space at $p \in \mathcal{M}$. We define, $T_p\mathcal{M}^* = \{\alpha_p : T_p\mathcal{M} \rightarrow \mathbb{R} \mid \alpha \text{ is linear}\}$.

If x is a local coordinate chart at p and $\frac{\partial}{\partial x^1}\big|_p, \frac{\partial}{\partial x^2}\big|_p, \dots, \frac{\partial}{\partial x^m}\big|_p$ is a basis for $T_p\mathcal{M}$ then we denote the dual basis $d_px^1, d_px^2, \dots, d_px^m$ where $d_px^i(\frac{\partial}{\partial x^k}\big|_p) = \delta_{ik}$. Moreover, if α is a covector at p then⁹:

$$\alpha = \sum_{k=1}^m \alpha_k dx^k$$

where $\alpha_k = \alpha\left(\frac{\partial}{\partial x^k}\bigg|_p\right)$ and dx^k is a short-hand for d_px^k . We should understand that covectors are defined at a point even if the point is not explicitly indicated in a particular context. This does lead to some ambiguity in the same way that the careless identification of the function f and it's value $f(x)$ does throughout calculus. That said, an abbreviated notation is often important to help us see through more difficult patterns without getting distracted by the minutia of the problem.

You might worry the notation used for the differential and our current notation for the dual basis of covectors is not consistent. After all, we have two rather different meanings for d_px^k at this time:

1. $x^k : V \rightarrow \mathbb{R}$ is a smooth function hence $d_px^k : T_p\mathcal{M} \rightarrow T_{x^k(p)}\mathbb{R}$ is defined as a *push-forward*, $d_px^k(X_p)(g) = X_p(g \circ x^k)$
2. $d_px^k : T_p\mathcal{M} \rightarrow \mathbb{R}$ where $d_px^k(\frac{\partial}{\partial x^j}\big|_p) = \delta_{jk}$

It is customary to identify $T_{x^k(p)}\mathbb{R}$ with \mathbb{R} hence there is no trouble. Let us examine how the dual-basis condition can be *derived* for the differential, suppose $g : \mathbb{R} \rightarrow \mathbb{R}$ hence $g \circ x^k : V \rightarrow \mathbb{R}$,

$$d_px^k\left(\frac{\partial}{\partial x^j}\bigg|_p\right)(g) = \frac{\partial}{\partial x^j}\bigg|_p(g(x^k)) = \underbrace{\frac{\partial x^k}{\partial x^j}\bigg|_p \frac{dg}{dt}\bigg|_{x^k(p)}}_{\text{chain rule}} = \delta_{jk} \frac{d}{dt}\bigg|_{x^k(p)}(g) = \delta_{jk}g$$

Where, we've made the identification $1 = \frac{d}{dt}\bigg|_{x^k(p)}$ (which is the nut and bolts of writing $T_{x^k(p)}\mathbb{R} = \mathbb{R}$) and hence have the beautiful identity:

$$d_px^k\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \delta_{jk}.$$

⁹we explained this for an arbitrary vector space V and its dual V^* in a previous chapter, we simply apply those results once more here in the particular context $V = T_p\mathcal{M}$

In contrast, there is no need to derive this for case (2.) since in that context this serves as the definition for the object. Personally, I find the multiple interpretations of objects in manifold theory is one of the most difficult aspects of the theory. On the other hand, the notation is really neat once you understand how subtly it assumes many theorems. You should understand the notation we enjoy at this time is the result of generations of mathematical thought. Following a similar derivation for an arbitrary vector $X_p \in T_p\mathcal{M}$ and $f : \mathcal{M} \rightarrow \mathbb{R}$ we find

$$\boxed{d_p f(X_p) = X_p(f)}$$

This notation is completely consistent with the *total differential* as commonly discussed in multivariate calculus. Recall that if $f : \mathbb{R}^m \rightarrow \mathbb{R}$ then we defined

$$df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \cdots + \frac{\partial f}{\partial x^m} dx^m.$$

Notice that the j -th component of df is simply $\frac{\partial f}{\partial x^j}$. Notice that the identity $d_p f(X_p) = X_p(f)$ gives us the same component if we simply evaluate the covector $d_p f$ on the coordinate basis $\frac{\partial}{\partial x^j} \Big|_p$,

$$d_p f \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \frac{\partial f}{\partial x^j} \Big|_p$$

8.5 tensors at a point

Given a smooth m -dimensional manifold \mathcal{M} and a point $p \in \mathcal{M}$ we have a tangent space $T_p\mathcal{M}$ and a cotangent space $T_p\mathcal{M}^*$. The set of tensors at $p \in \mathcal{M}$ is simply the set of all multilinear mappings on the tangent and cotangent space at p . We again define the set of all type (r, s) tensors to be $T_s^r\mathcal{M}_p$ meaning $L \in T_s^r\mathcal{M}_p$ iff L is a multilinear mapping of the form

$$L : \underbrace{T_p\mathcal{M} \times \cdots \times T_p\mathcal{M}}_{r \text{ copies}} \times \underbrace{T_p\mathcal{M}^* \times \cdots \times T_p\mathcal{M}^*}_{s \text{ copies}} \rightarrow \mathbb{R}.$$

Relative to a particular coordinate chart x at p we can build a basis for $T_s^r\mathcal{M}_p$ via the tensor product. In particular, for each $L \in T_s^r\mathcal{M}_p$ there exist constants $L_{i_1 i_2 \dots i_r}^{j_1 j_2 \dots j_s}$ such that

$$\boxed{L_p = \sum_{i_1, \dots, i_r, j_1, \dots, j_s=1}^m (L_{i_1 i_2 \dots i_r}^{j_1 j_2 \dots j_s})(p) d_p x^{i_1} \otimes \cdots \otimes d_p x^{i_r} \otimes \frac{\partial}{\partial x^{j_1}} \Big|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}} \Big|_p}.$$

The components can be calculated by contraction with the appropriate vectors and covectors:

$$(L_{i_1 i_2 \dots i_r}^{j_1 j_2 \dots j_s})(p) = L \left(\frac{\partial}{\partial x^{i_1}} \Big|_p, \dots, \frac{\partial}{\partial x^{i_r}} \Big|_p, d_p x^{j_1}, \dots, d_p x^{j_s} \right).$$

We can summarize the equations above with multi-index notation:

$$d_p x^I = d_p x^{i_1} \otimes d_p x^{i_2} \otimes \cdots \otimes d_p x^{i_r} \quad \text{and} \quad \frac{\partial}{\partial x^J} \Big|_p = \frac{\partial}{\partial x^{j_1}} \Big|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{j_s}} \Big|_p$$

Consequently, $\left[L_p = \sum_{I,J} L_I^J(p) d_p x^I \otimes \frac{\partial}{\partial x^J} \Big|_p \right]$. We may also construct wedge products and build the exterior algebra as we did for an arbitrary vector space. Given a metric $g_p \in T_2^0 \mathcal{M}_p$ we can calculate hodge duals in $\Lambda \mathcal{M}_p$. All these constructions are possible at each point in a smooth manifold¹⁰.

8.6 tensor fields

Since the tangent and cotangent space are defined at each point in a smooth manifold we can construct the tangent bundle and cotangent bundle by simply taking the union of all the tangent or cotangent spaces:

Definition 8.6.1. *tangent and cotangent bundles.*

Suppose \mathcal{M} is a smooth manifold the we define the **tangent bundle** $T\mathcal{M}$ and the **cotangent bundle** $T\mathcal{M}^*$ as follows:

$$T\mathcal{M} = \cup_{p \in \mathcal{M}} T_p \mathcal{M} \quad \text{and} \quad T\mathcal{M}^* = \cup_{p \in \mathcal{M}} T_p \mathcal{M}^*.$$

The canonical projections $\pi, \tilde{\pi}$ tell us where a particular vector or covector are found on the manifold:

$$\pi(X_p) = p \quad \text{and} \quad \tilde{\pi}(\alpha_p) = p$$

I usually picture this construction as follows:

XXX– add projection pictures

Notice the fibers of π and $\tilde{\pi}$ are $\pi^{-1}(p) = T_p \mathcal{M}$ and $\tilde{\pi}^{-1}(p) = T_p \mathcal{M}^*$. Generally a fiber bundle (E, \mathcal{M}, π) consists of a base manifold \mathcal{M} , a bundle space E and a projection $\pi : E \rightarrow \mathcal{M}$. A **local section** of E is a mapping $s : V \subseteq \mathcal{M} \rightarrow E$ such that $\pi \circ s$ is injective. In other words, the image of a section hits each fiber over its domain just once. A section selects a particular element of each fiber. Here's an abstract picture of section, I sometimes think of the section as its image although technically the section is actually a mapping:

XXX– add section picture–

Given the language above we find a natural language to define vector and covector-fields on a manifold. However, for reasons that become clear later, we call a covector-field a *differential one-form*.

Definition 8.6.2. *tensor fields.*

¹⁰I assume \mathcal{M} is Hausdorff and has a countable basis, see Burn's and Gidea Theorem 3.2.5 on page 116.

Let $V \subseteq \mathcal{M}$, we define:

1. X is a vector field on V iff X is a section of $T\mathcal{M}$ on V
2. α is a differential one-form on V iff α is a section of $T\mathcal{M}^*$ on V .
3. L is a type (r, s) tensor-field on V iff L is a section of $T_s^r\mathcal{M}$ on V .

We consider only **smooth sections** and it turns out this is equivalent¹¹ to the demand that the component functions of the fields above are smooth on V .

8.7 metric tensor

I'll begin by discussing briefly the informal concept of a metric. The calculations given in the first part of this section show you how to think for nice examples that are embedded in \mathbb{R}^m . In such cases the metric can be deduced by setting appropriate terms for the metric on \mathbb{R}^m to zero. The metric is then used to set-up arclength integrals over a curved space, see my Chapter on Variational Calculus from the previous notes if you want examples.

In the second part of this chapter I give the careful definition which applies to an arbitrary manifold. I include this whole section mostly for informational purposes. Our main thrust in this course is with the calculus of differential forms and the metric is actually, ignoring the task of hodge duals, not on the center stage. That said, any student of differential geometry will be interested in the metric. The problem of parallel transport¹², and the definition and calculation of geodesics¹³ are fascinating problems beyond this course.

8.7.1 classical metric notation in \mathbb{R}^m

Definition 8.7.1.

The Euclidean metric is $ds^2 = dx^2 + dy^2 + dz^2$. Generally, for orthogonal curvilinear coordinates u, v, w we calculate $ds^2 = \frac{1}{\|\nabla u\|^2} du^2 + \frac{1}{\|\nabla v\|^2} dv^2 + \frac{1}{\|\nabla w\|^2} dw^2$.

The beauty of the metric is that it allows us to calculate in other coordinates, consider

$$x = r \cos(\theta) \quad y = r \sin(\theta)$$

For which we have implicit inverse coordinate transformations $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$. From these inverse formulas we calculate:

$$\nabla r = \langle x/r, y/r \rangle \quad \nabla \theta = \langle -y/r^2, x/r^2 \rangle$$

¹¹all the bundles above are themselves manifolds, for example $T\mathcal{M}$ is a $2m$ -dimensional manifold, and as such the term smooth has already been defined. I do not intend to delve into that aspect of the theory here. See any text on manifold theory for details.

¹²how to move vectors around in a curved manifold

¹³curve of shortest distance on a curved space, basically they are the lines on a manifold

Thus, $\|\nabla r\| = 1$ whereas $\|\nabla \theta\| = 1/r$. We find that the metric in polar coordinates takes the form:

$$ds^2 = dr^2 + r^2 d\theta^2$$

Physicists and engineers tend to like to think of these as arising from calculating the length of infinitesimal displacements in the r or θ directions. Generically, for u, v, w coordinates

$$dl_u = \frac{1}{\|\nabla u\|} du \quad dl_v = \frac{1}{\|\nabla v\|} dv \quad dl_w = \frac{1}{\|\nabla w\|} dw$$

and $ds^2 = dl_u^2 + dl_v^2 + dl_w^2$. So in that notation we just found $dl_r = dr$ and $dl_\theta = r d\theta$. Notice then that cylindric coordinates have the metric,

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2.$$

For spherical coordinates $x = r \cos(\phi) \sin(\theta)$, $y = r \sin(\phi) \sin(\theta)$ and $z = r \cos(\theta)$ (here $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$, physics notation). Calculation of the metric follows from the line elements,

$$dl_r = dr \quad dl_\phi = r \sin(\theta) d\phi \quad dl_\theta = r d\theta$$

Thus,

$$ds^2 = dr^2 + r^2 \sin^2(\theta) d\phi^2 + r^2 d\theta^2.$$

We now have all the tools we need for examples in spherical or cylindrical coordinates. What about other cases? In general, given some p -manifold embedded in \mathbb{R}^n how does one find the metric on that manifold? If we are to follow the approach of this section we'll need to find coordinates on \mathbb{R}^n such that the manifold S is described by setting all but p of the coordinates to a constant. For example, in \mathbb{R}^4 we have generalized cylindric coordinates (r, ϕ, z, t) defined implicitly by the equations below

$$x = r \cos(\phi), \quad y = r \sin(\phi), \quad z = z, \quad t = t$$

On the hyper-cylinder $r = R$ we have the metric $ds^2 = R^2 d\theta^2 + dz^2 + dw^2$. There are mathematicians/physicists whose careers are founded upon the discovery of a metric for some manifold. This is generally a difficult task.

8.7.2 metric tensor on a smooth manifold

A metric on a smooth manifold \mathcal{M} is a type $(2,0)$ tensor field on \mathcal{M} which is at each point p a metric on $T_p \mathcal{M}$. In particular, g is a metric iff g makes the assignment $p \rightarrow g_p$ for each $p \in \mathcal{M}$ where the mapping $g_p : T_p \mathcal{M} \times T_p \mathcal{M} \rightarrow \mathbb{R}$ is a metric. Recall the means g_p is a symmetric, nondegenerate bilinear form on $T_p \mathcal{M}$. Relative to a particular coordinate system x at p we write

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$$

In this context $g_{ij} : V \rightarrow \mathbb{R}$ are assumed to be smooth functions, the values may vary from point to point in V . Furthermore, we know that $g_{ij} = g_{ji}$ for all $i, j \in \mathbb{N}_m$ and the matrix $[g_{ij}]$ is invertible

by the nondegeneracy of g . Recall we use the notation g^{ij} for components of the inverse matrix, in particular we suppose that $\sum_{k=1}^m g_{ik}g^{kj} = \delta_{ij}$.

Recall that according to Sylvester's theorem we can choose coordinates at some point p which will diagonalize the metric and leave $\text{diag}(g_{ij}) = \{-1, -1, \dots, -1, 1, 1, \dots, 1\}$. In other words, we can orthogonalize the coordinate basis at a particular point p . The interesting feature of a curved manifold \mathcal{M} is that as we travel away from the point where we straightened the coordinates it is generally the case the components of the metric will not stay diagonal and constant over the whole coordinate chart. If it is possible to choose coordinates centered on V such that the coordinates are constantly orthogonal with respect to the metric over V then the manifold \mathcal{M} is said to be **flat** on V . Examples of flat manifolds include \mathbb{R}^m , cylinders and even cones without their point. A manifold is said to be curved if it is not flat. The definition I gave just now is not probably one you'll find in a mathematics text¹⁴. Instead, the curvature of a manifold is quantified through various tensors which are derived from the metric and its derivatives. In particular, the Ricci and Riemann tensors are used to carefully characterize the geometry of a manifold. It is very tempting to say more about the general theory of curvature, but I will resist. If you would like to do further study I can recommend a few books. We will consider some geometry of embedded two-dimensional manifolds in \mathbb{R}^3 . That particular case was studied in the 19-th century by Gauss and others and some of the notation below goes back to that time.

Example 8.7.2. Consider a regular surface S which has a global parametrization $\phi : U \subseteq \mathbb{R}^2 \rightarrow S \subseteq \mathbb{R}^3$. In the usual notation in \mathbb{R}^3 ,

$$\phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

Consider a curve $\gamma : [0, 1] \rightarrow S$ we can calculate the arclength of γ via the usual calculation in \mathbb{R}^3 . The magnitude of velocity $\gamma'(t)$ is $\|\gamma'(t)\|$ and naturally this gives us $\frac{ds}{dt}$ hence $ds = \|\gamma'(t)\|dt$ and the following integral calculates the length of γ ,

$$s_\gamma = \int_0^1 \|\gamma'(t)\| dt$$

Since $\gamma[0, 1] \subset S$ it follows there must exist some two-dimensional curve $t \rightarrow (u(t), v(t))$ for which $\gamma(t) = \phi(u(t), v(t))$. Observe by the chain rule that

$$\gamma'(t) = \left(\frac{\partial x}{\partial u} \frac{du}{dt} + \frac{\partial x}{\partial v} \frac{dv}{dt}, \frac{\partial y}{\partial u} \frac{du}{dt} + \frac{\partial y}{\partial v} \frac{dv}{dt}, \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} \right)$$

We can calculate the square of the speed in view of the formula above, let $\frac{du}{dt} = \dot{u}$ and $\frac{dv}{dt} = \dot{v}$,

$$\begin{aligned} \|\gamma'(t)\|^2 = & \left(x_u^2 \dot{u}^2 + 2x_u x_v \dot{u} \dot{v} + x_v^2 \dot{v}^2, \right. \\ & y_u^2 \dot{u}^2 + 2y_u y_v \dot{u} \dot{v} + y_v^2 \dot{v}^2, \\ & \left. z_u^2 \dot{u}^2 + 2z_u z_v \dot{u} \dot{v} + z_v^2 \dot{v}^2 \right) \end{aligned} \quad (8.3)$$

¹⁴this was the definition given in a general relativity course I took with the physicist Martin Rocek of SUNY Stony Brook. He then introduced non-coordinate form-fields which kept the metric constant. I may find a way to show you some of those calculations at the end of this course.

Collecting together terms which share either \dot{u}^2 , $\dot{u}\dot{v}$ or \dot{v}^2 and noting that $x_u^2 + y_u^2 + z_u^2 = \phi_u \cdot \phi_u$, $x_u x_v + y_u y_v + z_u z_v = \phi_u \cdot \phi_v$ and $x_v^2 + y_v^2 + z_v^2 = \phi_v \cdot \phi_v$ we obtain:

$$||\gamma'(t)||^2 = \phi_u \cdot \phi_u \dot{u}^2 + 2\phi_u \cdot \phi_v \dot{u}\dot{v} + \phi_v \cdot \phi_v \dot{v}^2$$

Or, in the notation of Gauss, $\phi_u \cdot \phi_u = E$, $\phi_u \cdot \phi_v = F$ and $\phi_v \cdot \phi_v = G$ hence the arclength on S is given by

$$s_\gamma = \int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} dt$$

We discover that on S there is a metric induced from the ambient euclidean metric. In the current coordinates, using $(u, v) = \phi^{-1}$,

$$g = Edu \otimes du + 2Fdu \otimes dv + Gdv \otimes dv$$

hence the length of a tangent vector is defined via $||X|| = \sqrt{g(X, X)}$, we calculate the length of a curve by integrating its speed along its extent and the speed is simply the magnitude of the tangent vector at each point. The new thing here is that we judge the magnitude on the basis of a metric which is intrinsic to the surface.

If arclength on S is given by Gauss' E, F, G then what about surface area?. We know the magnitude of the cross product of the tangent vectors ϕ_u, ϕ_v on S will give us the area of a tiny parallelogram corresponding to a change du in u and dv in v . Thus:

$$dA = \sqrt{||\phi_u \times \phi_v||^2} du dv$$

However, Lagrange's identity says $||\phi_u \times \phi_v||^2 = ||\phi_u||^2 ||\phi_v||^2 - \phi_u \cdot \phi_v$ hence $dA = \sqrt{EF - G^2} du dv$ and we can calculate surface area (if this integral exists!) via

$$\text{Area}(S) = \int_U \sqrt{EG - F^2} du dv.$$

I make use of the standard notation for double integrals from multivariate calculus and the integration is to be taken over the domain of the parametrization of S .

Many additional formulas are known for E, F, G and there are entire texts devoted to exploring the geometric intricacies of surfaces in \mathbb{R}^3 . For example, John Oprea's *Differential Geometry and its Applications*. Theorem 4.1 of that text is the celebrated Theorem Egregium of Gauss which states the curvature of a surface depends only on the metric of the surface as given by E, F, G . In particular,

$$K = \frac{-1}{2\sqrt{EG}} \left(\frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) \right).$$

Where **curvature** at p is defined by $K(p) = \det(S_p)$ and S_p is the **shape operator** is defined by the covariant derivative $S_p(v) = -\nabla_v U = -(v(U_1), v(U_2), v(U_3))$ and U is simply the **normal vector field** to S defined by $U(u, v) = \phi_u \times \phi_v$ in our current notation.

It turns out there is an easier way to calculate curvature via wedge products. I will hopefully show how that is done in the next chapter. However, I do not attempt to motivate why the curvature is called curvature. You really should read something like Oprea if you want those thoughts.

Example 8.7.3. Let $\mathcal{M} = \mathbb{R}^4$ and choose an atlas of charts which are all inertially related to the standard Cartesian coordinates on \mathbb{R}^4 . In other words, we allow coordinates \bar{x} which can be obtained from a Lorentz transformation; $\bar{x} = \Lambda x$ and $\Lambda \in \mathbb{R}^{4 \times 4}$ such that $\Lambda^T \eta \Lambda = \eta$. Define $g = \sum_{\mu, \nu=0}^3 \eta_{\mu\nu} dx^\mu \otimes dx^\nu$ for the standard Cartesian coordinates on \mathbb{R}^4 . We can show that the metric is invariant as we change coordinates, if you calculate the components of g in some other coordinate system then you will once more obtain $\eta_{\mu\nu}$ as the components. This means that if we can write the equation for the interval between events in one coordinate system then that interval equation must also hold true in any other inertial coordinate system. In particle physics this is a very useful observation because it means if we want to analyze an relativistic interaction then we can study the problem in the frame of reference which makes the problem simplest to understand.

In physics a coordinate system is also called a "frame of reference", technically there is something missing from our construction of \mathcal{M} from a relativity perspective. As a mathematical model of spacetime \mathbb{R}^4 is not quite right. Why? Because Einstein's first axiom or postulate of special relativity is that there is no "preferred frame of reference". With \mathbb{R}^4 there certainly is a preferred frame, it's implicit within the very definition of the set \mathbb{R}^4 , we get Cartesian coordinates for free. To eliminate this convenient set of, according to Einstein, unphysical coordinates you have to consider an affine space which is diffeomorphic to \mathbb{R}^4 . If you take modern geometry you'll learn all about affine space. I will not pursue it further here, and as a bad habit I tend to say \mathcal{M} paired with η is "minkowski space". Technically this is not quite right for the reasons I just explained.

8.8 on boundaries and submanifolds

A manifold with boundary¹⁵ is basically just a manifold which has an edge which is also a manifold. The boundary of a disk is a circle. In fact, in general, a closed ball in $(n+1)$ -dimensional euclidean space has a boundary which is the S_n sphere.

The boundary of quadrants I and II of the xy -plane is the x -axis. Or, to generalize this example, we define the upper-half of \mathbb{R}^n as follows:

$$\mathbb{H}^n = \{(x^1, x^2, \dots, x^{n-1}, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}.$$

The boundary of \mathbb{H}^n is the $x^1 x^2 \dots x^{n-1}$ -hyperplane which is the solution set of $x^n = 0$ in \mathbb{R}^n ; we can denote the boundary by $\partial \mathbb{H}^n$ hence, $\partial \mathbb{H}^n = \mathbb{R}^{n-1} \times \{0\}$. Furthermore, we define

$$\mathbb{H}_+^n = \{(x^1, x^2, \dots, x^{n-1}, x^n) \in \mathbb{R}^n \mid x^n > 0\}.$$

¹⁵I am glossing over some analytical details here concerning extensions and continuity, smoothness etc... see section 24 of Munkres a bit more detail in the embedded case.

It follows that $\mathbb{H}^n = \mathbb{H}_+^n \cup \mathbb{R}^{n-1} \times \{0\}$. Note that a subset U of \mathbb{H}^n is said to be **open in \mathbb{H}^n** iff there exists some open set $U' \subseteq \mathbb{R}^n$ such that $U' \cap \mathbb{H}^n = U$. For example, if we consider \mathbb{R}^3 then the open sets in the xy -plane are formed from intersecting open sets in \mathbb{R}^3 with the plane; an open ball intersects to give an open disk on the plane. Or for \mathbb{R}^2 an open disks intersected with the x -axis give open intervals.

Definition 8.8.1.

We say \mathcal{M} is a **smooth m -dimensional manifold with boundary** iff there exists a family $\{U_i\}$ of open subsets of \mathbb{R}^m or \mathbb{H}^m and **local parameterizations** $\phi_i : U_i \rightarrow V_i \subseteq \mathcal{M}$ such that the following criteria hold:

1. each map $\phi_i : U_i \rightarrow V_i$ is injective
2. if $V_i \cap V_j \neq \emptyset$ then there exists a smooth mapping

$$\theta_{ij} : \phi_j^{-1}(V_i \cap V_j) \rightarrow \phi_i^{-1}(V_i \cap V_j)$$

such that $\phi_j = \phi_i \circ \theta_{ij}$

3. $M = \cup_i \phi_i(U_i)$

We again refer to the inverse of a local parameterization as a **coordinate chart** and often use the notation $\phi^{-1}(p) = (x^1(p), x^2(p), \dots, x^m(p))$. If there exists U open in \mathbb{R}^m such that $\phi : U \rightarrow V$ is a local parametrization with $p \in V$ then p is an **interior point**. Any point $p \in \mathcal{M}$ which is not an interior point is a boundary point. The set of all boundary points is called **boundary** of \mathcal{M} is denoted $\partial\mathcal{M}$.

A more pragmatic characterization¹⁶ of a boundary point is that $p \in \partial\mathcal{M}$ iff there exists a chart at p such that $x^m(p) = 0$. A manifold without boundary is simply a manifold in our definition since the definitions match precisely if there are no half-space-type charts. In the case that $\partial\mathcal{M}$ is nonempty we can show that it forms a manifold without boundary. Moreover, the atlas for $\partial\mathcal{M}$ is naturally induced from that of \mathcal{M} by restriction.

Proposition 8.8.2.

Suppose \mathcal{M} is a smooth m -dimensional manifold with boundary $\partial\mathcal{M} \neq \emptyset$ then $\partial\mathcal{M}$ is a smooth manifold of dimension $m-1$. In other words, $\partial\mathcal{M}$ is an $m-1$ dimensional manifold with boundary and $\partial(\partial\mathcal{M}) = \emptyset$.

Proof: Let $p \in \partial\mathcal{M}$ and suppose $\phi : U \subseteq \mathbb{H}^m \rightarrow V \subseteq \mathcal{M}$ is a local parametrization containing $p \in V$. It follows $\phi^{-1} = (x^1, x^2, \dots, x^{m-1}, x^m) : V \rightarrow U$ is a chart at p with $x^m(p) = 0$. Define the **restriction** of ϕ^{-1} to $x^m = 0$ by $\psi : U' \rightarrow V \cap (\partial\mathcal{M})$ by $\psi(u) = \phi(u, 0)$ where $U' = \{(u^1, \dots, u^{m-1}) \in \mathbb{R}^{m-1} \mid (u^1, \dots, u^{m-1}, u^m) \in U\}$. It follows that $\psi^{-1} : V \cap (\partial\mathcal{M}) \rightarrow U' \subseteq \mathbb{R}^{m-1}$ is just the first $m-1$ coordinates of the chart ϕ^{-1} which is to say $\psi^{-1} = (x^1, x^2, \dots, x^{m-1})$. We

¹⁶I leave it to the reader to show this follows from the words in green.

construct charts in this fashion at each point in $\partial\mathcal{M}$. Note that U' is open in \mathbb{R}^{m-1} hence the manifold $\partial\mathcal{M}$ only has interior points. There is no parametrization in $\partial\mathcal{M}$ which takes a boundary-type subset half-plane as its domain. It follows that $\partial(\partial\mathcal{M}) = \emptyset$. I leave compatibility and smoothness of the restricted charts on $\partial\mathcal{M}$ to the reader. \square

Given the terminology in this section we should note that there are shapes of interest which simply do not fit our terminology. For example, a rectangle $R = [a, b] \times [c, d]$ is not a manifold with boundary since if it were we would have a boundary with sharp edges (which is not a smooth manifold!).

I have not included a full discussion of **submanifolds** in these notes. However, I would like to give you some brief comments concerning how they arise from particular functions. In short, a submanifold is a subset of a manifold which also a manifold in a natural manner. Burns and Gidea define for a smooth mapping f from a manifold \mathcal{M} to another manifold \mathcal{N} that

- a $p \in \mathcal{M}$ is a **critical point** of f if $d_p f : T_p \mathcal{M} \rightarrow T_{f(p)} \mathcal{N}$ is not surjective. Moreover, the image $f(p)$ is called the critical value of f .
- b $p \in \mathcal{M}$ is a **regular point** of f if p is not critical. Moreover, $q \in \mathcal{N}$ is called a **regular value** of f iff $f^{-1}\{q\}$ contains no critical points.

It turns out that:

Theorem 8.8.3.

If $f : \mathcal{M} \rightarrow \mathcal{N}$ is a smooth function on smooth manifolds \mathcal{M} , \mathcal{N} of dimensions m, n respectively and $q \in \mathcal{N}$ is a regular value of f with nonempty fiber $f^{-1}\{q\}$ then the fiber $f^{-1}\{q\}$ is a submanifold of \mathcal{M} of dimension $(m - n)$.

Proof: see page 46 of Burns and Gidea. \square .

The idea of this theorem is a variant of the implicit function theorem. Recall if we are given $G : \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ then the local solution $y = h(x)$ of $G(x, y) = k$ exists provided $\frac{\partial G}{\partial y}$ is invertible. But, this local solution suitably restricted is injective and hence the mapping $\phi(x) = (x, h(x))$ is a local parametrization of a manifold in $\mathbb{R}^k \times \mathbb{R}^n$. In fact, the graph $y = h(x)$ gives k -dimensional submanifold of the manifold $\mathbb{R}^k \times \mathbb{R}^n$. (think of $\mathcal{M} = \mathbb{R}^k \times \mathbb{R}^n$ hence $m = k + n$ and $m - n = k$ so we find agreement with the theorem above at least in the concrete case of level-sets)

Example 8.8.4. Consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = x^2 + y^2$. Calculate $df = 2xdx + 2ydy$ we find that the only critical value of f is $(0, 0)$ since otherwise either x or y is nonzero and as a consequence df is surjective. It follows that $f^{-1}\{R^2\}$ is a submanifold of \mathbb{R}^2 for any $R > 0$. I think you've seen these submanifolds before. What are they?

Example 8.8.5. Consider $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $f(x, y, z) = z^2 - x^2 - y^2$ calculate that $df = -2xdx - 2ydy + 2zdz$. Note $(0, 0, 0)$ is a critical value of f . Furthermore, note $f^{-1}\{0\}$ is the cone $z^2 = x^2 + y^2$ which is not a submanifold of \mathbb{R}^3 . It turns out that in general just about anything

can arise as the inverse image of a critical value. It could happen that the inverse image is a submanifold, it's just not a given.

Theorem 8.8.6.

If \mathcal{M} be a smooth manifold without boundary and $f : \mathcal{M} \rightarrow \mathbb{R}$ is a smooth function with a regular value $a \in \mathbb{R}$ then $f^{-1}(-\infty, a]$ is a smooth manifold with boundary $f^{-1}\{a\}$.

Proof: see page 50 of Burns and Gidea. \square .

Example 8.8.7. Suppose $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is defined by $f(x) = ||x||^2$ then $x = 0$ is the only critical value of f and we find $f^{-1}(-\infty, R^2]$ is a submanifold with boundary $f^{-1}\{R^2\}$. Note that $f^{-1}(-\infty, 0) = \emptyset$ in this case. However, perhaps you also see $B^m = f^{-1}[0, R^2]$ is the closed m -ball and $\partial B^m = S_{m-1}(R)$ is the $(m-1)$ -sphere of radius R .

Theorem 8.8.8.

Let \mathcal{M} be a smooth manifold with boundary $\partial\mathcal{M}$ and \mathcal{N} a smooth manifold without boundary. If $f : \mathcal{M} \rightarrow \mathcal{N}$ and $f|_{\partial\mathcal{M}} : \partial\mathcal{M} \rightarrow \mathcal{N}$ have regular value $q \in \mathcal{N}$ then $f^{-1}\{q\}$ is a smooth $(m-n)$ -dimensional manifold with boundary $f^{-1}\{q\} \cap \partial\mathcal{M}$.

Proof: see page 50 of Burns and Gidea. \square .

This theorem would seem to give us a generalization of the implicit function theorem for some closed sets. Interesting. Finally, I should mention that it is customary to also allow use the set $\mathbb{L}^1 = \{x \in \mathbb{R} \mid x \leq 0\}$ as the domain of a parametrization in the case of one-dimensional manifolds.

Chapter 9

differential forms

9.1 algebra of differential forms

In this section we apply the results of the previous section on exterior algebra to the vector space $V = T_p M$. Recall that $\{\frac{\partial}{\partial x^i}|_p\}$ is a basis of $T_p M$ and thus the basis $\{e_i\}$ of V utilized throughout the previous section on exterior algebra will be taken to be

$$e_i = \frac{\partial}{\partial x^i}|_p, \quad 1 \leq i \leq n$$

in this section. Also recall that the set of covectors $\{dx^i\}$ is a basis of $T_p^* M$ which is dual to $\{\frac{\partial}{\partial x^i}|_p\}$ and consequently the $\{e^j\}$ in the previous section is taken to be

$$e^j = dx^j, \quad 1 \leq j \leq n$$

in the present context. With these choices the machinery of the previous section takes over and one obtains a vector space $\wedge^k(T_p M)$ for each $1 \leq k$ and for arbitrary $p \in M$. We write $\wedge^k TM$ for the set of ordered pairs (p, α) where $p \in M$ and $\alpha \in \wedge^k(T_p M)$ and we refer to $\wedge^k(TM)$ as the k -th exterior power of the tangent bundle TM . There is a projection $\pi : \wedge^k(TM) \rightarrow M$ defined by $\pi(p, \alpha) = p$ for $(p, \alpha) \in \wedge^k(TM)$. One refers to $(\wedge^k TM, \pi)$ as a vector bundle for reasons we do not pursue at this point. To say that $\hat{\alpha}$ is a section of this vector bundle means that $\hat{\alpha} : M \rightarrow \wedge^k(TM)$ is a (smooth) function such that $\hat{\alpha}(p) \in \wedge^k(T_p M)$ for all $p \in M$. Such functions are also called differential forms, or in this case, k-forms.

Definition 9.1.1. *vector field on open subset of \mathbb{R}^n .*

To say that X is a vector field on an open subset U of M means that

$$X = X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2} + \cdots + X^n \frac{\partial}{\partial x^n}$$

where X^1, X^2, \dots, X^n are smooth functions from U into \mathbf{R} .

Note that in this context we implicitly require that differential forms be smooth. To explain this we write out the requirements more fully below.

If β is a function with domain M such that for each $p \in M$, $\beta(p) \in \wedge^k(T_p M)$ then β is called a differential k-form on M if for all local vector fields X_1, X_2, \dots, X_k defined on an arbitrary open subset U of M it follows that the map defined by

$$p \rightarrow \beta_p(X_1(p), X_2(p), \dots, X_k(p))$$

is smooth on U . For example if (x^1, x^2, \dots, x^n) is a chart then its domain $V \subset M$ is open in M and the map

$$p \rightarrow d_p x^i$$

is a differential 1-form on U . Similarly the map

$$p \rightarrow d_p x^i \wedge d_p x^j$$

is a differential 2-form on U . Generally if β is a 1-form and (x^1, x^2, \dots, x^n) is a chart then there are functions (b_i) defined on the domain of x such that

$$\beta(q) = \sum_{\mu} b_{\mu}(q) d_q x^{\mu}$$

for all q in the domain of x .

Similarly if γ is a 2-form on M and $x = (x^1, x^2, \dots, x^n)$ is any chart on M then there are smooth functions c_{ij} on $\text{dom}(x)$ such that

$$\gamma_p = \frac{1}{2} \sum_{i,j=1}^n c_{ij}(p) (d_p x^i \wedge d_p x^j)$$

and such that $c_{ij}(p) = -c_{ji}(p)$ for all $p \in \text{dom}(x)$.

Generally if α is a k -form and x is a chart then on $\text{dom}(x)$

$$\alpha_p = \sum \frac{1}{k!} a_{i_1 i_2 \dots i_k}(p) (d_p x^{i_1} \wedge \dots \wedge d_p x^{i_k})$$

where the $\{a_{i_1 i_2 \dots i_k}\}$ are smooth real-valued functions on $U = \text{dom}(x)$ and $\alpha_{i_{\sigma_1} i_{\sigma_2} \dots i_{\sigma_k}} = \text{sgn}(\sigma) a_{i_1 i_2 \dots i_k}$, for every permutation σ . (*this is just a fancy way of saying if you switch any pair of indices it generates a minus sign*).

The algebra of differential forms follows the same rules as the exterior algebra we previously discussed. Remember, a differential form evaluated at a particular point gives us a wedge product of a bunch of dual vectors. It follows that the differential form in total also follows the general properties of the exterior algebra.

Theorem 9.1.2.

If α is a p -form, β is a k -form, and γ is a l -form on M then

1. $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$
2. $\alpha \wedge \beta = (-1)^{pk}(\beta \wedge \alpha)$
3. $\alpha \wedge (a\beta + b\gamma) = a(\alpha \wedge \beta) + b(\alpha \wedge \gamma) \quad a, b \in \mathbb{R}$

Notice that in \mathbb{R}^3 the set of differential forms

$$\mathcal{B} = \{1, dx, dy, dz, dy \wedge dz, dz \wedge dx, dx \wedge dy, dx \wedge dy \wedge dz\}$$

is a basis of the space of differential forms in the sense that every form on \mathbb{R}^3 is a linear combination of the forms in \mathcal{B} with smooth real-valued functions on \mathbb{R}^3 as coefficients.

Example 9.1.3. Let $\alpha = f dx + g dy$ and let $\beta = 3 dx + dz$ where f, g are functions. Find $\alpha \wedge \beta$, write the answer in terms of the basis defined in the Remark above,

$$\begin{aligned} \alpha \wedge \beta &= (f dx + g dy) \wedge (3 dx + dz) \\ &= f dx \wedge (3 dx + dz) + g dy \wedge (3 dx + dz) \\ &= 3 f dx \wedge dx + f dx \wedge dz + 3 g dy \wedge dx + g dy \wedge dz \\ &= -g dy \wedge dz - f dz \wedge dx - 3 g dx \wedge dy \end{aligned} \tag{9.1}$$

Example 9.1.4. Top form: Let $\alpha = dx \wedge dy \wedge dz$ and let β be any other form with degree $p > 0$. We argue that $\alpha \wedge \beta = 0$. Notice that if $p > 0$ then there must be at least one differential inside β so if that differential is dx^k we can rewrite $\beta = dx^k \wedge \gamma$ for some γ . Then consider,

$$\alpha \wedge \beta = dx \wedge dy \wedge dz \wedge dx^k \wedge \gamma \tag{9.2}$$

now k has to be either 1, 2 or 3 therefore we will have dx^k repeated, thus the wedge product will be zero. (can you prove this?).

9.2 exterior derivatives: the calculus of forms

The operation \wedge depends only on the values of the forms point by point. We define an operator d on differential forms which depends not only on the value of the differential form at a point but on its value in an entire neighborhood of the point. Thus if β is a k -form then to define $d\beta$ at a point p we need to know not only the value of β at p but we also need to know its value at every q in a neighborhood of p .

You might note the derivative below does not directly involve the construction of differential forms from tensors. Also, the rule given below is easily taken as a starting point for formal calculations. In other words, even if you don't understand the nuts and bolts of manifold theory you can still calculate with differential forms. In the same sense that highschool students "do" calculus, you can "do" differential form calculations. I don't believe this is a futile exercise so long as you understand you have more to learn. Which is not to say we don't know some things!

Definition 9.2.1. *the exterior derivative.*

If β is a k -form and $x = (x^1, x^2, \dots, x^n)$ is a chart and $\beta = \sum_I \frac{1}{k!} \beta_I dx^I$ and we define a $(k+1)$ -form $d\beta$ to be the form

$$d\beta = \sum_I \frac{1}{k!} d\beta_I \wedge dx^I .$$

Where $d\beta_I$ is defined as it was in calculus III,

$$d\beta_I = \sum_{j=1}^n \frac{\partial \beta_I}{\partial x_j} dx^j .$$

Note that $d\beta_I$ is well-defined as

$$\beta_I = \beta_{i_1 i_2 \dots i_k}$$

is just a real-valued function on $\text{dom}(x)$. The definition in an expanded form is given by

$$d_p \beta = \frac{1}{k!} \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_k=1}^n (d_p \beta_{i_1 i_2 \dots i_k}) \wedge d_p x^{i_1} \wedge \cdots \wedge d_p x^{i_k}$$

where

$$\beta_q = \frac{1}{k!} \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_k=1}^n \beta_{i_1 i_2 \dots i_k}(q) d_q x^{i_1} \wedge \cdots \wedge d_q x^{i_k} .$$

9.2.1 coordinate independence of exterior derivative

The Einstein summation convention is used in this section and throughout the remainder of this chapter, please feel free to email me if it confuses you somewhere. When an index is repeated in a single summand it is implicitly assumed there is a sum over all values of that index

It must be shown that this definition is independent of the chart used to define $d\beta$. Suppose for example, that

$$\beta_q = \bar{\beta}_J(q) (d_q \bar{x}^{j_1} \wedge \cdots \wedge d_q \bar{x}^{j_k})$$

for all q in the domain of a chart $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ where

$$\text{dom}(x) \cap \text{dom}(\bar{x}), \neq \emptyset .$$

We assume, of course that the coefficients $\{\bar{\beta}_J(q)\}$ are skew-symmetric in J for all q . We will have defined $d\beta$ in this chart by

$$d\beta = d\bar{\beta}_J \wedge d\bar{x}^J .$$

We need to show that $d_p \bar{\beta}_J \wedge d_p \bar{x}^J = d_p \beta_I \wedge d_p x^I$ for all $p \in \text{dom}(x) \cap \text{dom}(\bar{x})$ if this definition is to be meaningful. Since β is given to be a well-defined form we know

$$\beta_I(p) d_p x^I = \beta_p = \bar{\beta}_J(p) d_p \bar{x}^J.$$

Using the identities

$$d\bar{x}^j = \frac{\partial \bar{x}^j}{\partial x^i} dx^i$$

we have

$$\beta_I dx^I = \bar{\beta}_J \frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{j_k}}{\partial x^{i_k}} \cdots \frac{\partial \bar{x}^{j_k}}{\partial x^{i_k}} dx^I$$

so that

$$\beta_I = \bar{\beta}_J \left(\frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}} \frac{\partial \bar{x}^{j_2}}{\partial x^{i_2}} \cdots \frac{\partial \bar{x}^{j_k}}{\partial x^{i_k}} \right).$$

Consequently,

$$\begin{aligned} d\beta_J \wedge dx^J &= \frac{\partial \beta_J}{\partial x^\lambda} (dx^\lambda \wedge dx^J) = \frac{\partial}{\partial x^\lambda} [\bar{\beta}_I \left(\frac{\partial \bar{x}^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial \bar{x}^{i_k}}{\partial x^{j_k}} \right)] (dx^\lambda \wedge dx^J) \\ &\stackrel{*}{=} \frac{\partial \bar{\beta}_I}{\partial x^\lambda} \left(\frac{\partial \bar{x}^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial \bar{x}^{i_k}}{\partial x^{j_k}} \right) (dx^\lambda \wedge dx^J) \\ &\quad + \bar{\beta}_I \sum_r \left(\frac{\partial \bar{x}^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial^2 \bar{x}^{i_r}}{\partial x^\lambda \partial x^{j_r}} \cdots \frac{\partial \bar{x}^{i_k}}{\partial x^{j_k}} \right) (dx^\lambda \wedge dx^J) \\ &= \frac{\partial \bar{\beta}_I}{\partial x^\lambda} dx^\lambda \wedge \left(\frac{\partial \bar{x}^{i_1}}{\partial x^{j_1}} dx^{j_1} \right) \wedge \cdots \wedge \left(\frac{\partial \bar{x}^{i_k}}{\partial x^{j_k}} dx^{j_k} \right) \\ &= \frac{\partial \bar{\beta}_I}{\partial \bar{x}^p} \left[\left(\frac{\partial \bar{x}^p}{\partial x^\lambda} dx^\lambda \right) \wedge d\bar{x}^{i_1} \wedge \cdots \wedge d\bar{x}^{i_k} \right] \\ &= d\bar{\beta}_I \wedge d\bar{x}^I \end{aligned}$$

where in (*) the sum \sum_r is zero since:

$$\frac{\partial^2 \bar{x}^{i_r}}{\partial x^\lambda \partial x^{j_r}} (dx^\lambda \wedge dx^J) = \pm \frac{\partial^2 \bar{x}^{i_r}}{\partial x^\lambda \partial x^{j_r}} [(dx^\lambda \wedge dx^{j_r}) \wedge dx^{j_1} \wedge \cdots \wedge \widehat{dx^{j_r}} \wedge \cdots \wedge dx^{j_k}] = 0.$$

It follows that $d\beta$ is independent of the coordinates used to define it.

Consequently we see that for each k the operator d maps $\wedge^k(M)$ into $\wedge^{k+1}(M)$ and has the following properties:

Theorem 9.2.2. *properties of the exterior derivative.*

If $\alpha \in \wedge^k(M)$, $\beta \in \wedge^l(M)$ and $a, b \in \mathbf{R}$ then

1. $d(a\alpha + b\beta) = a(d\alpha) + b(d\beta)$
2. $d(\alpha \wedge \beta) = (d\alpha \wedge \beta) + (-1)^k(\alpha \wedge d\beta)$
3. $d(d\alpha) = 0$

Proof: The proof of (1) is obvious. To prove (2), let $x = (x^1, \dots, x^n)$ be a chart on M then (ignoring the factorial coefficients)

$$\begin{aligned}
 d(\alpha \wedge \beta) &= d(\alpha_I \beta_J) \wedge dx^I \wedge dx^J = (\alpha_I d\beta_J + \beta_J d\alpha_I) \wedge dx^I \wedge dx^J \\
 &= \alpha_I (d\beta_J \wedge dx^I \wedge dx^J) \\
 &\quad + \beta_J (d\alpha_I \wedge dx^I \wedge dx^J) \\
 &= \alpha_I (dx^I \wedge (-1)^k (d\beta_J \wedge dx^J)) \\
 &\quad + \beta_J ((d\alpha_I \wedge dx^I) \wedge dx^J) \\
 &= (\alpha \wedge (-1)^k d\beta) + \beta_J (d\alpha \wedge dx^J) \\
 &= d\alpha \wedge \beta + (-1)^k (\alpha \wedge d\beta) .
 \end{aligned}$$

9.2.2 exterior derivatives on \mathbb{R}^3

We begin by noting that vector fields may correspond either to a one-form or to a two-form.

Definition 9.2.3. *dictionary of vectors versus forms on \mathbb{R}^3 .*

Let $\vec{A} = (A^1, A^2, A^3)$ denote a vector field in \mathbb{R}^3 . Define then,

$$\omega_A = \delta_{ij} A^i dx^j = A_i dx^i$$

which we will call the **work-form** of \vec{A} . Also define

$$\Phi_A = \frac{1}{2} \delta_{ik} A^k \epsilon_{ijk} (dx^i \wedge dx^j) = \frac{1}{2} A_i \epsilon_{ijk} (dx^i \wedge dx^j)$$

which we will call the **flux-form** of \vec{A} .

If you accept the primacy of differential forms, then you can see that vector calculus confuses two separate objects. Apparently there are two types of vector fields. In fact, if you have studied coordinate change for vector fields deeply then you will encounter the qualifiers **axial** or **polar** vector fields. Those fields which are axial correspond directly to two-forms whereas those correspondant to one-forms are called polar. As an example, the magnetic field is axial whereas the electric field is polar.

Example 9.2.4. Gradient: Consider three-dimensional Euclidean space. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ then

$$df = \frac{\partial f}{\partial x^i} dx^i = \omega_{\nabla f}$$

which gives the one-form corresponding to ∇f .

Example 9.2.5. Curl: Consider three-dimensional Euclidean space. Let \vec{F} be a vector field and let $\omega_F = F_i dx^i$ be the corresponding one-form then

$$\begin{aligned}
 d\omega_F &= dF_i \wedge dx^i \\
 &= \partial_j F_i dx^j \wedge dx^i \\
 &= \partial_x F_y dx \wedge dy + \partial_y F_x dy \wedge dx + \partial_z F_x dz \wedge dx + \partial_x F_z dx \wedge dz + \partial_y F_z dy \wedge dz + \partial_z F_y dz \wedge dy \\
 &= (\partial_x F_y - \partial_y F_x) dx \wedge dy + (\partial_z F_x - \partial_x F_z) dz \wedge dx + (\partial_y F_z - \partial_z F_y) dy \wedge dz \\
 &= \Phi_{\nabla \times \vec{F}} .
 \end{aligned}$$

Thus we recover the curl.

Example 9.2.6. Divergence: Consider three-dimensional Euclidean space. Let \vec{G} be a vector field and let $\Phi_G = \frac{1}{2}\epsilon_{ijk}G_i dx^j \wedge dx^k$ be the corresponding two-form then

$$\begin{aligned} d\Phi_G &= d(\frac{1}{2}\epsilon_{ijk}G_i) \wedge dx^j \wedge dx^k \\ &= \frac{1}{2}\epsilon_{ijk}(\partial_m G_i) dx^m \wedge dx^j \wedge dx^k \\ &= \frac{1}{2}\epsilon_{ijk}(\partial_m G_i)\epsilon_{mjk} dx \wedge dy \wedge dz \\ &= \frac{1}{2}2\delta_{im}(\partial_m G_i) dx \wedge dy \wedge dz \\ &= \partial_i G_i dx \wedge dy \wedge dz \\ &= (\nabla \cdot \vec{G}) dx \wedge dy \wedge dz \end{aligned}$$

and in this way we recover the divergence.

9.3 pullbacks

Another important operation one can perform on differential forms is the “pull-back” of a form under a map¹. The definition is constructed in large part by a sneaky application of the push-forward (aka differential) discussed in the preceding chapter.

Definition 9.3.1. *pull-back of a differential form.*

If $f : \mathcal{M} \rightarrow \mathcal{N}$ is a smooth map and $\omega \in \wedge^k(N)$ then $f^*\omega$ is the form on \mathcal{M} defined by

$$(f^*\omega)_p(X_1, \dots, X_k) = \omega_{f(p)}(d_p f(X_1), d_p f(X_2), \dots, d_p f(X_k)) .$$

for each $p \in \mathcal{M}$ and $X_1, X_2, \dots, X_k \in T_p \mathcal{M}$. Moreover, in the case $k = 0$ we have a smooth function $\omega : \mathcal{N} \rightarrow \mathbb{R}$ and the pull-back is accomplished by composition $(f^*\omega)(p) = (\omega \circ f)(p)$ for all $p \in \mathcal{M}$.

This operation is linear on forms and commutes with the wedge product and exterior derivative:

Theorem 9.3.2. *properties of the pull-back.*

If $f : M \rightarrow N$ is a C^1 -map and $\omega \in \wedge^k(N)$, $\tau \in \wedge^l(N)$ then

1. $f^*(a\omega + b\tau) = a(f^*\omega) + b(f^*\tau) \quad a, b \in \mathbb{R}$
2. $f^*(\omega \wedge \tau) = f^*\omega \wedge (f^*\tau)$
3. $f^*(d\omega) = d(f^*\omega)$

¹thanks to my advisor R.O. Fulp for the arguments that follow

Proof: The proof of (1) is clear. We now prove (2).

$$\begin{aligned}
f^*(\omega \wedge \tau)]_p(X_1, \dots, X_{k+l}) &= (\omega \wedge \tau)_{f(p)}(d_p f(X_1), \dots, d_p f(X_{k+l})) \\
&= \sum_{\sigma} (\text{sgn } \sigma) (\omega \otimes \tau)_{f(p)}(d_p f(X_{\sigma(1)}), \dots, d_p f(X_{\sigma(k+l)})) \\
&= \sum_{\sigma} \text{sgn}(\sigma) \omega(d_p f(X_{\sigma(1)}), \dots, d_p f(X_{\sigma(k)})) \tau(df(X_{\sigma(k+1)}) \cdots df(X_{\sigma(k+l)})) \\
&= \sum_{\sigma} \text{sgn}(\sigma) (f^* \omega)_p(X_{\sigma(1)}, \dots, X_{\sigma(k)}) (f^* \tau)_p(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}) \\
&= [(f^* \omega) \wedge (f^* \tau)]_p(X_1, X_2, \dots, X_{(k+l)})
\end{aligned}$$

Finally we prove (3).

$$\begin{aligned}
f^*(d\omega)]_p(X_1, X_2, \dots, X_{k+1}) &= (d\omega)_{f(p)}(df(X_1), \dots, df(X_{k+1})) \\
&= (d\omega_I \wedge dx^I)_{f(p)}(df(X_1), \dots, df(X_{k+1})) \\
&= \left(\frac{\partial \omega_I}{\partial x^\lambda} \Big|_{f(p)} \right) (dx^\lambda \wedge dx^I)_{f(p)}(df(X_1), \dots, df(X_{k+1})) \\
&= \left(\frac{\partial \omega_I}{\partial x^\lambda} \Big|_{f(p)} \right) [d_p(x^\lambda \circ f) \wedge d_p(x^I \circ f)](X_1, \dots, X_{k+1}) \\
&= [d(\omega_I \circ f) \wedge d(x^I \circ f)](X_1, \dots, X_{k+1}) \\
&= d[(\omega_I \circ f)_p d_p(x^I \circ f)](X_1, \dots, X_{k+1}) \\
&= d(f^* \omega)_p(X_1, \dots, X_{k+1}).
\end{aligned}$$

The theorem follows. \square .

We saw that one important application of the push-forward was to change coordinates for a given vector. Similar comments apply here. If we wish to change coordinates on a given differential form then we can use the pull-back. However, given the direction of the operation we need to use the inverse coordinate transformation to pull forms forward. Let me mirror the example from the last chapter for forms on \mathbb{R}^2 . We wish to convert from r, θ to x, y notation.

Example 9.3.3. Suppose $F : \mathbb{R}_{r,\theta}^2 \rightarrow \mathbb{R}_{x,y}^2$ is the polar coordinate transformation. In particular,

$$F(r, \theta) = (r \cos \theta, r \sin \theta)$$

The inverse transformation, at least for appropriate angles, is given by

$$F^{-1}(x, y) = \left(\sqrt{x^2 + y^2}, \tan^{-1}(y/x) \right).$$

Let calculate the pull-back of dr under F^{-1} : let $p = F^{-1}(q)$

$$F^{-1*}(dr)_q = d_p r(\partial_x|_p) d_p x + d_p r(\partial_y|_p) d_p y$$

Again, drop the annoying point-dependence to see this clearly:

$$F^{-1*}(dr) = dr(\partial_x) dx + dr(\partial_y) dy = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy$$

Likewise,

$$F^{-1*}(d\theta) = d\theta(\partial_x)dx + d\theta(\partial_y)dy = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy$$

Note that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ have the following partial derivatives:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} & \text{and} & & \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r} \\ \frac{\partial \theta}{\partial x} &= \frac{-y}{x^2 + y^2} = \frac{-y}{r^2} & \text{and} & & \frac{\partial \theta}{\partial y} &= \frac{x}{x^2 + y^2} = \frac{x}{r^2} \end{aligned}$$

Of course the expressions using r are pretty, but to make the point, we have changed into x, y -notation via the pull-back of the inverse transformation as advertised. We find:

$$\boxed{dr = \frac{xdx + ydy}{\sqrt{x^2 + y^2}} \quad \text{and} \quad d\theta = \frac{-ydx + xdy}{x^2 + y^2}.}$$

Once again we have found results with the pull-back that we might previously have chalked up to substitution in multivariate calculus. That's often the idea behind an application of the pull-back. It's just a formal language to be precise about a substitution. It takes us past simple symbol pushing and helps us think about where things are defined and how we may recast them to work together with other objects. I leave it at that for here.

9.4 integration of differential forms

The general strategy is generally as follows:

- (i) there is a natural way to calculate the integral of a k -form on a subset of \mathbb{R}^k
- (ii) given a k -form on a manifold we can locally pull it back to a subset of \mathbb{R}^k provided the manifold is an oriented² k -dimensional and thus by the previous idea we have an integral.
- (iii) globally we should expect that we can add the results from various local charts and arrive at a total value for the manifold, assuming of course the integral in each chart is finite.

We will only investigate items (i.) and (ii.) in these notes. There are many other excellent texts which take great effort to carefully expand on point (iii.) and I do not wish to replicate that effort here. You can read Edwards and see about *pavings*, or read Munkres' where he has at least 100 pages devoted to the careful study of multivariate integration. I do not get into those topics in my notes because we simply do not have sufficient analytical power to do them justice. I would encourage the student interested in deeper ideas of integration to find time to talk to Dr. Skoumbourdis, he has thought a long time about these matters and he really understands integration in a way we dare not cover in the calculus sequence. You really should have that conversation after you've taken

²we will discuss this as the section progresses

real analysis and have gained a better sense of what analysis' purpose is in mathematics. That said, what we do cover in this section and the next is fascinating whether or not we understand all the analytical underpinnings of the subject!

9.4.1 integration of k -form on \mathbb{R}^k

Note that on $U \subseteq \mathbb{R}^k$ a k -form α is the top form thus there exists some smooth function f on U such that $\alpha_x = f(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k$ for all $x \in U$. If D is a subset of U then we define the integral of α over D via the corresponding integral of k -variables in \mathbb{R}^k . In particular,

$$\int_D \alpha = \int_D f(x) d^k x$$

where on the r.h.s. the symbol $d^k x$ is meant to denote the usual integral of k -variables on \mathbb{R}^k . It is sometimes convenient to write such an integral as:

$$\int_D f(x) d^k x = \int_D f(x) dx^1 dx^2 \cdots dx^k$$

but, to be more careful, the integration of f over D is a quantity which is independent of the particular order in which the variables on \mathbb{R}^k are assigned. On the other hand, the order of the variables in the formula for α certainly can introduce signs. Note

$$\alpha_x = -f(x) dx^2 \wedge dx^1 \wedge \cdots \wedge dx^k.$$

How can we reasonably maintain the integral proposed above? Well, the answer is to make a convention that we must write the form to match the standard orientation of \mathbb{R}^k . The standard **orientation** of \mathbb{R}^k is given by $Vol_k = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k$. If the given form is written $\alpha_x = f(x)Vol_k$ then we define $\int_D \alpha = \int_D f(x) d^k x$. Since it is always possible to write a k -form as a function multiplying Vol_k on \mathbb{R}^k this definition suffices to cover all possible k -forms. I expand a few basic cases below:

Suppose $\alpha_x = f(x)dx$ on some subset $D = [a, b]$ of \mathbb{R} ,

$$\int_D \alpha = \int_D f(x)dx = \int_a^b f(x)dx.$$

Or, if $\alpha_{(x,y)} = f(x,y)dx \wedge dy$ then for D a subset of \mathbb{R}^2 ,

$$\int_D \alpha = \int_D f(x,y)dx dy = \int_D f dA.$$

If $\alpha_{(x,y,z)} = f(x,y,z)dx \wedge dy \wedge dz$ then for D a subset of \mathbb{R}^3 ,

$$\int_D \alpha = \int_D f(x,y,z)dx dy dz = \int_D f dV.$$

In practice we tend to break the integrals above down into an iterated integral thanks to Fubini's theorems. The integrals $\int_D f dA$ and $\int_D f dV$ are not in and of themselves dependent on orientation. However the set D may be oriented the value of those integrals are the same for a fixed function f . The orientation dependence of the form integral is completely wrapped up in our rule that the form must be written as a multiple of the volume form on the given space.

9.4.2 orientations and submanifolds

Given a k -manifold \mathcal{M} we say it is an **oriented** manifold iff all coordinates on \mathcal{M} are **consistently oriented**. If we make a choice and say $\phi_0 : U_0 \rightarrow V_0$ is **right-handed** then any overlapping patch $\phi_1 : U_1 \rightarrow V_1$ is said to be **right-handed** iff $\det(d\theta_{01}) > 0$. Otherwise, if $\det(d\theta_{01}) < 0$ then the patch $\phi_1 : U_1 \rightarrow V_1$ is said to be **left-handed**. If the manifold is **orientable** then as we continue to travel across the manifold we can choose coordinates such that on each overlap the transition functions satisfy $\det(d\theta_{ij}) > 0$. In this way we find an atlas for an orientable \mathcal{M} which is **right-handed**.

We can also say \mathcal{M} is oriented if there exists a nonzero volume-form on \mathcal{M} . If \mathcal{M} is k -dimensional then a volume form Vol is simply a nonzero k -form. At each point $p \in \mathcal{M}$ we can judge if a given coordinate system is left or right handed. We have to make a convention to be precise and I do so at this point. We assume Vol is positive and we say a coordinate system with chart (x^1, x^2, \dots, x^k) is **positively oriented** iff $Vol(\partial_1|_p, \partial_2|_p, \dots, \partial_k|_p) > 0$. If a coordinate system is not positively oriented then it is said to be negatively oriented and we will find $Vol(\partial_1|_p, \partial_2|_p, \dots, \partial_k|_p) < 0$ in that case. It is important that we suppose $Vol_p \neq 0$ at each $p \in \mathcal{M}$ since that is what allows us to demarcate coordinate systems as positive or negatively oriented.

Naturally, you are probably wondering: is a positively oriented coordinate system is the same idea as a right-handed coordinate system as defined above? To answer that we should analyze how the Vol changes coordinates on an overlap. Suppose we are given a positive volume form Vol and a point $p \in \mathcal{M}$ where two coordinate systems x and y are both defined. There must exist some function f such that

$$Vol_x = f(x) dx^1 \wedge dx^2 \wedge \dots \wedge dx^k$$

To change coordinates recall $dx^j = \sum_{j=1}^k \frac{\partial x^j}{\partial y^i} dy^i$ and substitute,

$$\begin{aligned} Vol &= \sum_{j_1, \dots, j_k=1}^k (f \circ x \circ y^{-1})(y) \frac{\partial x^1}{\partial y^{j_1}} \frac{\partial x^2}{\partial y^{j_2}} \dots \frac{\partial x^k}{\partial y^{j_k}} dy^{j_1} \wedge dy^{j_2} \wedge \dots \wedge dy^{j_k} \\ &= (f \circ x \circ y^{-1})(y) \det \left[\frac{\partial x}{\partial y} \right] dy^1 \wedge dy^2 \wedge \dots \wedge dy^k \end{aligned} \quad (9.3)$$

If you calculate the value of Vol on $\partial_{x^I}|_p = \partial_{x_1}|_p, \partial_{x_2}|_p, \dots, \partial_{x_k}|_p$ you'll find $Vol(\partial_{x^I}|_p) = f(x(p))$. Whereas, if you evaluate Vol on $\partial_{y^I}|_p = \partial_{y_1}|_p, \partial_{y_2}|_p, \dots, \partial_{y_k}|_p$ then the value is $Vol(\partial_{y^I}|_p) = f(x(p)) \det \left[\frac{\partial x}{\partial y}(p) \right]$. But, we should recognize that $\det \left[\frac{\partial x}{\partial y} \right] = \det(d\theta_{ij})$ hence two coordinate systems which are positively oriented must also be consistently oriented. Why? Assume $Vol(\partial_{x^I}|_p) =$

$f(x(p)) > 0$ then $Vol(\partial_{y^I}|_p) = f(x(p))det\left[\frac{\partial x}{\partial y}(p)\right] > 0$ iff $det\left[\frac{\partial x}{\partial y}(p)\right] > 0$ hence y is positively oriented if we are given that x is positively oriented **and** $det\left[\frac{\partial x}{\partial y}\right] > 0$.

Let \mathcal{M} be an oriented k -manifold with orientation given by the volume form Vol and an associated atlas of positively oriented charts. Furthermore, let α be a p -form defined on $V \subseteq \mathcal{M}$. Suppose there exists a local parametrization $\phi : U \subseteq \mathbb{R}^k \rightarrow V \subseteq \mathcal{M}$ and $D \subset V$ then there is a smooth function h such that $\alpha_q = h(q)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k$ for each $q \in V$. We define the integral of α over D as follows:

$$\int_D \alpha = \int_{\phi^{-1}(D)} h(\phi(x)) d^k x \quad \leftarrow [\star_x]$$

Is this definition dependent on the coordinate system $\phi : U \subseteq \mathbb{R}^k \rightarrow V \subseteq \mathcal{M}$? If we instead used coordinate system $\psi : \bar{U} \subseteq \mathbb{R}^k \rightarrow \bar{V} \subseteq \mathcal{M}$ where coordinates y^1, y^2, \dots, y^k on \bar{V} then the given form α has a different coefficient of $dy^1 \wedge dy^2 \wedge \cdots \wedge dy^k$

$$\alpha = h(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k = (h \circ x \circ y^{-1})(y)det\left[\frac{\partial x}{\partial y}\right] dy^1 \wedge dy^2 \wedge \cdots \wedge dy^k$$

Thus, as we change over to y coordinates the function picks up a factor which is precisely the determinant of the derivative of the transition functions.

$$\begin{aligned} \int_D \alpha &= \int_D (h \circ x \circ y^{-1})(y)det\left[\frac{\partial x}{\partial y}\right] dy^1 \wedge dy^2 \wedge \cdots \wedge dy^k \\ &= \int_{\psi^{-1}(D)} (h \circ x \circ y^{-1})(y)det\left[\frac{\partial x}{\partial y}\right] d^k y \quad \leftarrow [\star_y] \end{aligned}$$

We need $\star_x = \star_y$ in order for the integral $\int_D \alpha$ to be well-defined. Fortunately, the needed equality is *almost* provided by the change of variables theorem for multivariate integrals on \mathbb{R}^k . Recall,

$$\int_R f(x) d^k x = \int_{\bar{R}} \tilde{f}(y) \left| det \frac{\partial x}{\partial y} \right| d^k y$$

where \tilde{f} is more pedantically written as $\tilde{f} = f \circ y^{-1}$, notation aside its just the function f written in terms of the new y -coordinates. Likewise, \bar{R} limits y -coordinates so that the corresponding x -coordinates are found in R . Applying this theorem to our pull-back expression,

$$\int_{\phi^{-1}(D)} h(\phi(x)) d^k x = \int_{\psi^{-1}(D)} (h \circ x \circ y^{-1})(y) \left| det \left[\frac{\partial x}{\partial y} \right] \right| d^k y.$$

Equality of \star_x and \star_y follows from the fact that \mathcal{M} is oriented and has transition functions³ θ_{ij} which satisfy $det(d\theta_{ij}) > 0$. We see that this integral to be well-defined only for oriented manifolds. To integrate over manifolds without an orientation additional ideas are needed, but it is possible.

³once more recall the notation $\frac{\partial x}{\partial y}$ is just the matrix of the linear transformation $d\theta_{ij}$ and the determinant of a linear transformation is the determinant of the matrix of the transformation

Perhaps the most interesting case to consider is that of an embedded k -manifold in \mathbb{R}^n . In this context we must deal with both the coordinates of the ambient \mathbb{R}^n and the local parametrizations of the k -manifold. In multivariate calculus we often consider vector fields which are defined on an open subset of \mathbb{R}^3 and then we calculate the flux over a surfaces or the work along a curve. What we have defined thus-far is in essence like definition how to integrate a vector field on a surface or a vector field along a curve, no mention of the vector field off the domain of integration was made. We supposed the forms were already defined on the oriented manifold, but, what if we are instead given a formula for a differential form on \mathbb{R}^n ? How can we restrict that differential form to a surface or line or more generally a parametrized k -dimensional submanifold of \mathbb{R}^n ? That is the problem we concern ourselves with for the remainder of this section.

Let's begin with a simple object. Consider a one-form $\alpha = \sum_{i=1}^n \alpha_i dx^i$ where the function $p \rightarrow \alpha_i(p)$ is smooth on some subset of \mathbb{R}^n . Suppose C is a curve parametrized by $X : D \subseteq \mathbb{R} \rightarrow C \subseteq \mathbb{R}^n$ then the natural chart on C is provided by the parameter t in particular we have $T_p C = \text{span}\{\frac{\partial}{\partial t}|_{t_o}\}$ where $X(t_o) = p$ and $T_p C^* = \text{span}\{d_{t_o} t\}$ hence a vector field along C has the form $f(t)\frac{\partial}{\partial t}$ and a differential form has the form $g(t)dt$. How can we use the one-form α on \mathbb{R}^n to naturally obtain a one-form defined along C ? I propose:

$$\alpha|_C(t) = \sum_{i=1}^n \alpha_i(X(t)) \frac{\partial X^i}{\partial t} dt$$

It can be shown that $\alpha|_C$ is a one-form on C . If we change coordinates on the curve by reparametrizing $t \rightarrow s$ it then the component relative to s vs. the component relative to t are related:

$$\sum_{i=1}^n \alpha_i(X(t(s))) \frac{\partial X^i}{\partial s} = \sum_{i=1}^n \alpha_i(X(t)) \frac{dt}{ds} \frac{\partial X^i}{\partial t} = \frac{dt}{ds} \left(\sum_{i=1}^n \alpha_i(X(t)) \frac{\partial X^i}{\partial t} \right)$$

This is precisely the transformation rule we want for the components of a one-form.

Example 9.4.1. Suppose $\alpha = dx + 3x^2 dy + y dz$ and C is the curve $X : \mathbb{R} \rightarrow C \subseteq \mathbb{R}^3$ defined by $X(t) = (1, t, t^2)$ we have $x = 1$, $y = t$ and $z = t^2$ hence $dx = 0$, $dy = dt$ and $dz = 2t dt$ on C hence $\alpha|_C = 0 + 3dt + t(2t dt) = (3 + 2t^2)dt$.

Next, consider a two-form $\beta = \sum_{i,j=1}^n \frac{1}{2} \beta_{ij} dx^i \wedge dx^j$. Once more we consider a parametrized submanifold of \mathbb{R}^n . In particular use the notation $X : D \subseteq \mathbb{R}^2 \rightarrow S \subseteq \mathbb{R}^n$ where u, v serve as coordinates on the surface S . We can write an arbitrary two-form on S in the form $h(u, v) du \wedge dv$ where $h : S \rightarrow \mathbb{R}$ is a smooth function on S . How should we construct $h(u, v)$ given β ? Again, I think the following formula is quite natural, honestly, what else would you do⁴?

$$\beta|_S(u, v) = \sum_{i,j=1}^n \beta_{ij}(X(u, v)) \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v} du \wedge dv$$

⁴include the $\frac{1}{2}$ you say?, we'll see why not soon enough

The coefficient function of $du \wedge dv$ is smooth because we assume β_{ij} is smooth on \mathbb{R}^n and the local parametrization is also assumed smooth so the functions $\frac{\partial X^i}{\partial u}$ and $\frac{\partial X^i}{\partial v}$ are smooth. Moreover, the component function has the desired coordinate change property with respect to a reparametrization of S . Suppose we reparametrize by s, t , then suppressing the point-dependence of β_{ij} ,

$$\beta|_S = \sum_{i,j=1}^n \beta_{ij} \frac{\partial Y^i}{\partial s} \frac{\partial Y^j}{\partial t} ds \wedge dt = \frac{du}{ds} \frac{dv}{dt} \sum_{i,j=1}^n \beta_{ij} \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v} ds \wedge dt = \sum_{i,j=1}^n \beta_{ij} \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v} du \wedge dv.$$

Therefore, the restriction of β to S is coordinate independent and we have thus constructed a two-form on a surface from the two-form in the ambient space.

Example 9.4.2. Consider $\beta = y^2 dt \wedge dx + z dx \wedge dy + (x + y + z + t) dt \wedge dz$. Suppose $S \subseteq \mathbb{R}^4$ is parametrized by

$$X(u, v) = (1, u^2 v^2, 3u, v)$$

In other words, we are given that

$$t = 1, x = u^2 v^2, y = 3u, z = v$$

Hence, $dt = 0$, $dx = 2uv^2 du + 2u^2 v dv$, $dy = 3du$ and $dz = dv$. Computing $\beta|_S$ is just a matter of substituting in all the formulas above, fortunately $dt = 0$ so only the $z dx \wedge dy$ term is nontrivial:

$$\beta|_S = v(2uv^2 du + 2u^2 v dv) \wedge (3du) = 6u^2 v^2 dv \wedge du = -6u^2 v^2 du \wedge dv.$$

It is fairly clear that we can restrict any p -form on \mathbb{R}^n to a p -dimensional parametrized submanifold by the procedure we explained above for $p = 1, 2$. That is the underlying idea in the definitions which follow. Beyond that, once we have restricted the p -form β on \mathbb{R}^n to $\beta|_{\mathcal{M}}$ then we pull-back the restricted form to an open subset of \mathbb{R}^p and reduce the problem to an ordinary multivariate integral.

Our goal in the remainder of the section is to make contact with the⁵ integrals we study in calculus. Note that an embedded manifold with a single patch is almost trivially oriented since there is no overlap to consider. In particular, if $\phi : U \subseteq \mathbb{R}^k \rightarrow \mathcal{M} \subseteq \mathbb{R}^n$ is a local parametrization with $\phi^{-1} = (u^1, u^2, \dots, u^k)$ then $du^1 \wedge du^2 \wedge \dots \wedge du^k$ is a volume form for \mathcal{M} . This is the natural generalization of the normal-vector field construction for surfaces in \mathbb{R}^3 .

Definition 9.4.3. integral of one-form along oriented curve:

Let $\alpha = \alpha_i dx^i$ be a one form and let C be an oriented curve with parametrization $X(t) : [a, b] \rightarrow C$ then we define the integral of the one-form α along the curve C as follows,

$$\int_C \alpha \equiv \int_a^b \alpha_i(X(t)) \frac{dX^i}{dt}(t) dt$$

where $X(t) = (X^1(t), X^2(t), \dots, X^n(t))$ so we mean X^i to be the i^{th} component of $X(t)$. Moreover, the indices are understood to range over the dimension of the ambient space, if we consider forms in \mathbb{R}^2 then $i = 1, 2$ if in \mathbb{R}^3 then $i = 1, 2, 3$ if in Minkowski \mathbb{R}^4 then i should be replaced with $\mu = 0, 1, 2, 3$ and so on.

⁵hopefully known to you already from multivariate calculus

Example 9.4.4. One form integrals vs. line integrals of vector fields: We begin with a vector field \vec{F} and construct the corresponding one-form $\omega_{\vec{F}} = F_i dx^i$. Next let C be an oriented curve with parametrization $X : [a, b] \subset \mathbb{R} \rightarrow C \subset \mathbb{R}^n$, observe

$$\int_C \omega_{\vec{F}} = \int_a^b F_i(X(t)) \frac{dX^i}{dt}(t) dt = \int_C \vec{F} \cdot d\vec{l}$$

You may note that the definition of a line integral of a vector field is not special to three dimensions, we can clearly construct the line integral in n -dimensions, likewise the correspondance ω can be written between one-forms and vector fields in any dimension, provided we have a metric to lower the index of the vector field components. The same cannot be said of the flux-form correspondance, it is special to three dimensions for reasons we have explored previously.

Definition 9.4.5. integral of two-form over an oriented surface:

Let $\beta = \frac{1}{2} \beta_{ij} dx^i \wedge dx^j$ be a two-form and let S be an oriented piecewise smooth surface with parametrization $X(u, v) : D_2 \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^n$ then we define the integral of the two-form β over the surface S as follows,

$$\int_S \beta \equiv \int_{D_2} \beta_{ij}(X(u, v)) \frac{\partial X^i}{\partial u}(u, v) \frac{\partial X^j}{\partial v}(u, v) du dv$$

where $X(u, v) = (X^1(u, v), X^2(u, v), \dots, X^n(u, v))$ so we mean X^i to be the i^{th} component of $X(u, v)$. Moreover, the indices are understood to range over the dimension of the ambient space, if we consider forms in \mathbb{R}^2 then $i, j = 1, 2$ if in \mathbb{R}^3 then $i, j = 1, 2, 3$ if in Minkowski \mathbb{R}^4 then i, j should be replaced with $\mu, \nu = 0, 1, 2, 3$ and so on.

Example 9.4.6. Two-form integrals vs. surface integrals of vector fields in \mathbb{R}^3 : We begin with a vector field \vec{F} and construct the corresponding two-form $\Phi_{\vec{F}} = \frac{1}{2} \epsilon_{ijk} F_k dx^i \wedge dx^j$ which is to say $\Phi_{\vec{F}} = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$. Next let S be an oriented piecewise smooth surface with parametrization $X : D \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$, then

$$\int_S \Phi_{\vec{F}} = \int_S \vec{F} \cdot d\vec{A}$$

Proof: Recall that the normal to the surface S has the form,

$$N(u, v) = \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} = \epsilon_{ijk} \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v} e_k$$

at the point $X(u, v)$. This gives us a vector which points along the outward normal to the surface and it is nonvanishing throughout the whole surface by our assumption that S is oriented. Moreover the vector surface integral of \vec{F} over S was defined by the formula,

$$\int_S \vec{F} \cdot d\vec{A} \equiv \iint_D \vec{F}(X(u, v)) \cdot \vec{N}(u, v) du dv.$$

now that the reader is reminded what's what, lets prove the proposition, dropping the (u,v) dependence to reduce clutter we find,

$$\begin{aligned}
 \int_S \vec{F} \cdot d\vec{A} &= \int \int_D \vec{F} \cdot \vec{N} \, dudv \\
 &= \int \int_D F_k N_k \, dudv \\
 &= \int \int_D F_k \epsilon_{ijk} \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v} \, dudv \\
 &= \int \int_D (\Phi_{\vec{F}})_{ij} \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v} \, dudv \\
 &= \int_S \Phi_{\vec{F}}
 \end{aligned}$$

notice that we have again used our convention that $(\Phi_{\vec{F}})_{ij}$ refers to the tensor components of the 2-form $\Phi_{\vec{F}}$ meaning we have $\Phi_{\vec{F}} = (\Phi_{\vec{F}})_{ij} dx^i \otimes dx^j$ whereas with the wedge product $\Phi_{\vec{F}} = \frac{1}{2}(\Phi_{\vec{F}})_{ij} dx^i \wedge dx^j$, I mention this in case you are concerned there is a half in $\Phi_{\vec{F}}$ yet we never found a half in the integral. Well, we don't expect to because we defined the integral of the form with respect to the tensor components of the form, again they don't contain the half.

Example 9.4.7. Consider the vector field $\vec{F} = (0, 0, 3)$ then the corresponding two-form is simply $\Phi_F = 3dx \wedge dy$. Lets calculate the surface integral and two-form integrals over the square $D = [0, 1] \times [0, 1]$ in the xy -plane, in this case the parameters can be taken to be x and y so $X(x, y) = (x, y)$ and,

$$N(x, y) = \frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$$

which is nice. Now calculate,

$$\begin{aligned}
 \int_S \vec{F} \cdot d\vec{A} &= \int \int_D \vec{F} \cdot \vec{N} \, dxdy \\
 &= \int \int_D (0, 0, 3) \cdot (0, 0, 1) dxdy \\
 &= \int_0^1 \int_0^1 3 dxdy \\
 &= 3.
 \end{aligned}$$

Consider that $\Phi_F = 3dx \wedge dy = 3dx \otimes dy - 3dy \otimes dx$ therefore we may read directly that $(\Phi_F)_{12} =$

$-(\Phi_F)_{21} = 3$ and all other components are zero,

$$\begin{aligned}
 \int_S \Phi_F &= \iint_D (\Phi_F)_{ij} \frac{\partial X^i}{\partial x} \frac{\partial X^j}{\partial y} dx dy \\
 &= \iint_D \left(3 \frac{\partial X^1}{\partial x} \frac{\partial X^2}{\partial y} - 3 \frac{\partial X^2}{\partial x} \frac{\partial X^1}{\partial y} \right) dx dy \\
 &= \int_0^1 \int_0^1 \left(3 \frac{\partial x}{\partial x} \frac{\partial y}{\partial y} - 3 \frac{\partial y}{\partial x} \frac{\partial x}{\partial y} \right) dx dy \\
 &= 3.
 \end{aligned}$$

Definition 9.4.8. integral of a three-form over an oriented volume:

Let $\gamma = \frac{1}{6} \beta_{ijk} dx^i \wedge dx^j \wedge dx^k$ be a three-form and let V be an oriented piecewise smooth volume with parametrization $X(u, v, w) : D_3 \subset \mathbb{R}^3 \rightarrow V \subset \mathbb{R}^n$ then we define the integral of the three-form γ in the volume V as follows,

$$\int_V \gamma \equiv \int_{D_3} \gamma_{ijk}(X(u, v, w)) \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v} \frac{\partial X^k}{\partial w} du dv dw$$

where $X(u, v, w) = (X^1(u, v, w), X^2(u, v, w), \dots, X^n(u, v, w))$ so we mean X^i to be the i^{th} component of $X(u, v, w)$. Moreover, the indices are understood to range over the dimension of the ambient space, if we consider forms in \mathbb{R}^3 then $i, j, k = 1, 2, 3$ if in Minkowski \mathbb{R}^4 then i, j, k should be replaced with $\mu, \nu, \sigma = 0, 1, 2, 3$ and so on.

Finally we define the integral of a p -form over an p -dimensional subspace of \mathbb{R} , we assume that $p \leq n$ so that it is possible to embed such a subspace in \mathbb{R} ,

Definition 9.4.9. integral of a p-form over an oriented volume:

Let $\gamma = \frac{1}{p!} \beta_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$ be a p -form and let S be an oriented piecewise smooth subspace with parametrization $X(u_1, \dots, u_p) : D_p \subset \mathbb{R}^p \rightarrow S \subset \mathbb{R}^n$ (for $n \geq p$) then we define the integral of the p -form γ in the subspace S as follows,

$$\int_S \gamma \equiv \int_{D_p} \beta_{i_1 \dots i_p}(X(u_1, \dots, u_p)) \frac{\partial X^{i_1}}{\partial u_1} \dots \frac{\partial X^{i_p}}{\partial u_p} du_1 \dots du_p$$

where $X(u_1, \dots, u_p) = (X^1(u_1, \dots, u_p), X^2(u_1, \dots, u_p), \dots, X^n(u_1, \dots, u_p))$ so we mean X^i to be the i^{th} component of $X(u_1, \dots, u_p)$. Moreover, the indices are understood to range over the dimension of the ambient space.

Integrals of forms play an important role in modern physics. I hope you can begin to appreciate that forms recover all the formulas we learned in multivariate calculus and give us a way to extend calculation into higher dimensions with ease. I include a toy example at the conclusion of this chapter just to show you how electromagnetism is easily translated into higher dimensions.

9.5 Generalized Stokes Theorem

The generalized Stokes theorem contains within it most of the main theorems of integral calculus, namely the fundamental theorem of calculus, the fundamental theorem of line integrals (a.k.a the FTC in three dimensions), Greene's Theorem in the plane, Gauss' Theorem and also Stokes Theorem, not to mention a myriad of higher dimensional not so commonly named theorems. The breadth of its application is hard to overstate, yet the statement of the theorem is simple,

Theorem 9.5.1. *Generalized Stokes Theorem:*

Let S be an oriented, piecewise smooth $(p+1)$ -dimensional subspace of \mathbb{R}^n where $n \geq p+1$ and let ∂S be its boundary which is consistently oriented then for a p -form α which behaves reasonably on S we have that

$$\int_S d\alpha = \int_{\partial S} \alpha$$

The proof of this theorem (and a more careful statement of it) can be found in a number of places, Susan Colley's *Vector Calculus* or Steven H. Weintraub's *Differential Forms: A Complement to Vector Calculus* or Spivak's *Calculus on Manifolds* just to name a few. I believe the argument in Edward's text is quite complete. In any event, you should already be familiar with the idea from the usual Stokes Theorem where we must insist the boundary curve to the surface is related to the surface's normal field according to the right-hand-rule. Explaining how to orient the boundary ∂M given an oriented M is the problem of generalizing the right-hand-rule to many dimensions. I leave it to your homework for the time being.

Lets work out how this theorem reproduces the main integral theorems of calculus.

Example 9.5.2. Fundamental Theorem of Calculus in \mathbb{R} : Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a zero-form then consider the interval $[a, b]$ in \mathbb{R} . If we let $S = [a, b]$ then $\partial S = \{a, b\}$. Further observe that $df = f'(x)dx$. Notice by the definition of one-form integration

$$\int_S df = \int_a^b f'(x)dx$$

However on the other hand we find (the integral over a zero-form is taken to be the evaluation map, perhaps we should have defined this earlier, oops., but its only going to come up here so I'm leaving it.)

$$\int_{\partial S} f = f(b) - f(a)$$

Hence in view of the definition above we find that

$$\int_a^b f'(x)dx = f(b) - f(a) \iff \int_S df = \int_{\partial S} f$$

Example 9.5.3. Fundamental Theorem of Calculus in \mathbb{R}^3 : Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a zero-form then consider a curve C from $p \in \mathbb{R}^3$ to $q \in \mathbb{R}^3$ parametrized by $\phi : [a, b] \rightarrow \mathbb{R}^3$. Note that $\partial C = \{\phi(a) = p, \phi(b) = q\}$. Next note that

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Then consider that the exterior derivative of a function corresponds to the gradient of the function thus we are not to surprised to find that

$$\int_C df = \int_a^b \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} dt = \int_C (\nabla f) \cdot d\vec{l}$$

On the other hand, we use the definition of the integral over a two point set again to find

$$\int_{\partial C} f = f(q) - f(p)$$

Hence if the Generalized Stokes Theorem is true then so is the FTC in three dimensions,

$$\int_C (\nabla f) \cdot d\vec{l} = f(q) - f(p) \iff \int_C df = \int_{\partial C} f$$

another popular title for this theorem is the "fundamental theorem for line integrals". As a final thought here we notice that this calculation easily generalizes to 2,4,5,6,... dimensions.

Example 9.5.4. Greene's Theorem: Let us recall the statement of Greene's Theorem as I have not replicated it yet in the notes, let D be a region in the xy -plane and let ∂D be its consistently oriented boundary then if $\vec{F} = (M(x, y), N(x, y), 0)$ is well behaved on D

$$\int_{\partial D} Mdx + Ndy = \int \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

We begin by finding the one-form corresponding to \vec{F} namely $\omega_F = Mdx + Ndy$ consider then that

$$d\omega_F = d(Mdx + Ndy) = dM \wedge dx + dN \wedge dy = \frac{\partial M}{\partial y} dy \wedge dx + \frac{\partial N}{\partial x} dx \wedge dy$$

which simplifies to,

$$d\omega_F = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \wedge dy = \Phi_{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \hat{k}}$$

Thus, using our discussion in the last section we recall

$$\int_{\partial D} \omega_F = \int_{\partial D} \vec{F} \cdot d\vec{l} = \int_{\partial D} Mdx + Ndy$$

where we have reminded the reader that the notation in the rightmost expression is just another way of denoting the line integral in question. Next observe,

$$\int_D d\omega_F = \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \hat{k} \cdot d\vec{A}$$

And clearly, since $d\vec{A} = \hat{k}dxdy$ we have

$$\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \hat{k} \cdot d\vec{A} = \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Therefore,

$$\int_{\partial D} Mdx + Ndy = \int \int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \iff \int_D d\omega_F = \int_{\partial D} \omega_F$$

Example 9.5.5. Gauss Theorem: Let us recall Gauss Theorem to begin, for suitably defined \vec{F} and V ,

$$\int_{\partial V} \vec{F} \cdot d\vec{A} = \int_V \nabla \cdot \vec{F} d\tau$$

First we recall our earlier result that

$$d(\Phi_F) = (\nabla \cdot \vec{F}) dx \wedge dy \wedge dz$$

Now note that we may integrate the three form over a volume,

$$\int_V d(\Phi_F) = \int_V (\nabla \cdot \vec{F}) dxdydz$$

whereas,

$$\int_{\partial V} \Phi_F = \int_{\partial V} \vec{F} \cdot d\vec{A}$$

so there it is,

$$\int_V (\nabla \cdot \vec{F}) d\tau = \int_{\partial V} \vec{F} \cdot d\vec{A} \iff \int_V d(\Phi_F) = \int_{\partial V} \Phi_F$$

I have left a little detail out here, I may assign it for homework.

Example 9.5.6. Stokes Theorem: Let us recall Stokes Theorem to begin, for suitably defined \vec{F} and S ,

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{A} = \int_{\partial S} \vec{F} \cdot d\vec{l}$$

Next recall we have shown in the last chapter that,

$$d(\omega_F) = \Phi_{\nabla \times \vec{F}}$$

Hence,

$$\int_S d(\omega_F) = \int_S (\nabla \times \vec{F}) \cdot d\vec{A}$$

whereas,

$$\int_{\partial S} \omega_F = \int_{\partial S} \vec{F} \cdot d\vec{l}$$

which tells us that,

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{A} = \int_{\partial S} \vec{F} \cdot d\vec{l} \iff \int_S d(\omega_F) = \int_{\partial S} \omega_F$$

The Generalized Stokes Theorem is perhaps the most persuasive argument for mathematicians to be aware of differential forms, it is clear they allow for more deep and sweeping statements of the calculus. The generality of differential forms is what drives modern physicists to work with them, string theorists for example examine higher dimensional theories so they are forced to use a language more general than that of the conventional vector calculus. See the end of the next chapter for an example of such thinking.

9.6 poincaré's lemma and converse

This section is in large part inspired by M. Gockeler and T. Schucker's *Differential geometry, gauge theories, and gravity* page 20-22. The converse calculation is modelled on the argument found in H. Flanders *Differential Forms with Applications to the Physical Sciences*. The original work was done around the dawn of the twentieth century and can be found in many texts besides the two I mentioned here.

9.6.1 exact forms are closed

Proposition 9.6.1.

The exterior derivative of the exterior derivative is zero. $d^2 = 0$

Proof: Let α be an arbitrary p -form then

$$d\alpha = \frac{1}{p!}(\partial_m \alpha_{i_1 i_2 \dots i_p}) dx^m \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \quad (9.4)$$

then differentiate again,

$$\begin{aligned} d(d\alpha) &= d \left[\frac{1}{p!}(\partial_m \alpha_{i_1 i_2 \dots i_p}) dx^m \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \right] \\ &= \frac{1}{p!}(\partial_k \partial_m \alpha_{i_1 i_2 \dots i_p}) dx^k \wedge dx^m \wedge dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \\ &= 0 \end{aligned} \quad (9.5)$$

since the partial derivatives commute whereas the wedge product anticommutes so we note that the pair of indices (k,m) is symmetric for the derivatives but antisymmetric for the wedge, as we know the sum of symmetric against antisymmetric vanishes (see equation ?? part *iv* if you forgot.)

Definition 9.6.2.

A differential form α is **closed** iff $d\alpha = 0$. A differential form β is **exact** iff there exists γ such that $\beta = d\gamma$.

Proposition 9.6.3.

All exact forms are closed. However, there exist closed forms which are not exact.

Proof: Exact implies closed is easy, let β be exact such that $\beta = d\gamma$ then

$$d\beta = d(d\gamma) = 0$$

using the theorem $d^2 = 0$. To prove that there exists a closed form which is not exact it suffices to give an example. A popular example (due to its physical significance to magnetic monopoles, Dirac Strings and the like..) is the following differential form in \mathbb{R}^2

$$\phi = \frac{1}{x^2 + y^2}(xdy - ydx) \quad (9.6)$$

You may verify that $d\phi = 0$ in homework. Observe that if ϕ were exact then there would exist f such that $\phi = df$ meaning that

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2 + y^2}, \quad \frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2}$$

which are solved by $f = \arctan(y/x) + c$ where c is arbitrary. Observe that f is ill-defined along the y -axis $x = 0$ (this is the Dirac String if we put things in context), however the natural domain of ϕ is $\mathbb{R}^{n \times n} - \{(0, 0)\}$.

9.6.2 potentials for closed forms

Poincare' suggested the following partial converse, he said closed implies exact provided we place a topological restriction on the domain of the form. In particular, if the domain of a closed form is smoothly deformable to a point then each closed form is exact. We'll work out a proof of that result for a subset of \mathbb{R}^n . Be patient, we have to build some toys before we play.

Suppose $U \subseteq \mathbb{R}^n$ and $I = [0, 1]$ we denote a typical point in $I \times U$ as (t, x) where $t \in I$ and $x \in \mathbb{R}^n$. Define maps,

$$J_1 : U \rightarrow I \times U, \quad J_0 : U \rightarrow I \times U$$

by $J_1(x) = (1, x)$ and $J_0(x) = (0, x)$ for each $x \in U$. Flanders encourages us to view $I \times U$ as a cylinder and where the map J_1 maps U to the top and J_0 maps U to the base. We can pull-back forms on the cylinder to the U on the top ($t = 1$) or to the base ($t = 0$). For instance, if we consider $\omega = (x + t)dx + dt$ for the case $n = 1$ then

$$J_0^*\omega = xdx \quad J_1^*\omega = (x + 1)dx.$$

Define a smooth mapping K of $(p + 1)$ forms on $I \times U$ to p -forms on U as follows:

$$(1.) K(a(t, x)dx^I) = 0, \quad (2.) K(a(t, x)dt \wedge dx^J) = \left(\int_0^1 a(t, x)dt \right) dx^J$$

for multi-indices I of length $(p+1)$ and J of length p . The cases (1.) and (2.) simply divide the possible monomial⁶ inputs from $\Lambda^{p+1}(I \times U)$ into forms which have dt and those which don't. Then K is defined for a general $(p+1)$ -form on $I \times U$ by linearly extending the formulas above to multinomials of the basic monomials.

It turns out that the following identity holds for K :

Lemma 9.6.4. *the K -lemma.*

If ω is a differential form on $I \times U$ then

$$K(d\omega) + d(K(\omega)) = J_1^* \omega - J_0^* \omega.$$

Proof: Since the equation is given for linear operations it suffices to check the formula for monomials since we can extend the result linearly once those are affirmed. As in the definition of K there are two basic categories of forms on $I \times U$:

Case 1: If $\omega = a(t, x)dx^I$ then clearly $K(\omega) = 0$ hence $d(K(\omega)) = 0$. Observe,

$$d\omega = da \wedge dx^I = \sum_j \frac{\partial a}{\partial x^j} dx^j \wedge dx^I + \frac{\partial a}{\partial t} dt \wedge dx^I$$

Hence $K(d\omega)$ is calculated as follows:

$$\begin{aligned} K(d\omega) &= K\left(\sum_j \frac{\partial a}{\partial x^j} dx^j \wedge dx^I\right) + K\left(\frac{\partial a}{\partial t} dt \wedge dx^I\right) \\ &= \left(\int_0^1 \frac{\partial a}{\partial t} dt\right) dx^I \\ &= [a(x, 1) - a(x, 0)] dx^I \\ &= J_1^* \omega - J_0^* \omega \end{aligned} \tag{9.7}$$

where we used the FTC in the next to last step. The pull-backs in this case just amount to evaluation at $t = 0$ or $t = 1$ as there is no dt -type term to squash in ω . The identity follows.

Case 2: Suppose $\omega = a(t, x)dt \wedge dx^J$. Calculate,

$$d\omega = \sum_j \frac{\partial a}{\partial x^k} dx^k \wedge dt \wedge dx^J + \frac{\partial a}{\partial t} \underbrace{dt \wedge dt}_{\text{zero!}} \wedge dx^J$$

⁶ $dx \wedge dy$ is a monomial whereas $dx + dy$ is a binomial in this context

Thus, using $dx^k \wedge dt = -dt \wedge dx^k$, we calculate:

$$\begin{aligned} K(d\omega) &= K\left(-\sum_k \frac{\partial a}{\partial x^k} dt \wedge dx^k \wedge dx^I\right) \\ &= -\sum_k \left(\int_0^1 \frac{\partial a}{\partial x^k} dt\right) dx^k \wedge dx^I \end{aligned}$$

at which point we cannot proceed further since a is an arbitrary function which can include a nontrivial time-dependence. We turn to the calculation of $d(K(\omega))$. Recall we defined

$$K(\omega) = \left(\int_0^1 a(t, x) dt\right) dx^J.$$

We calculate the exterior derivative of $K(\omega)$:

$$\begin{aligned} d(K(\omega)) &= d\left(\int_0^1 a(t, x) dt\right) \wedge dx^J \\ &= \left(\frac{\partial}{\partial t} \underbrace{\left[\int_0^1 a(\tau, x) d\tau\right]}_{\text{constant in } t} dt + \sum_k \frac{\partial}{\partial x^k} \left[\int_0^1 a(t, x) dt\right] dx^k\right) \wedge dx^J \\ &= \sum_k \left(\int_0^1 \frac{\partial a}{\partial x^k} dt\right) dx^k \wedge dx^J. \end{aligned} \tag{9.8}$$

Therefore, $K(d\omega) + d(K(\omega)) = 0$ and clearly $J_0^*\omega = J_1^*\omega = 0$ in this case since the pull-backs squash the dt to zero. The lemma follows. \square .

Definition 9.6.5.

A subset $U \subseteq \mathbb{R}^n$ is deformable to a point P if there exists a smooth mapping $G : I \times U \rightarrow U$ such that $G(1, x) = x$ and $G(0, x) = P$ for all $x \in U$.

The map G deforms U smoothly into the point P . Recall that $J_1(x) = (1, x)$ and $J_0(x) = (0, x)$ hence the conditions on the deformation can be expressed as:

$$G(J_1(x)) = x \quad G(J_0(x)) = P$$

Denoting Id for the identity on U and P as the constant mapping with value P on U we have

$$G \circ J_1 = Id \quad G \circ J_0 = P$$

Therefore, if γ is a $(p+1)$ -form on U we calculate,

$$(G \circ J_1)^*\gamma = Id^*\gamma \Rightarrow J_1^*[G^*\gamma] = \gamma$$

whereas,

$$(G \circ J_0)^* \gamma = P^* \gamma = 0 \Rightarrow J_0^*[G^* \gamma] = 0$$

Apply the K -lemma to the form $\omega = G^* \gamma$ on $I \times U$ and we find:

$$K(d(G^* \gamma)) + d(K(G^* \gamma)) = \gamma.$$

However, recall that we proved that pull-backs and exterior derivatives commute thus

$$d(G^* \gamma) = G^*(d\gamma)$$

and we find an extremely interesting identity,

$$K(G^*(d\gamma)) + d(K(G^* \gamma)) = \gamma.$$

Proposition 9.6.6.

If $U \subseteq \mathbb{R}$ is deformable to a point P then a p -form γ on U is closed iff ϕ is exact.

Proof: Suppose γ is exact then $\gamma = d\beta$ for some $(p-1)$ -form β on U hence $d\gamma = d(d\beta) = 0$ by Proposition 9.6.1 hence γ is closed. Conversely, suppose γ is closed. Apply the boxed consequence of the K -lemma, note that $K(G^*(0)) = 0$ since we assume $d\gamma = 0$. We find,

$$d(K(G^* \gamma)) = \gamma$$

identify that $G^* \gamma$ is a p -form on $I \times U$ whereas $K(G^* \gamma)$ is a $(p-1)$ -form on U by the very construction of K . It follows that γ is exact since we have shown how it is obtained as the exterior derivative of another differential form of one degree less. \square

Where was deformability to a point P used in the proof above? The key is the existence of the mapping G . In other words, if you have a space which is not deformable to a point then no deformation map G is available and the construction via K breaks down. Basically, if the space has a hole which you get stuck on as you deform loops to a point then it is not deformable to a point. Often we call such spaces *simply connected*. Careful definition of these terms is too difficult for calculus, deformation of loops and higher dimensional objects is properly covered in algebraic topology. In any event, the connection of the deformation and exactness of closed forms allows topologists to use differential forms detect holes in spaces. In particular:

Definition 9.6.7. de Rham cohomology:

We define several real vector spaces of differential forms over some subset U of \mathbb{R} ,

$$Z^p(U) \equiv \{\phi \in \Lambda^p U \mid \phi \text{ closed}\}$$

the space of closed p -forms on U . Then,

$$B^p(U) \equiv \{\phi \in \Lambda^p U \mid \phi \text{ exact}\}$$

the space of exact p -forms where by convention $B^0(U) = \{0\}$. The de Rham cohomology groups are defined by the quotient of closed/exact,

$$H^p(U) \equiv Z^p(U)/B^p(U).$$

the $\dim(H^p U) = p^{\text{th}}$ Betti number of U .

We observe that simply connected regions have all the Betti numbers zero since $Z^p(U) = B^p(U)$ implies that $H^p(U) = \{0\}$. Of course there is much more to say about Cohomology, I just wanted to give you a taste and alert you to the fact that differential forms can be used to reveal aspects of topology. Not all algebraic topology uses differential forms though, there are several other calculational schemes based on triangulation of the space, or studying singular simplexes. One important event in 20-th century mathematics was the discovery that all these schemes described the same homology groups. Steenrod reduced the problem to a few central axioms and it was shown that all the calculational schemes adhere to that same set of axioms.

One interesting aspect of the proof we (copied from Flanders ⁷) is that it is not a mere existence proof. It actually lays out how to calculate the form which provides exactness. Let's call β the potential form of γ if $\gamma = d\beta$. Notice this is totally reasonable language since in the case of classical mechanics we consider conservative forces \vec{F} which are derivable from a scalar potential V by $\vec{F} = -\nabla V$. Translated into differential forms we have $\omega_{\vec{F}} = -dV$. Let's explore how the K -mapping and proof indicate the potential of a vector field ought to be calculated.

Suppose U is deformable to a point and F is a smooth conservative vector field on U . Perhaps you recall that for conservative F are irrotational hence $\nabla \times F = 0$. Recall that $d\omega_F = \Phi_{\nabla \times F} = \Phi_0 = 0$ thus the one-form corresponding to a conservative vector field is a closed form. Apply the identity: let $G: I \times U \rightarrow U \subseteq \mathbb{R}^3$ be the deformation of U to a point P ,

$$d(K(G^*\omega_F)) = \omega_F$$

Hence, including the minus to make energy conservation natural,

$$V = -K(G^*\omega_F)$$

For convenience, let's suppose the space considered is the unit-ball B and let's use a deformation to the origin. Explicitly, $G(t, r) = tr$ for all $r \in \mathbb{R}^3$ such that $\|r\| \leq 1$. Note that clearly $G(0, r) = 0$

⁷I don't know the complete history of this calculation at the present. It would be nice to find it since I doubt Flanders is the originator.

whereas $G(1, r) = r$ and G has a nice formula so it's smooth⁸. We wish to calculate the pull-back of $\omega_F = Pdx + Qdy + Rdz$ under G , from the definition of pull-back we have

$$(G^*\omega_F)(X) = \omega_F(dG(X))$$

for each smooth vector field X on $I \times B$. Differential forms on $I \times B$ are written as linear combinations of dt, dx, dy, dz with smooth functions as coefficients. We can calculate the coefficients by evaluation on the corresponding vector fields $\partial_t, \partial_x, \partial_y, \partial_z$. Observe, since $G(t, x, y, z) = (tx, ty, tz)$ we have

$$dG(\partial_t) = \frac{\partial G^1}{\partial t} \frac{\partial}{\partial x} + \frac{\partial G^2}{\partial t} \frac{\partial}{\partial y} + \frac{\partial G^3}{\partial t} \frac{\partial}{\partial z} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$$

whereas,

$$dG(\partial_x) = \frac{\partial G^1}{\partial x} \frac{\partial}{\partial x} + \frac{\partial G^2}{\partial x} \frac{\partial}{\partial y} + \frac{\partial G^3}{\partial x} \frac{\partial}{\partial z} = t \frac{\partial}{\partial x}$$

and similarly,

$$dG(\partial_y) = t \frac{\partial}{\partial y} \quad dG(\partial_z) = t \frac{\partial}{\partial z}$$

Furthermore,

$$\omega_F(dG(\partial_t)) = \omega_F(x\partial_x + y\partial_y + z\partial_z) = xP + yQ + zR$$

$$\omega_F(dG(\partial_x)) = \omega_F(t\partial_x) = tP, \quad \omega_F(dG(\partial_y)) = \omega_F(t\partial_y) = tQ, \quad \omega_F(dG(\partial_z)) = \omega_F(t\partial_z) = tR$$

Therefore,

$$G^*\omega_F = (xP + yQ + zR)dt + tPdx + tQdy + tRdz = (xP + yQ + zR)dt + t\omega_F$$

Now we can calculate $K(G^*\omega_F)$ and hence V . Note that only the coefficient of dt gives a nontrivial contribution so in retrospect we did a bit more calculation than necessary. That said, I'll just keep it as a celebration of extreme youth for calculation. Also, I've been a bit careless in omitting the point up to this point, let's include the point dependence since it will be critical to properly understand the formula.

$$K(G^*\omega_F)(t, x, y, z) = K\left((xP(tx, ty, tz) + yQ(tx, ty, tz) + zR(tx, ty, tz))dt\right)$$

Therefore,

$$V(x, y, z) = -K(G^*\omega_F) = -\int_0^1 (xP(tx, ty, tz) + yQ(tx, ty, tz) + zR(tx, ty, tz))dt$$

Notice this is precisely the line-integral of $F = \langle P, Q, R \rangle$ along the line C with direction $\langle x, y, z \rangle$ from the origin to (x, y, z) . In particular, if $\vec{r}(t) = \langle tx, ty, tz \rangle$ then $\frac{d\vec{r}}{dt} = \langle x, y, z \rangle$ hence we identify

$$V(x, y, z) = -\int_0^1 \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt = -\int_C \vec{F} \cdot d\vec{r}$$

⁸there is of course a deeper meaning to the word, but, for brevity I gloss over this.

Perhaps you recall this is precisely how we calculate the potential function for a conservative vector field provided we take the origin as the zero for the potential.

Actually, this calculation is quite interesting. Suppose we used a different deformation $\tilde{G}: I \times U \rightarrow U$. For fixed point Q we travel to from the origin to the point by the path $t \mapsto \tilde{G}(t, Q)$. Of course this path need not be a line. The space considered might look like a snake where a line cannot reach from the base point P to the point Q . But, the same potential is derived. Why? Path independence of the vector field is one answer. The criteria $\nabla \times F = 0$ suffices for a simply connected region. However, we see something deeper. The criteria of a closed form paired with a simply connected (deformable) domain suffices to construct a potential for the given form. This result reproduces the familiar case of conservative vector fields derived from scalar potentials and *much more*. In Flanders he calculates the potential for a closed two-form. This ought to be the mathematics underlying the construction of the so-called **vector potential** of magnetism. In junior-level electromagnetism⁹ the magnetic field B satisfies $\nabla \cdot B = 0$ and thus the two-form Φ_B has exterior derivative $d\Phi_B = \nabla \cdot B dx \wedge dy \wedge dz = 0$. The magnetic field corresponds to a closed form. Poincaré's lemma shows that there exists a one-form ω_A such that $d\omega_A = \Phi_B$. But this means $\Phi_{\nabla \times A} = \Phi_B$ hence in the language of vector fields we expect the vector potential A generated the magnetic field B through the curl $B = \nabla \times A$. Indeed, this is precisely what a typical junior level physics student learns in the magnetostatic case. Appreciate that it goes deeper still, the Poincaré lemma holds for p -forms which correspond to objects which don't match up with the quaint vector fields of 19-th century physics. We can be confident to find potential for 3-form fluxes in a 10-dimensional space, or wherever our imagination takes us. I explain at the end of this chapter how to translate electromagnetics into the language of differential forms, it may well be that in the future we think about forms the way we currently think about vectors. This is one of the reasons I like Flanders text, he really sticks with the language of differential forms throughout. In contrast to these notes, he just does what is most interesting. I think undergraduates need to see more detail and not just the most clever calculations, but, I can hardly blame Flanders! He makes no claim to be an undergraduate work.

Finally, I should at least mention that though we can derive a potential β for a given closed form α on a simply connected domain it need not be unique. In fact, it will not be unique unless we add further criteria for the potential. This ambiguity is called **gauge freedom** in physics. Mathematically it's really simple give form language. If $\alpha = d\beta$ where β is a $(p-1)$ -form then we can take any smooth $(p-2)$ form and calculate that

$$d(\alpha + d\lambda) = d\beta + d^2\lambda = d\beta = \alpha$$

Therefore, if β is a potential-form for α then $\beta + d\lambda$ is also a potential-form for α .

⁹just discussing magnetostatic case here to keep it simple

9.7 classical differential geometry in forms

XXX- already worked it out, copy from my notepad.

9.8 E & M in differential form

Warning: I will use Einstein's implicit summation convention throughout this section. I have made a point of abstaining from Einstein's convention in these notes up to this point. However, I just can't bear the summations in this section. They're just too ugly.

9.8.1 differential forms in Minkowski space

The logic here follows fairly close to the last section, however the wrinkle is that the metric here demands more attention. We must take care to raise the indices on the forms when we Hodge dual them. First we list the basis differential forms, we have to add time to the mix (again $c = 1$ so $x^0 = ct = t$ if you worried about it)

Name	Degree	Typical Element	"Basis" for $\Lambda^p(\mathbb{R}^4)$
function	$p = 0$	f	1
one-form	$p = 1$	$\alpha = \alpha_\mu dx^\mu$	dt, dx, dy, dz
two-form	$p = 2$	$\beta = \frac{1}{2}\beta_{\mu\nu}dx^\mu \wedge dx^\nu$	$dy \wedge dz, dz \wedge dx, dx \wedge dy$
three-form	$p = 3$	$\gamma = \frac{1}{3!}\gamma_{\mu\nu\alpha}dx^\mu \wedge dx^\nu \wedge dx^\alpha$	$dt \wedge dx, dt \wedge dy, dt \wedge dz$
four-form	$p = 4$	$gdt \wedge dx \wedge dy \wedge dz$	$dx \wedge dy \wedge dz, dt \wedge dy \wedge dz$ $dt \wedge dx \wedge dz, dt \wedge dx \wedge dy$ $dt \wedge dx \wedge dy \wedge dz$

Greek indices are defined to range over 0, 1, 2, 3. Here the top form is degree four since in four dimensions we can have four differentials without a repeat. Wedge products work the same as they have before, just now we have dt to play with. Hodge duality may offer some surprises though.

Definition 9.8.1. The antisymmetric symbol in flat \mathbb{R}^4 is denoted $\epsilon_{\mu\nu\alpha\beta}$ and it is defined by the value

$$\epsilon_{0123} = 1$$

plus the demand that it be completely antisymmetric.

We must not assume that this symbol is invariant under a cyclic exchange of indices. Consider,

$$\begin{aligned}
 \epsilon_{0123} &= -\epsilon_{1023} && \text{flipped (01)} \\
 &= +\epsilon_{1203} && \text{flipped (02)} \\
 &= -\epsilon_{1230} && \text{flipped (03)}.
 \end{aligned} \tag{9.9}$$

Example 9.8.2. We now compute the Hodge dual of $\gamma = dx$ with respect to the Minkowski metric $\eta_{\mu\nu}$. First notice that dx has components $\gamma_\mu = \delta_\mu^1$ as is readily verified by the equation $dx = \delta_\mu^1 dx^\mu$. We raise the index using η , as follows

$$\gamma^\mu = \eta^{\mu\nu} \gamma_\nu = \eta^{\mu\nu} \delta_\nu^1 = \eta^{1\mu} = \delta^{1\mu}.$$

Beginning with the definition of the Hodge dual we calculate

$$\begin{aligned}
*(dx) &= \frac{1}{(4-1)!} \delta^{1\mu} \epsilon_{\mu\nu\alpha\beta} dx^\nu \wedge dx^\alpha \wedge dx^\beta \\
&= (1/6) \epsilon_{1\nu\alpha\beta} dx^\nu \wedge dx^\alpha \wedge dx^\beta \\
&= (1/6) [\epsilon_{1023} dt \wedge dy \wedge dz + \epsilon_{1230} dy \wedge dz \wedge dt + \epsilon_{1302} dz \wedge dt \wedge dy \\
&\quad + \epsilon_{1320} dz \wedge dy \wedge dt + \epsilon_{1203} dy \wedge dt \wedge dz + \epsilon_{1032} dt \wedge dz \wedge dy] \\
&= (1/6) [-dt \wedge dy \wedge dz - dy \wedge dz \wedge dt - dz \wedge dt \wedge dy \\
&\quad + dz \wedge dy \wedge dt + dy \wedge dt \wedge dz + dt \wedge dz \wedge dy] \\
&= -dy \wedge dz \wedge dt.
\end{aligned} \tag{9.10}$$

The difference between the three and four dimensional Hodge dual arises from two sources, for one we are using the Minkowski metric so indices up or down makes a difference, and second the antisymmetric symbol has more possibilities than before because the Greek indices take four values.

Example 9.8.3. We find the Hodge dual of $\gamma = dt$ with respect to the Minkowski metric $\eta_{\mu\nu}$. Notice that dt has components $\gamma_\mu = \delta_\mu^0$ as is easily seen using the equation $dt = \delta_\mu^0 dx^\mu$. Raising the index using η as usual, we have

$$\gamma^\mu = \eta^{\mu\nu} \gamma_\nu = \eta^{\mu\nu} \delta_\nu^0 = -\eta^{0\mu} = -\delta^{0\mu}$$

where the minus sign is due to the Minkowski metric. Starting with the definition of Hodge duality we calculate

$$\begin{aligned}
*(dt) &= -(1/6) \delta^{0\mu} \epsilon_{\mu\nu\alpha\beta} dx^\nu \wedge dx^\alpha \wedge dx^\beta \\
&= -(1/6) \epsilon_{0\nu\alpha\beta} dx^\nu \wedge dx^\alpha \wedge dx^\beta \\
&= -(1/6) \epsilon_{0ijk} dx^i \wedge dx^j \wedge dx^k \\
&= -(1/6) \epsilon_{ijk} dx^i \wedge dx^j \wedge dx^k \\
&= -dx \wedge dy \wedge dz.
\end{aligned} \tag{9.11}$$

for the case here we are able to use some of our old three dimensional ideas. The Hodge dual of dt cannot have a dt in it which means our answer will only have dx, dy, dz in it and that is why we were able to shortcut some of the work, (compared to the previous example).

Example 9.8.4. Finally, we find the Hodge dual of $\gamma = dt \wedge dx$ with respect to the Minkowski metric $\eta_{\mu\nu}$. Recall that $*(dt \wedge dx) = \frac{1}{(4-2)!} \epsilon_{01\mu\nu} \gamma^{01} (dx^\mu \wedge dx^\nu)$ and that $\gamma^{01} = \eta^{0\lambda} \eta^{1\rho} \gamma_{\lambda\rho} = (-1)(1) \gamma_{01} = -1$. Thus

$$\begin{aligned}
*(dt \wedge dx) &= -(1/2) \epsilon_{01\mu\nu} dx^\mu \wedge dx^\nu \\
&= -(1/2) [\epsilon_{0123} dy \wedge dz + \epsilon_{0132} dz \wedge dy] \\
&= -dy \wedge dz.
\end{aligned} \tag{9.12}$$

Notice also that since $dt \wedge dx = -dx \wedge dt$ we find $*(dx \wedge dt) = dy \wedge dz$

$*1 = dt \wedge dx \wedge dy \wedge dz$	$*(dt \wedge dx \wedge dy \wedge dz) = -1$
$*(dx \wedge dy \wedge dz) = -dt$	$*dt = -dx \wedge dy \wedge dz$
$*(dt \wedge dy \wedge dz) = -dx$	$*dx = -dy \wedge dz \wedge dt$
$*(dt \wedge dz \wedge dx) = -dy$	$*dy = -dz \wedge dx \wedge dt$
$*(dt \wedge dx \wedge dy) = -dz$	$*dz = -dx \wedge dy \wedge dt$
$*(dz \wedge dt) = dx \wedge dy$	$*(dx \wedge dy) = -dz \wedge dt$
$*(dx \wedge dt) = dy \wedge dz$	$*(dy \wedge dz) = -dx \wedge dt$
$*(dy \wedge dt) = dz \wedge dx$	$*(dz \wedge dx) = -dy \wedge dt$

The other Hodge duals of the basic two-forms follow from similar calculations. Here is a table of all the basic Hodge dualities in Minkowski space, In the table the terms are grouped as they are to emphasize the isomorphisms between the one-dimensional $\Lambda^0(M)$ and $\Lambda^4(M)$, the four-dimensional $\Lambda^1(M)$ and $\Lambda^3(M)$, the six-dimensional $\Lambda^2(M)$ and itself. Notice that the dimension of $\Lambda(M)$ is 16 which just happens to be 2^4 .

Now that we've established how the Hodge dual works on the differentials we can easily take the Hodge dual of arbitrary differential forms on Minkowski space. We begin with the example of the 4-current \mathcal{J}

Example 9.8.5. Four Current: *often in relativistic physics we would even just call the four current simply the current, however it actually includes the charge density ρ and current density \vec{J} . Consequently, we define,*

$$(\mathcal{J}^\mu) \equiv (\rho, \vec{J}),$$

moreover if we lower the index we obtain,

$$(\mathcal{J}_\mu) = (-\rho, \vec{J})$$

which are the components of the current one-form,

$$\mathcal{J} = \mathcal{J}_\mu dx^\mu = -\rho dt + J_x dx + J_y dy + J_z dz$$

This equation could be taken as the definition of the current as it is equivalent to the vector definition. Now we can rewrite the last equation using the vectors \mapsto forms mapping as,

$$\mathcal{J} = -\rho dt + \omega_{\vec{J}}.$$

Consider the Hodge dual of \mathcal{J} ,

$$\begin{aligned}
*\mathcal{J} &= *(-\rho dt + J_x dx + J_y dy + J_z dz) \\
&= -\rho *dt + J_x *dx + J_y *dy + J_z *dz \\
&= \rho dx \wedge dy \wedge dz - J_x dy \wedge dz \wedge dt - J_y dz \wedge dx \wedge dt - J_z dx \wedge dy \wedge dt \\
&= \rho dx \wedge dy \wedge dz - \Phi_{\vec{J}} \wedge dt.
\end{aligned} \tag{9.13}$$

we will find it useful to appeal to this calculation in a later section.

Example 9.8.6. Four Potential: *often in relativistic physics we would call the four potential simply the potential, however it actually includes the scalar potential V and the vector potential \vec{A} (discussed at the end of chapter 3). To be precise we define,*

$$(A^\mu) \equiv (V, \vec{A})$$

we can lower the index to obtain,

$$(A_\mu) = (-V, \vec{A})$$

which are the components of the current one-form,

$$A = A_\mu dx^\mu = -Vdt + A_x dx + A_y dy + A_z dz$$

Sometimes this equation is taken as the definition of the four potential. We can rewrite the four potential vector field using the vectors \mapsto forms mapping as,

$$A = -Vdt + \omega_{\vec{A}}.$$

The Hodge dual of A is

$$*A = Vdx \wedge dy \wedge dz - \Phi_{\vec{A}} \wedge dt. \quad (9.14)$$

Several steps were omitted because they are identical to the calculation of the dual of the 4-current above.

Definition 9.8.7. Faraday tensor.

Given an electric field $\vec{E} = (E_1, E_2, E_3)$ and a magnetic field $\vec{B} = (B_1, B_2, B_3)$ we define a 2-form F by

$$F = \omega_E \wedge dt + \Phi_B.$$

This 2-form is often called the **electromagnetic field tensor or the Faraday tensor**. If we write it in tensor components as $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ and then consider its matrix $(F_{\mu\nu})$ of components then it is easy to see that

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad (9.15)$$

Convention: Notice that when we write the matrix version of the tensor components we take the first index to be the row index and the second index to be the column index, that means $F_{01} = -E_1$ whereas $F_{10} = E_1$.

Example 9.8.8. *In this example we demonstrate various conventions which show how one can transform the field tensor to other type tensors. Define a type $(1,1)$ tensor by raising the first index by the inverse metric $\eta^{\alpha\mu}$ as follows,*

$$F^\alpha{}_\nu = \eta^{\alpha\mu} F_{\mu\nu}$$

The zeroth row,

$$(F^0{}_\nu) = (\eta^{0\mu} F_{\mu\nu}) = (0, E_1, E_2, E_3)$$

Then row one is unchanged since $\eta^{1\mu} = \delta^{1\mu}$,

$$(F^1{}_\nu) = (\eta^{1\mu} F_{\mu\nu}) = (E_1, 0, B_3, -B_2)$$

and likewise for rows two and three. In summary the $(1,1)$ tensor $F' = F^\alpha_\nu (\frac{\partial}{\partial x^\alpha} \otimes dx^\nu)$ has the components below

$$(F^\alpha{}_\nu) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \quad (9.16)$$

At this point we raise the other index to create a $(2,0)$ tensor,

$$F^{\alpha\beta} = \eta^{\alpha\mu} \eta^{\beta\nu} F_{\mu\nu} \quad (9.17)$$

and we see that it takes one copy of the inverse metric to raise each index and $F^{\alpha\beta} = \eta^{\beta\nu} F^\alpha{}_\nu$ so we can pick up where we left off in the $(1,1)$ case. We could proceed case by case like we did with the $(1,1)$ case but it is better to use matrix multiplication. Notice that $\eta^{\beta\nu} F^\alpha{}_\nu = F^\alpha{}_\nu \eta^{\nu\beta}$ is just the (α, β) component of the following matrix product,

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \quad (9.18)$$

So we find a $(2,0)$ tensor $F'' = F^{\alpha\beta} (\frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta})$. Other books might even use the same symbol F for F' and F'' , it is in fact typically clear from the context which version of F one is thinking about. Pragmatically physicists just write the components so its not even an issue for them.

Example 9.8.9. Field tensor's dual: We now calculate the Hodge dual of the field tensor,

$$\begin{aligned} *F &= *(\omega_E \wedge dt + \Phi_B) \\ &= E_x^*(dx \wedge dt) + E_y^*(dy \wedge dt) + E_z^*(dz \wedge dt) \\ &\quad + B_x^*(dy \wedge dz) + B_y^*(dz \wedge dx) + B_z^*(dx \wedge dy) \\ &= E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy \\ &\quad - B_x dx \wedge dt - B_y dy \wedge dt - B_z dz \wedge dt \\ &= \Phi_E - \omega_B \wedge dt. \end{aligned}$$

we can also write the components of $*F$ in matrix form:

$$(*F_{\mu\nu}) = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix} \quad (9.19)$$

Notice that the net-effect of Hodge duality on the field tensor was to make the exchanges $\vec{E} \mapsto -\vec{B}$ and $\vec{B} \mapsto \vec{E}$.

9.8.2 exterior derivatives of charge forms, field tensors, and their duals

In the last chapter we found that the single operation of the exterior differentiation reproduces the gradient, curl and divergence of vector calculus provided we make the appropriate identifications under the "work" and "flux" form mappings. We now move on to some four dimensional examples.

Example 9.8.10. Charge conservation: *Consider the 4-current we introduced in example 9.8.5. Take the exterior derivative of the dual of the current to get,*

$$\begin{aligned}
 d(*\mathcal{J}) &= d(\rho dx \wedge dy \wedge dz - \Phi_{\vec{J}} \wedge dt) \\
 &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - d[(J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy) \wedge dt] \\
 &= d\rho \wedge dx \wedge dy \wedge dz \\
 &\quad - \partial_x J_x dx \wedge dy \wedge dz \wedge dt - \partial_y J_y dy \wedge dz \wedge dx \wedge dt - \partial_z J_z dz \wedge dx \wedge dy \wedge dt \\
 &= (\partial_t \rho + \nabla \cdot \vec{J}) dt \wedge dx \wedge dy \wedge dz.
 \end{aligned}$$

We work through the same calculation using index techniques,

$$\begin{aligned}
 d(*\mathcal{J}) &= d(\rho dx \wedge dy \wedge dz - \Phi_{\vec{J}} \wedge dt) \\
 &= d(\rho) \wedge dx \wedge dy \wedge dz - d[\frac{1}{2} \epsilon_{ijk} J_i dx^j \wedge dx^k \wedge dt] \\
 &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - \frac{1}{2} \epsilon_{ijk} \partial_\mu J_i dx^\mu \wedge dx^j \wedge dx^k \wedge dt \\
 &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - \frac{1}{2} \epsilon_{ijk} \partial_m J_i dx^m \wedge dx^j \wedge dx^k \wedge dt \\
 &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - \frac{1}{2} \epsilon_{ijk} \epsilon_{mjk} \partial_m J_i dx \wedge dy \wedge dz \wedge dt \\
 &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - \frac{1}{2} 2 \delta_{im} \partial_m J_i dx \wedge dy \wedge dz \wedge dt \\
 &= (\partial_t \rho + \nabla \cdot \vec{J}) dt \wedge dx \wedge dy \wedge dz.
 \end{aligned}$$

Observe that we can now phrase charge conservation by the following equation

$$d(*\mathcal{J}) = 0 \quad \Longleftrightarrow \quad \partial_t \rho + \nabla \cdot \vec{J} = 0.$$

In the classical scheme of things this was a derived consequence of the equations of electromagnetism, however it is possible to build the theory regarding this equation as fundamental. Rindler describes that formal approach in a late chapter of "Introduction to Special Relativity".

Proposition 9.8.11.

If $(A_\mu) = (-V, \vec{A})$ is the vector potential (which gives the magnetic field) and $A = -V dt + \omega_{\vec{A}}$, then $dA = \omega_{\vec{E}} + \Phi_{\vec{B}} = F$ where F is the electromagnetic field tensor. Moreover, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Proof: The proof uses the definitions $\vec{E} = -\nabla V - \partial_t \vec{A}$ and $\vec{B} = \nabla \times \vec{A}$ and some vector identities:

$$\begin{aligned}
 dA &= d(-Vdt + \omega_{\vec{A}}) \\
 &= -dV \wedge dt + d(\omega_{\vec{A}}) \\
 &= -dV \wedge dt + (\partial_t A_i)dt \wedge dx^i + (\partial_j A_i)dx^j \wedge dx^i \\
 &= \omega_{(-\nabla V)} \wedge dt - \omega_{\partial_t \vec{A}} \wedge dt + \Phi_{\nabla \times \vec{A}} \\
 &= (\omega_{(-\nabla V)} - \omega_{\partial_t \vec{A}}) \wedge dt + \Phi_{\nabla \times \vec{A}} \\
 &= \omega_{(-\nabla V - \partial_t \vec{A})} \wedge dt + \Phi_{\nabla \times \vec{A}} \\
 &= \omega_{\vec{E}} \wedge dt + \Phi_{\vec{B}} \\
 &= F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.
 \end{aligned}$$

Moreover we also have:

$$\begin{aligned}
 dA &= d(A_\nu) \wedge dx^\nu \\
 &= \partial_\mu A_\nu dx^\mu \wedge dx^\nu \\
 &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)dx^\mu \wedge dx^\nu + \frac{1}{2}(\partial_\mu A_\nu + \partial_\nu A_\mu)dx^\mu \wedge dx^\nu \\
 &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)dx^\mu \wedge dx^\nu.
 \end{aligned}$$

Comparing the two identities we see that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the proposition follows.

Example 9.8.12. Exterior derivative of the field tensor: *We have just seen that the field tensor is the exterior derivative of the potential one-form. We now compute the exterior derivative of the field tensor expecting to find Maxwell's equations since the derivative of the fields are governed by Maxwell's equations,*

$$\begin{aligned}
 dF &= d(E_i dx^i \wedge dt) + d(\Phi_{\vec{B}}) \\
 &= \partial_m E_i (dx^m \wedge dx^i \wedge dt) + (\nabla \cdot \vec{B}) dx \wedge dy \wedge dz + \frac{1}{2} \epsilon_{ijk} (\partial_t B_i) (dt \wedge dx^j \wedge dx^k).
 \end{aligned} \tag{9.20}$$

We pause here to explain our logic. In the above we dropped the $\partial_t E_i dt \wedge \dots$ term because it was wedged with another dt in the term so it vanished. Also we broke up the exterior derivative on the flux form of \vec{B} into the space and then time derivative terms and used our work in example 9.2.6. Continuing the calculation,

$$\begin{aligned}
 dF &= [\partial_j E_k + \frac{1}{2} \epsilon_{ijk} (\partial_t B_i)] dx^j \wedge dx^k \wedge dt + (\nabla \cdot \vec{B}) dx \wedge dy \wedge dz \\
 &= [\partial_x E_y - \partial_y E_x + \epsilon_{12} (\partial_t B_i)] dx \wedge dy \wedge dt \\
 &\quad + [\partial_z E_x - \partial_x E_z + \epsilon_{31} (\partial_t B_i)] dz \wedge dx \wedge dt \\
 &\quad + [\partial_y E_z - \partial_z E_y + \epsilon_{23} (\partial_t B_i)] dy \wedge dz \wedge dt \\
 &\quad + (\nabla \cdot \vec{B}) dx \wedge dy \wedge dz \\
 &= (\nabla \times \vec{E} + \partial_t \vec{B})_i \Phi_{e_i} \wedge dt + (\nabla \cdot \vec{B}) dx \wedge dy \wedge dz \\
 &= \Phi_{\nabla \times \vec{E} + \partial_t \vec{B}} \wedge dt + (\nabla \cdot \vec{B}) dx \wedge dy \wedge dz
 \end{aligned} \tag{9.21}$$

where we used the fact that Φ is an isomorphism of vector spaces (at a point) and $\Phi_{e_1} = dy \wedge dz$, $\Phi_{e_2} = dz \wedge dx$, and $\Phi_{e_3} = dx \wedge dy$. Behold, we can state two of Maxwell's equations as

$$\boxed{dF = 0 \quad \Longleftrightarrow \quad \nabla \times \vec{E} + \partial_t \vec{B} = 0, \quad \nabla \cdot \vec{B} = 0} \quad (9.22)$$

Example 9.8.13. We now compute the exterior derivative of the dual to the field tensor:

$$\begin{aligned} d^*F &= d(-B_i dx^i \wedge dt) + d(\Phi_{\vec{E}}) \\ &= \Phi_{-\nabla \times \vec{B} + \partial_t \vec{E}} \wedge dt + (\nabla \cdot \vec{E}) dx \wedge dy \wedge dz \end{aligned} \quad (9.23)$$

This follows directly from the last example by replacing $\vec{E} \mapsto -\vec{B}$ and $\vec{B} \mapsto \vec{E}$. We obtain the two inhomogeneous Maxwell's equations by setting d^*F equal to the Hodge dual of the 4-current,

$$\boxed{d^*F = \mu_o {}^*\mathcal{J} \quad \Longleftrightarrow \quad -\nabla \times \vec{B} + \partial_t \vec{E} = -\mu_o \vec{J}, \quad \nabla \cdot \vec{E} = \rho} \quad (9.24)$$

Here we have used example 9.8.5 to find the RHS of the Maxwell equations.

We now know how to write Maxwell's equations via differential forms. The stage is set to prove that Maxwell's equations are Lorentz covariant, that is they have the same form in all inertial frames.

9.8.3 coderivatives and comparing to Griffith's relativistic E & M

Optional section, for those who wish to compare our tensorial E & M with that of Griffith's, you may skip ahead to the next section if not interested

I should mention that this is not the only way to phrase Maxwell's equations in terms of differential forms. If you try to see how what we have done here compares with the equations presented in Griffith's text it is not immediately obvious. He works with $F^{\mu\nu}$ and $G^{\mu\nu}$ and J^μ none of which are the components of differential forms. Nevertheless he recovers Maxwell's equations as $\partial_\mu F^{\mu\nu} = J^\nu$ and $\partial_\mu G^{\mu\nu} = 0$. If we compare the components of *F with equation 12.119 (the matrix form of $G^{\mu\nu}$) in Griffith's text,

$$(G^{\mu\nu}(c=1)) = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & -E_3 & 0 & -E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix} = -({}^*F^{\mu\nu}). \quad (9.25)$$

we find that we obtain the negative of Griffith's "dual tensor" (recall that raising the indices has the net-effect of multiplying the zeroth row and column by -1). The equation $\partial_\mu F^{\mu\nu} = J^\nu$ does not follow directly from an exterior derivative, rather it is the component form of a "coderivative". The coderivative is defined $\delta = {}^*d^*$, it takes a p -form to an $(n-p)$ -form then d makes it a $(n-p+1)$ -form then finally the second Hodge dual takes it to an $(n-(n-p+1))$ -form. That is δ takes a p -form to a $p-1$ -form. We stated Maxwell's equations as

$$dF = 0 \quad d^*F = {}^*\mathcal{J}$$

Now we can take the Hodge dual of the inhomogeneous equation to obtain,

$$*d^*F = \delta F = **\mathcal{J} = \pm \mathcal{J}$$

where I leave the sign for you to figure out. Then the other equation

$$\partial_\mu G^{\mu\nu} = 0$$

can be understood as the component form of $\delta^*F = 0$ but this is really $dF = 0$ in disguise,

$$0 = \delta^*F = *d^{**}F = \pm^*dF \iff dF = 0$$

so even though it looks like Griffith's is using the dual field tensor for the homogeneous Maxwell's equations and the field tensor for the inhomogeneous Maxwell's equations it is in fact not the case. The key point is that there are **coderivatives** implicit within Griffith's equations, so you have to read between the lines a little to see how it matched up with what we've done here. I have not entirely proved it here, to be complete we should look at the component form of $\delta F = \mathcal{J}$ and explicitly show that this gives us $\partial_\mu F^{\mu\nu} = J^\nu$, I don't think it is terribly difficult but I'll leave it to the reader.

Comparing with Griffith's is fairly straightforward because he uses the same metric as we have. Other texts use the mostly negative metric, its just a convention. If you try to compare to such a book you'll find that our equations are almost the same up to a sign. One good careful book is Reinhold A. Bertlmann's *Anomalies in Quantum Field Theory* you will find much of what we have done here done there with respect to the other metric. Another good book which shares our conventions is Sean M. Carroll's *An Introduction to General Relativity: Spacetime and Geometry*, that text has a no-nonsense introduction to tensors forms and much more over a curved space (in contrast to our approach which has been over a vector space which is flat). By now there are probably thousands of texts on tensors; these are a few we have found useful here.

9.8.4 Maxwell's equations are relativistically covariant

Let us begin with the definition of the field tensor once more. We define the components of the field tensor in terms of the 4-potentials as we take the view-point those are the basic objects (not the fields). If

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu,$$

then the field tensor $F = F_{\mu\nu} dx^\mu \otimes dx^\nu$ is a tensor, or is it ? We should check that the components transform as they ought according to the discussion in section ?? . Let $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ then we observe,

$$\begin{aligned} (1.) \quad \bar{A}_\mu &= (\Lambda^{-1})^\alpha_\mu A_\alpha \\ (2.) \quad \frac{\partial}{\partial \bar{x}^\nu} &= \frac{\partial x^\beta}{\partial \bar{x}^\nu} \frac{\partial}{\partial x^\beta} = (\Lambda^{-1})^\beta_\nu \frac{\partial}{\partial x^\beta} \end{aligned} \tag{9.26}$$

where (2.) is simply the chain rule of multivariate calculus and (1.) is not at all obvious. We will assume that (1.) holds, that is we assume that the 4-potential transforms in the appropriate way for a one-form. In principle one could prove that from more base assumptions. After all electromagnetism is the study of the interaction of charged objects, we should hope that the potentials

are derivable from the source charge distribution. Indeed, there exist formulas to calculate the potentials for moving distributions of charge. We could take those as definitions for the potentials, then it would be possible to actually calculate if (1.) is true. We'd just change coordinates via a Lorentz transformation and verify (1.). For the sake of brevity we will just assume that (1.) holds. We should mention that alternatively one can show the electric and magnetic fields transform as to make $F_{\mu\nu}$ a tensor. Those derivations assume that charge is an invariant quantity and just apply Lorentz transformations to special physical situations to deduce the field transformation rules. See Griffith's chapter on special relativity or look in Resnick for example.

Let us find how the field tensor transforms assuming that (1.) and (2.) hold, again we consider $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$,

$$\begin{aligned}\bar{F}_{\mu\nu} &= \bar{\partial}_\mu \bar{A}_\nu - \bar{\partial}_\nu \bar{A}_\mu \\ &= (\Lambda^{-1})^\alpha_\mu \partial_\alpha ((\Lambda^{-1})^\beta_\nu A_\beta) - (\Lambda^{-1})^\beta_\nu \partial_\beta ((\Lambda^{-1})^\alpha_\mu A_\alpha) \\ &= (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &= (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu F_{\alpha\beta}.\end{aligned}\tag{9.27}$$

therefore the field tensor really is a tensor over Minkowski space.

Proposition 9.8.14.

The dual to the field tensor is a tensor over Minkowski space. For a given Lorentz transformation $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ it follows that

$$*\bar{F}_{\mu\nu} = (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu *F_{\alpha\beta}$$

Proof: homework (just kidding in 2010), it follows quickly from the definition and the fact we already know that the field tensor is a tensor.

Proposition 9.8.15.

The four-current is a four-vector. That is under the Lorentz transformation $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ we can show,

$$\bar{\mathcal{J}}_\mu = (\Lambda^{-1})^\alpha_\mu \mathcal{J}_\alpha$$

Proof: follows from arguments involving the invariance of charge, time dilation and length contraction. See Griffith's for details, sorry we have no time.

Corollary 9.8.16.

The dual to the four current transforms as a 3-form. That is under the Lorentz transformation $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ we can show,

$$*\bar{\mathcal{J}}_{\mu\nu\sigma} = (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu (\Lambda^{-1})^\gamma_\sigma \mathcal{J}_{\alpha\beta\gamma}$$

Up to now the content of this section is simply an admission that we have been a little careless in defining things upto this point. The main point is that if we say that something is a tensor then we need to make sure that is in fact the case. With the knowledge that our tensors are indeed tensors the proof of the covariance of Maxwell's equations is trivial.

$$dF = 0 \qquad d^*F = *\mathcal{J}$$

are coordinate invariant expressions which we have already proved give Maxwell's equations in one frame of reference, thus they must give Maxwell's equations in all frames of reference.

The essential point is simply that

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}\bar{F}_{\mu\nu}d\bar{x}^\mu \wedge d\bar{x}^\nu$$

Again, we have no hope for the equation above to be true unless we know that $\bar{F}_{\mu\nu} = (\Lambda^{-1})_\mu^\alpha (\Lambda^{-1})_\nu^\beta F_{\alpha\beta}$. That transformation follows from the fact that the four-potential is a four-vector. It should be mentioned that others prefer to "prove" the field tensor is a tensor by studying how the electric and magnetic fields transform under a Lorentz transformation. We in contrast have derived the field transforms based ultimately on the seemingly innocuous assumption that the four-potential transforms according to $\bar{A}_\mu = (\Lambda^{-1})_\mu^\alpha A_\alpha$. OK enough about that.

So the fact that Maxwell's equations have the same form in all **relativistically inertial** frames of reference simply stems from the fact that we found Maxwell's equation were given by an arbitrary frame, and the field tensor looks the same in the new barred frame so we can again go through all the same arguments with barred coordinates. Thus we find that Maxwell's equations are the same in all relativistic frames of reference, that is if they hold in one inertial frame then they will hold in any other frame which is related by a Lorentz transformation.

9.8.5 Electrostatics in Five dimensions

We will endeavor to determine the electric field of a point charge in 5 dimensions where we are thinking of adding an extra spatial dimension. Lets call the fourth spatial dimension the w -direction so that a typical point in space time will be (t, x, y, z, w) . First we note that the electromagnetic field tensor can still be derived from a one-form potential,

$$A = -\rho dt + A_1 dx + A_2 dy + A_3 dz + A_4 dw$$

we will find it convenient to make our convention for this section that $\mu, \nu, \dots = 0, 1, 2, 3, 4$ whereas $m, n, \dots = 1, 2, 3, 4$ so we can rewrite the potential one-form as,

$$A = -\rho dt + A_m dx^m$$

This is derived from the vector potential $A^\mu = (\rho, A^m)$ under the assumption we use the natural generalization of the Minkowski metric, namely the 5 by 5 matrix,

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (\eta^{\mu\nu}) \quad (9.28)$$

we could study the linear isometries of this metric, they would form the group $O(1,4)$. Now we form the field tensor by taking the exterior derivative of the one-form potential,

$$F = dA = \frac{1}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) dx^\mu \wedge dx^\nu$$

now we would like to find the electric and magnetic "fields" in 4 dimensions. Perhaps we should say 4+1 dimensions, just understand that I take there to be 4 spatial directions throughout this discussion if in doubt. Note that we are faced with a dilemma of interpretation. There are 10 independent components of a 5 by 5 antisymmetric tensor, naively we would expect that the electric and magnetic fields each would have 4 components, but that is not possible, we'd be missing two components. The solution is this, the time components of the field tensor are understood to correspond to the electric part of the fields whereas the remaining 6 components are said to be magnetic. This aligns with what we found in 3 dimensions, its just in 3 dimensions we had the fortunate quirk that the number of linearly independent one and two forms were equal at any point. This definition means that the magnetic field will in general not be a vector field but rather a "flux" encoded by a 2-form.

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z & -E_w \\ E_x & 0 & B_z & -B_y & H_1 \\ E_y & -B_z & 0 & B_x & H_2 \\ E_z & B_y & -B_x & 0 & H_3 \\ E_w & -H_1 & -H_2 & -H_3 & 0 \end{pmatrix} \quad (9.29)$$

Now we can write this compactly via the following equation,

$$F = E \wedge dt + B$$

I admit there are subtle points about how exactly we should interpret the magnetic field, however I'm going to leave that to your imagination and instead focus on the electric sector. What is the generalized Maxwell's equation that E must satisfy?

$$d^*F = \mu_o^* \mathcal{J} \implies d^*(E \wedge dt + B) = \mu_o^* \mathcal{J}$$

where $\mathcal{J} = -\rho dt + J_m dx^m$ so the 5 dimensional Hodge dual will give us a $5 - 1 = 4$ form, in particular we will be interested in just the term stemming from the dual of dt ,

$$*(-\rho dt) = \rho dx \wedge dy \wedge dz \wedge dw$$

the corresponding term in d^*F is $d^*(E \wedge dt)$ thus, using $\mu_o = \frac{1}{\epsilon_o}$,

$$d^*(E \wedge dt) = \frac{1}{\epsilon_o} \rho dx \wedge dy \wedge dz \wedge dw \quad (9.30)$$

is the 4-dimensional Gauss's equation. Now consider the case we have an isolated point charge which has somehow always existed at the origin. Moreover consider a 3-sphere that surrounds the charge. We wish to determine the generalized Coulomb field due to the point charge. First we note that the solid 3-sphere is a 4-dimensional object, it the set of all $(x, y, z, w) \in \mathbb{R}^4$ such that

$$x^2 + y^2 + z^2 + w^2 \leq r^2$$

We may parametrize a three-sphere of radius r via generalized spherical coordinates,

$$\begin{aligned} x &= r \sin(\theta) \cos(\phi) \sin(\psi) \\ y &= r \sin(\theta) \sin(\phi) \sin(\psi) \\ z &= r \cos(\theta) \sin(\psi) \\ w &= r \cos(\psi) \end{aligned} \quad (9.31)$$

Now it can be shown that the volume and surface area of the radius r three-sphere are as follows,

$$\text{vol}(S^3) = \frac{\pi^2}{2} r^4 \quad \text{area}(S^3) = 2\pi^2 r^3$$

We may write the charge density of a smeared out point charge q as,

$$\rho = \begin{cases} 2q/\pi^2 a^4, & 0 \leq r \leq a \\ 0, & r > a \end{cases} \quad (9.32)$$

Notice that if we integrate ρ over any four-dimensional region which contains the solid three sphere of radius a will give the enclosed charge to be q . Then integrate over the Gaussian 3-sphere S^3 with radius r call it M ,

$$\int_M d^*(E \wedge dt) = \frac{1}{\epsilon_o} \int_M \rho dx \wedge dy \wedge dz \wedge dw$$

now use the Generalized Stokes Theorem to deduce,

$$\int_{\partial M} *(E \wedge dt) = \frac{q}{\epsilon_o}$$

but by the "spherical" symmetry of the problem we find that E must be independent of the direction it points, this means that it can only have a radial component. Thus we may calculate the integral with respect to generalized spherical coordinates and we will find that it is the product of $E_r \equiv E$ and the surface volume of the four dimensional solid three sphere. That is,

$$\int_{\partial M} *(E \wedge dt) = 2\pi^2 r^3 E = \frac{q}{\epsilon_o}$$

Thus,

$$E = \frac{q}{2\pi^2\epsilon_0 r^3}$$

the Coulomb field is weaker if it were to propagate in 4 spatial dimensions. Qualitatively what has happened is that they have taken the same net flux and spread it out over an additional dimension, this means it thins out quicker. A very similar idea is used in some *brane world* scenarios. String theorists posit that the gravitational field spreads out in more than four dimensions while in contrast the standard model fields of electromagnetism, and the strong and weak forces are confined to a four-dimensional brane. That sort of model attempts an explanation as to why gravity is so weak in comparison to the other forces. Also it gives large scale corrections to gravity that some hope will match observations which at present don't seem to fit the standard gravitational models.

This example is but a taste of the theoretical discussion that differential forms allow. As a final comment I remind the reader that we have done things for flat space for the most part in this course, when considering a curved space there are a few extra considerations that must enter. Coordinate vector fields e_i must be thought of as derivations $\partial/\partial x^\mu$ for one. Also the metric is not a constant tensor like δ_{ij} or $\eta_{\mu\nu}$ rather it depends on position, this means Hodge duality acquires a coordinate dependence as well. Doubtless I have forgotten something else in this brief warning. One more advanced treatment of many of our discussions is Dr. Fulp's Fiber Bundles 2001 notes which I have posted on my webpage. He uses the other metric but it is rather elegantly argued, all his arguments are coordinate independent. He also deals with the issue of the magnetic induction and the dielectric, issues which we have entirely ignored since we always have worked in free space.

References and Acknowledgements:

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Chapter 10

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