Lecture Notes for Advanced Calculus

James S. Cook
Liberty University
Department of Mathematics

Fall 2015

introduction and motivations for these notes

These notes are intended for someone who has completed the introductory calculus sequence and has some experience with matrices or linear algebra. The audience I intend has both students of pure mathematics and applied mathematics. For the applied crowd, there is much here I intend to offer: how to calculate derivatives in general, how to integrate in higher dimensions; what generalizes flux, variational calculus a quick introduction, how to find extrema locally and globally, how to calculate wedge products and exterior derivatives, and finally the tensor calculus and relativistic electrodynamics. For the pure student, we seek to understand topology and continuity in normed linear spaces, gain intuition about inverse and implicit function theorems, the contraction mapping principle, uniform convergence, the definition of differentiability in normed spaces, proofs of limit laws and rules of differentiation, the theory of quadratic forms, a study of multilinear mappings and the construction of the wedge product, the structure of the push-forward and pull-back, the generalized Stokes' theorem, the Poincare lemma, an introduction to DeRahm cohomology. In short, our goal is really to learn further techniques and intuition of calculus; this is our interpretation of advanced calculus.

There are many excellent texts on portions of this subject. However, the particular path I choose this semester is not quite in line with any particular text. I required the text on Advanced Calculus by Edwards because it contains all the major theorems that traditionally are covered in an Advanced Calculus course.

Linear algebra is not a prerequisite for this course. However, I will use linear algebra. Matrices, linear transformations and vector spaces are necessary ingredients for a proper discussion of advanced calculus. I believe an interested student can easily assimilate the needed tools as we go so I am not terribly worried if you have not had linear algebra previously. I will make a point to include some baby[1] linear exercises to make sure everyone who is working at this course keeps up with the story that unfolds.

Real analysis is also not a prerequisite for this course. However, I will present some proofs which properly fall in the category of real analysis. Some of these proofs involve a sophistication that is beyond the usual day-to-day business of the course. I include these thoughts for the sake of completeness. If I am to test them it's probably more on the question of were you paying attention as opposed to can you reconstruct the monster from scratch.

Certainly my main intent in this course is that you learn calculus more deeply. Yes we'll learn some new calculations, but I also hope that what we cover also gives you deeper insight into your previous experience with calculus.

Doing the homework is doing the course. I cannot overemphasize the importance of thinking through the homework. I would be happy if you left this course with a working knowledge of:

[1] if you view this as an insult then you haven't met the right babies yet. Baby exercises are cute.

- ✓ set-theoretic mapping langauge, fibers and images and how to picture relationships diagramatically.
- ✓ continuity in view of the metric topology in normed linear spaces.
- ✓ the concept and application of the derivative and differential of a mapping.
- ✓ continuous differentiability
- ✓ inverse function theorem
- ✓ implicit function theorem
- ✓ tangent space and normal space via gradients or derivatives of parametrizations
- ✓ extrema for multivariate functions, critical points and the Lagrange multiplier method
- ✓ multivariate Taylor series.
- ✓ quadratic forms
- ✓ critical point analysis for multivariate functions
- ✓ basic variational calculus (how to calculate the Euler-Lagrange equations for a given Lagrangian)
- ✓ dual space and the dual basis.
- ✓ multilinear algebra.
- ✓ musical morphisms and Hodge duality.
- ✓ the work and flux form mappings for $\mathbb{R}^3$.
- ✓ basic manifold theory
- ✓ vector fields as derivations.
- ✓ differential forms and the exterior derivative
- ✓ integration of forms
- ✓ generalized Stokes's Theorem.
- ✓ push-fowards and pull-backs

When I say *working knowledge* what I intend is that you have some sense of the problem and at least know where to start looking for a solution. Some of the topics above take a much longer time to understand deeply. I cover them to spark your interest and seed your intuition if all goes well.

Before we begin, I should warn you that I assume quite a few things from the reader. These notes are intended for someone who has already grappled with the problem of constructing proofs. I assume you know the difference between $\Rightarrow$ and $\Leftrightarrow$. I assume the phrase "iff" is known to you. I assume you are ready and willing to do a proof by induction, strong or weak. I assume you know what $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{Z}$ denote. I assume you know what a subset of a set is. I assume you know how to prove two sets are equal. I assume you are familar with basic set operations such as union and intersection (although we don't use those much). More importantly, I assume you have started to appreciate that mathematics is more than just calculations. Calculations without context, without theory, are doomed to failure. At a minimum theory and proper mathematics allows you to communicate analytical concepts to other like-educated individuals.

Some of the most seemingly basic objects in mathematics are insidiously complex. We've been taught they're simple since our childhood, but as adults, mathematically-minded adults, we find the actual definitions of such objects as $\mathbb{R}$ or $\mathbb{C}$ are rather involved. I will not attempt to provide foundational arguments to build numbers from basic set theory. I believe it is possible, I think it's well-thought-out mathematics, but we take the existence of the real numbers as an axiom for these notes. We assume that $\mathbb{R}$ exists and that the real numbers possess all their usual properties. In fact, I assume $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{N}$ and $\mathbb{Z}$ all exist complete with their standard properties. In short, I assume we have numbers to work with. We leave the rigorization of numbers to a different course.

I have avoided use of Einstein's implicit summation notation in the majority of these notes. This has introduced some clutter in calculations, but I hope the student finds the added detail helpful. Naturally if one goes on to study tensor calculations in physics then no such luxury is granted. In view of this, I left the more *physicsy* notation in the discussion of electromagnetism via differential forms.

These notes are a work in progress. There are a few sections I'd like to add, but, I don't think I have time to do it during this Fall 2015 Semester. The Course Planner shows you the path I intend and I will probably diverge from these notes considerably in certain lectures. In particular, I do intend to do a bit more about differentiation on a normed linear space. I'll likely define partial derivatives in a normed linear space in the lectures. I hope to prove a fairly general product rule which encompasses all the products you usually encounter. Also, I plan to give a lecture on the technique of swapping differentiation and integration paired with some brief introduction to the theory of sequences of functions. Somewhere I should discuss the operator norm and how continuous differentiability is indeed just that, I don't think that is currently in these notes although I covered it in the 2013 version of this course. Later in the course I intend to introduce the musical morphisms.

I recently obtained a copy of *Manifolds, Tensors, and Forms An Introduction for Mathematicians and Physicists* by Paul Renteln. I hope to deepen our treatment of several topics on the basis of his exposition.

I dedicate these notes to Hannah, Daniel, Isaac and Matthew:

Enjoy,

James Cook, August 29, 2015.

Contents

1 linear algebra **11**

1.1 matrices . . . 11
1.2 n-vectors and vector algebra . . . 16
1.3 vector spaces . . . 17
1.3.1 examples of vector spaces, bases and coordinate maps . . . 19
1.4 linear transformations . . . 21
1.4.1 matrix of a linear transformation . . . 22
1.4.2 change of coordinates . . . 22

2 topology and limits **25**

2.1 elementary topology and limits . . . 26
2.2 normed linear spaces . . . 36
2.3 limits and continuity in normed linear spaces . . . 37
2.4 intuitive theorems of calculus . . . 43
2.4.1 theorems for single-variable calculus . . . 43
2.4.2 theorems for multivariate calculus . . . 48

3 differentiation **49**

3.1 the Frechet differential . . . 50
3.2 partial derivatives and the Jacobian matrix . . . 56
3.2.1 directional derivatives . . . 56
3.3 derivatives of sum and scalar product . . . 62
3.4 a gallery of explicit derivatives . . . 63
3.5 chain rule . . . 67
3.6 common product rules . . . 70
3.6.1 scalar-vector product rule . . . 70
3.6.2 calculus of paths in $\mathbb{R}^3$ . . . 72
3.6.3 calculus of matrix-valued functions of a real variable . . . 74
3.6.4 calculus of complex-valued functions of a real variable . . . 75
3.7 continuous differentiability . . . 76
3.8 on why partial derivatives commute . . . 81
3.9 complex analysis in a nutshell . . . 84

3.9.1 harmonic functions . . . 89

4 inverse and implicit function theorems 91
4.1 inverse function theorem . . . 91
4.2 implicit function theorem . . . 96
4.3 implicit differentiation . . . 106
4.3.1 computational techniques for partial differentiation with side conditions . . . 109
4.4 the constant rank theorem . . . 112

5 two views of manifolds in $\mathbb{R}^n$ 117
5.1 definition of level set . . . 118
5.2 tangents and normals to a level set . . . 119
5.3 tangent and normal space from patches . . . 125
5.4 summary of tangent and normal spaces . . . 126
5.5 method of Lagrange mulitpliers . . . 127

6 critical point analysis for several variables 135
6.1 multivariate power series . . . 135
6.1.1 taylor's polynomial for one-variable . . . 135
6.1.2 taylor's multinomial for two-variables . . . 137
6.1.3 taylor's multinomial for many-variables . . . 140
6.2 a brief introduction to the theory of quadratic forms . . . 143
6.2.1 diagonalizing forms via eigenvectors . . . 146
6.3 second derivative test in many-variables . . . 153

7 convergence and estimation 159
7.1 sequences of functions . . . 159
7.2 power series . . . 159
7.3 matrix exponential . . . 159
7.4 uniform convergence . . . 159
7.5 differentiating under the integral . . . 159
7.6 contraction mappings and Newton's method . . . 159
7.7 multivariate mean value theorem . . . 159
7.8 proof of the inverse function theorem . . . 159

8 introduction to variational calculus 161
8.1 history . . . 161
8.2 the variational problem . . . 163
8.3 variational derivative . . . 164
8.4 Euler-Lagrange examples . . . 165
8.4.1 shortest distance between two points in plane . . . 166
8.4.2 surface of revolution with minimal area . . . 167
8.4.3 Braichistochrone . . . 168

8.5 Euler-Lagrange equations for several dependent variables . . . 170
8.5.1 free particle Lagrangian . . . 171
8.5.2 geodesics in $\mathbb{R}^3$ . . . 171
8.6 the Euclidean metric . . . 172
8.7 geodesics . . . 174
8.7.1 geodesic on cylinder . . . 174
8.7.2 geodesic on sphere . . . 175
8.8 Lagrangian mechanics . . . 175
8.8.1 basic equations of classical mechanics summarized . . . 175
8.8.2 kinetic and potential energy, formulating the Lagrangian . . . 176
8.8.3 easy physics examples . . . 178

9 multilinear algebra 179
9.1 dual space . . . 179
9.2 multilinearity and the tensor product . . . 181
9.2.1 bilinear maps . . . 182
9.2.2 trilinear maps . . . 186
9.2.3 multilinear maps . . . 189
9.3 wedge product . . . 192
9.3.1 wedge product of dual basis generates basis for ΛV . . . 192
9.3.2 the exterior algebra . . . 195
9.3.3 connecting vectors and forms in $\mathbb{R}^3$ . . . 200
9.4 bilinear forms and geometry, metric duality . . . 202
9.4.1 metric geometry . . . 202
9.4.2 metric duality for tensors . . . 204
9.4.3 inner products and induced norm . . . 207
9.5 hodge duality . . . 208
9.5.1 hodge duality in euclidean space $\mathbb{R}^3$ . . . 208
9.5.2 hodge duality in minkowski space $\mathbb{R}^4$ . . . 210
9.6 coordinate change . . . 214
9.6.1 coordinate change for $T^0_2(V)$ . . . 216

10 calculus with differential forms 217
10.1 an informal introduction to manifolds . . . 219
10.2 vectors as derivations . . . 223
10.2.1 concerning the geometry of derivations . . . 229
10.3 differential for manifolds, the push-forward . . . 229
10.3.1 intuition for the push-forward . . . 231
10.3.2 a pragmatic formula for the push-forward . . . 231
10.4 cotangent space . . . 234
10.5 differential forms . . . 235
10.6 the exterior derivative . . . 238
10.6.1 exterior derivatives on $\mathbb{R}^3$ . . . 240

10.6.2 coordinate independence of exterior derivative . . . 241
10.7 the pull-back . . . 242
10.7.1 intuitive computation of pull-backs . . . 244
10.7.2 implicit function theorem in view of forms . . . 247
10.8 integration of forms . . . 250
10.8.1 integration of k-form on $\mathbb{R}^k$ . . . 250
10.8.2 orientations and submanifolds . . . 251
10.9 Generalized Stokes Theorem . . . 259
10.10 Poincare lemma . . . 262
10.10.1 exact forms are closed . . . 262
10.10.2 potentials for closed forms . . . 263
10.11 introduction to geometric differential equations . . . 270
10.11.1 exact differential equations . . . 270
10.11.2 differential equations via forms . . . 271

11 Electromagnetism in differential form **273**
11.1 differential forms in Minkowski space . . . 273
11.2 exterior derivatives of charge forms, field tensors, and their duals . . . 278
11.3 coderivatives and comparing to Griffith's relativitic E & M . . . 280
11.4 Maxwell's equations are relativistically covariant . . . 281
11.5 Electrostatics in Five dimensions . . . 283

12 Geometry by frames and forms **289**
12.1 basic terminology and vector fields . . . 290
12.2 connection forms on $\mathbb{R}^3$ . . . 293
12.2.1 Frenet Serret equations . . . 297
12.3 structure equations for frame field on $\mathbb{R}^3$ . . . 298
12.4 surfaces in $\mathbb{R}^3$ . . . 300
12.5 isometry . . . 304

Chapter 1

linear algebra

This chapter probably needs a warning. Let me offer it in the form of an educational video:

concerning how to find a marble in oatmeal

Ok, with that settled, let us review the essential truths, notations, and motivations of linear algebra.

1.1 matrices

An $m \times n$ matrix is an array of objects with m rows and n columns. The elements in the array are called entries or components. If A is an $m \times n$ matrix then A_{ij} denotes the object in the i-th row and the j-th column. The label i is a row index and the index j is a column index in the preceding sentence. We usually denote $A = [A_{ij}]$. The set $m \times n$ of matrices with real number entries is denoted $\mathbb{R}^{m \times n}$. The set of $m \times n$ matrices with complex entries is $\mathbb{C}^{m \times n}$. Generally, is S is a set then $S^{m \times n}$ is the set of $m \times n$ arrays of objects from S. If a matrix has the same number of rows and columns then it is called a **square matrix**.

Two matrices A and B are equal if and only if they have the same size and $A_{ij} = B_{ij}$ for all i, j. Let $A, B \in \mathbb{R}^{m \times n}$ then $A + B \in \mathbb{R}^{m \times n}$ is defined by $(A + B)_{ij} = A_{ij} + B_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. If two matrices A, B are not of the same size then there sum is not defined[1]. Let $A, B \in \mathbb{R}^{m \times n}, c \in \mathbb{R}$ then $cA \in \mathbb{R}^{m \times n}$ is defined by $(cA)_{ij} = cA_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. We call the process of multiplying A by a number c **multiplication by a scalar**. We define $A - B \in \mathbb{R}^{m \times n}$ by $A - B = A + (-1)B$ which is equivalent to $(A - B)_{ij} = A_{ij} - B_{ij}$ for all i, j. The **zero matrix** in $\mathbb{R}^{m \times n}$ is denoted 0 and defined by $0_{ij} = 0$ for all i, j. The additive inverse of $A \in \mathbb{R}^{m \times n}$ is the matrix $-A$ such that $A + (-A) = 0$. The components of the additive inverse matrix are given by $(-A)_{ij} = -A_{ij}$ for all i, j.

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then the product of A and B is denoted by juxtaposition AB and $AB \in \mathbb{R}^{m \times p}$ is defined by: $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$ for each $1 \leq i \leq m$ and $1 \leq j \leq p$. In the case

[1] for now

$m = p = 1$ the indices i, j are omitted in the equation since the matrix product is simply a number which needs no index. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ then another way to understand the matrix multiplication of A and B is as the concatenation of several matrix-vector products,

$$AB = A[col_1(B)|col_2(B)|\cdots|col_p(B)] = [Acol_1(B)|Acol_2(B)|\cdots|Acol_p(B)] \tag{1.1}$$

I refer to this identity as the **concatenation proposition**. Similarly, for rows,

$$AB = \begin{bmatrix} row_1(A) \\ row_2(A) \\ \vdots \\ row_m(A) \end{bmatrix} B = \begin{bmatrix} row_1(A)B \\ row_2(A)B \\ \vdots \\ row_m(A)B \end{bmatrix}. \tag{1.2}$$

The **matrix-column product** is very important. Let $A \in \mathbb{R}^{m\times n}$ and $v \in \mathbb{R}^n$ then

$$Av = v_1 col_1(A) + v_2 col_2(A) + \cdots + v_n col_n(A). \tag{1.3}$$

The product of A and v is a linear combination of the columns of A.

The $n \times n$ **identity matrix** is denoted I or I_n if the size of the identity matrix needs emphasis.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.4}$$

Let $A \in \mathbb{R}^{n\times n}$. If there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = I$ and $BA = I$ then we say that A is **invertible** and $A^{-1} = B$. Invertible matrices are also called **nonsingular**. If a matrix has no inverse then it is called a **noninvertible** or **singular** matrix. For square A, we define $A^0 = I$, $A^1 = A$ and $A^m = AA^{m-1}$ for all $m \in \mathbb{N}$. If A is invertible then negative integer powers are similarly defined $A^{-p} = (A^{-1})^p$ for $p \in \mathbb{N}$. It can be shown,

$$(A^p)^q = A^{pq}, \quad A^pA^q = A^{p+q}. \tag{1.5}$$

However, $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 \neq A^2 + 2AB + B^2$ unless we have that the **commutator** $[A, B] = AB - BA$ vanishes. Generally, the bionomial theorem only holds for matrices A, B for which $AB = BA$. If A, B are invertible then we have the **socks-shoes** identity:

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{1.6}$$

Generally the calculation of A^{-1} is somewhat involved, but, for 2×2 matrices it's easy as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1.7}$$

Let $A \in \mathbb{R}^{m\times n}$ then a **submatrix** of A is a matrix which is made of some rectangle of elements in A. For example, rows and columns of a matrix are submatrices. Observe that any matrix can be decomposed into rows and columns:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} = \left[\begin{array}{c} row_1(A) \\ \hline row_2(A) \\ \hline \vdots \\ \hline row_m(A) \end{array}\right] = [col_1(A)|col_2(A)|\cdots|col_n(A)]\,. \tag{1.8}$$

Let $A \in \mathbb{R}^{m\times n}$ then $A^T \in \mathbb{R}^{n\times m}$ is called the **transpose** of A and is defined by $(A^T)_{ji} = A_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. It is simple to verify the following properties of transposition:

$$(A \pm B)^T = A^T \pm B^T, \quad (cA)^T = cA^T, \quad (AB)^T = B^T A^T, \quad (A^T)^T = A. \tag{1.9}$$

If A is invertible then $(A^{-1})^T = (A^T)^{-1}$. If $A^T = A$ then we say A is **symmetric** and if $A^T = -A$ then we say A is an **antisymmetric**.

It is sometimes convenient to view matrices as comprised of smaller submatrices. Such a decomposition is called a **block-decomposition**. For example, suppose $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ such that both A and B are partitioned as follows:

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad \text{and} \qquad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] \tag{1.10}$$

where A_{11} is an $m_1 \times n_1$ block, A_{12} is an $m_1 \times n_2$ block, A_{21} is an $m_2 \times n_1$ block and A_{22} is an $m_2 \times n_2$ block. Likewise, $B_{n_k p_k}$ is an $n_k \times p_k$ block for $k = 1, 2$. We insist that $m_1 + m_2 = m$ and $n_1 + n_2 = n$. If the partitions are compatible as decribed above then we may multiply A and B by multiplying the blocks as if they were scalars and we were computing the product of 2×2 matrices:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]. \tag{1.11}$$

A matrix A for which $A_{ij} = 0$ for $i \neq j$ is called a **diagonal matrix**. We say A is **block-diagonal** if A has a block-decomposition where the (i, j)-th-block $A_{ij} = 0$ whenever $i \neq j$. We can add matrices of differing sizes as follows: for A an $m_1 \times n_1$ and B an $m_2 \times n_2$ define the **direct sums** of A and B to be the $(m_1 + m_2) \times (n_1 + n_2)$ matrices $A \oplus B$ or $B \oplus A$ given by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \qquad \& \qquad B \oplus A = \begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}. \tag{1.12}$$

Notice, this is not really an **addition** since there is no reason to expect the operation $\oplus$ be commutative; $A \oplus B \neq B \oplus A$ (generally). That said, this is an interesting way of assembling new matrices block-by-block.

The **tensor product** or **Kronecker** product of an $m \times n$ matrix A and a $p \times q$ matrix B is a $(mp) \times (nq)$ matrix $A \otimes B$ defined by the following block-decomposition:

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix} \tag{1.13}$$

For example, $I_4 = I_2 \otimes I_2$. Or, $I_{2n} = I_2 \otimes I_n$. We also could calculate $I_{2n} = I_n \otimes I_2$. However, generally, $A \otimes B \neq B \otimes A$. We may investigate the properties of $\otimes$ and $\oplus$ in the homework. Hopefully what follows is not entirely new to the reader.

The symbol $\delta_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$ is called the **Kronecker delta**. Let $e_i \in \mathbb{R}^{n \times 1}$ be defined by $(e_i)_j = \delta_{ij}$. The size of the vector e_i is determined by context. We call e_i the i-th **standard basis vector**. In $\mathbb{R}$ we have $e_1 = (1) = 1$ (by convention we drop the brackets in this case). In $\mathbb{R}^2$ we have $e_1 = (1, 0)$ and $e_2 = (0, 1)$. In $\mathbb{R}^3$ we have $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. In $\mathbb{R}^4$ we have $e_1 = (1, 0, 0, 0)$ and $e_2 = (0, 1, 0, 0)$ and $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$.

The ij-th **standard basis matrix** for $\mathbb{R}^{m \times n}$ is denoted E_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$. The matrix E_{ij} is zero in all entries except for the (i, j)-th slot where it has a 1. In other words, we define $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. If $A \in \mathbb{R}^{m \times n}$ and $v \in \mathbb{R}^n$ then

$$v = \sum_{i=1}^{n} v_n e_n \qquad \& \qquad A = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} E_{ij}. \tag{1.14}$$

and we can select rows or columns or even components by

$$[e_i{}^T A] = row_i(A) \qquad \& \qquad [Ae_i] = col_i(A) \qquad \& \qquad A_{ij} = (e_i)^T A e_j. \tag{1.15}$$

If $E_{ij}, E_{kl} \in \mathbb{R}^{n \times n}$ and $e_i \in \mathbb{R}^n$ then

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \qquad \& \qquad E_{ij} = e_i e_j{}^T \qquad \& \qquad e_i{}^T e_j = \delta_{ij} \tag{1.16}$$

Let $\epsilon_{i_1 i_2 \ldots i_n}$ be defined to be the completely antisymmetric symbol in n-indices. We define $\epsilon_{12\ldots n} = 1$ then all other values are generated by demanding the interchange of any two indices is antisymmetric. This is also known as the **Levi-Civita** symbol. In view of this notation, we define the **determinant** of $A \in \mathbb{R}^{n \times n}$ as follows:

$$\det(A) = \sum_{i_1, i_2, \ldots, i_n} \epsilon_{i_1, i_2, \ldots, i_n} A_{1 i_1} A_{2 i_2} \cdots A_{n i_n}. \tag{1.17}$$

For $n = 2$ we have $\epsilon_{ij} \neq 0$ only for $\epsilon_{12} = 1 = -\epsilon_{21}$ hence for a 2×2 matrix A we calculate

$$\det(A) = \epsilon_{12} A_{11} A_{22} + \epsilon_{21} A_{21} A_{12} = A_{11} A_{22} - A_{12} A_{21}. \tag{1.18}$$

For a 3×3 matrix A we calculate $\det(A) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} A_{1i} A_{2j} A_{3k}$. Of the 27 terms in this triple-sum only 6 are nontrivial because if any index is repeated the ϵ_{ijk} is zero. The only nontrivial terms are $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$. Thus,

$$\begin{aligned} det(A) &= \epsilon_{123} A_{11} A_{22} A_{33} + \epsilon_{231} A_{12} A_{23} A_{31} + \epsilon_{312} A_{13} A_{21} A_{32} \\ &\quad + \epsilon_{321} A_{13} A_{22} A_{31} + \epsilon_{213} A_{12} A_{21} A_{33} + \epsilon_{132} A_{11} A_{23} A_{32} \\ &= A_{11} A_{22} A_{33} + A_{12} A_{23} A_{31} + A_{13} A_{21} A_{32} - A_{13} A_{22} A_{31} - A_{12} A_{21} A_{33} - A_{11} A_{23} A_{32} \end{aligned} \tag{1.19}$$

The identities below are not too easy to prove:

$$\det(AB) = \det(A)\det(B) \qquad \& \qquad \det(A^T) = \det(A). \tag{1.20}$$

Notice, $\det(I) = 1$ hence $AA^{-1} = I$ implies $\det(A)\det(A^{-1}) = 1$ thus $\det(A) \neq 0$ is a necessary condition for an invertible matrix. It is also sufficient as $A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)^T$ is a well-defined formula[2] for the inverse of A provided $\det(A) \neq 0$. If $Ax = b$ is a system of n-equations in n-unknowns where the coefficient matrix A is invertible then we can derive the unique solution is $x = A^{-1}b$. Furthermore, **Cramer's Rule** states: If $Ax = b$ is a linear system of equations with $x = [x_1 \; x_2 \; \cdots \; x_n]^T$ and $A \in \mathbb{R}^{n \times n}$ such that $det(A) \neq 0$ then we find solution:

$$x_1 = \frac{det(A_1)}{det(A)}, \quad x_2 = \frac{det(A_2)}{det(A)}, \quad \ldots, \quad x_n = \frac{det(A_n)}{det(A)} \tag{1.21}$$

where we define A_k to be the $n \times n$ matrix obtained by replacing the k-th column of A by the inhomogeneous term b. Of course, in general for over-determined ($m > n$) and underdetermined ($m < n$) systems of m-equations in n-unknowns there may be no solutions or infinitely many and the robust method to solve all systems is to perform Gaussian elimination on the augmented coefficient matrix. The solution set of $Ax = b$ is generally found by computation of $\mathrm{rref}[A|b]$.

The reduced row echelon form can efficiently determine linear dependencies amongst the columns of a given matrix. Let $A = [col_1(A)|\cdots|col_n(A)] \in \mathbb{R}^{m \times n}$ and $R = rref[A] = [col_1(R)|\cdots|col_n(R)]$. There exist constants $c_1, c_2, \ldots c_k$ such that $c_1 col_1(A) + c_2 col_2(A) + \cdots + c_k col_k(A) = 0$ if and only if $c_1 col_1(R) + c_2 col_2(R) + \cdots + c_k col_k(R) = 0$. If $col_j(rref[A])$ is a linear combination of other columns of $rref[A]$ then $col_j(A)$ is likewise the same linear combination of columns of A. Therefore, the non-pivot columns of a matrix can be written as linear combinations of the pivot columns and the pivot columns of the matrix are linearly independent.

Another significant aspect of the determinant is its behaviour under a similarity transformation. We say square matrices A and B are **similar matrices** iff there exists an invertible matrix P for which $B = P^{-1}AP$. Observe,

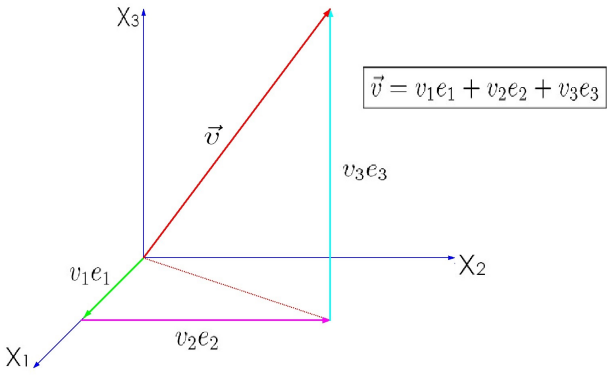
$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(A). \tag{1.22}$$

[2] the classical adjoint formula for the inverse in the 2×2 case yields $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Likewise, as a completely unecessary detail, $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{aei+bfg+cdh-gec-hfa-idb} \left[\begin{array}{c|c|c} ei-fh & ch-bi & bf-ce \\ \hline fg-di & ai-cg & cd-af \\ \hline dh-eg & bg-ah & ae-bd \end{array}\right]$

We should recall that $\text{trace}(A) = A_{11} + A_{22} + \cdots + A_{nn}$ is also invariant under a similarity transformation. The identity $\text{trace}(AB) = \text{trace}(BA)$ for $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ allows us to calculate $\text{trace}(P^{-1}AP) = \text{trace}(PP^{-1}A) = \text{trace}(A)$. We should recall, the invariance of determinant and trace under similarity transformations allows us to naturally define determinant and trace for linear transformations on a finite-dimensional vector space V.

1.2 n-vectors and vector algebra

Let $n \in \mathbb{N}$, we define[3] $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_j \in \mathbb{R} \text{ for } j = 1, 2, \ldots, n\}$. If $v \in \mathbb{R}^n$ then we say v is an **n-vector**. The numbers in the vector are called the **components**; $v = (v_1, v_2, \ldots, v_n)$ has j-th component v_j. A visualization in $n = 3$ is given below:



Geometrically, the addition of vectors can be imagined as a series of displacements where each new displacement begins where the previous terminated. Algebraically, this is simply defined for each $v, w \in \mathbb{R}^n$ by $(v + w)_j = v_j + w_j$ for all $j \in \{1, 2, \ldots, n\}$. Likewise, **scalar multiplication** is defined for $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$ by $(cv)_j = cv_j$ for all $j \in \{1, 2, \ldots, n\}$. If $c > 0$ then cv points in the same direction as v whereas $c < 0$ has cv with direction opposite that of v. In both cases, if $|c| > 1$ the vector stretches in length whereas $|c| < 1$ results in a shrunken vector.

If $v, w \in \mathbb{R}^n$ then we define the **dot-product** of v and w to by $v \bullet w = v^T w$. Likewise, the **euclidean norm** or **euclidean length** of v is defined by $||v|| = \sqrt{v \bullet v}$. Homogeneity $||cv|| = |c|||v||$ and the triangle-inequality $||v + w|| \leq ||v|| + ||w||$ are easy to understand geometrically. In addition, it is

[3] In principle one can use column vectors for everything or row vectors for everything. I choose a subtle notation that allows us to use both. We adopt the following **convention**

$$(v_1, v_2, \ldots, v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

If I want to denote a real row vector then we will just write $[v_1, v_2, \ldots, v_n]$.

simple to confirm $||v|| = 0$ iff $v = 0$. The **Cauchy-Schwarz** inequality connects the norm and dot-product: $|v \bullet w| \leq ||v||\,||w||$. For $v, w \neq 0$ the Cauchy-Schwarz inequality yields $-1 \leq \frac{v \bullet w}{||v||\,||w||} \leq 1$ thus we define the **angle** between v and w by $\theta \in [0, \pi]$ for which $v \bullet w = ||v||\,||w|| \cos\theta$. If $v \bullet w = 0$ then we say v and w are **orthogonal**. Furthermore, $S = \{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ is an **orthogonal set** iff $v_i \bullet v_j = 0$ whenever $i \neq j$. If an orthogonal set S also has vectors of unit-length then we say S is **orthonormal**. For example, the standard basis $\{e_1, \ldots, e_n\}$ has $e_i \bullet e_j = \delta_{ij}$ which shows $\{e_1, \ldots, e_n\}$ is orthonormal. Notice,

$$v \bullet e_j = (v_1 e_1 + \cdots + v_i e_i + \cdots + v_n e_n) \bullet e_j = 0 + \cdots + v_j + \cdots + 0 = v_j \tag{1.23}$$

hence $v = (v \bullet e_1)e_1 + \cdots + (v \bullet e_n)e_n$. There is also a generalized **pythagorean theorem** for $v \bullet w = 0$ we can show $||v + w||^2 = ||v||^2 + ||w||^2$. If $\vec{v} \neq 0$ and we denote $v = ||v||$ then the **unit-vector** $\hat{v}$ is defined by $\hat{v} = \frac{1}{v}\vec{v}$. In this notation, we have $\vec{v} = v\hat{v}$; a nonzero vector is decribed by its magnitude (v) and direction ($\hat{v}$).

The **cross-product** is special to $n = 3$. Define $v \times w = \sum_{i,j,k=1}^{3} \epsilon_{ijk} v_i w_j e_k$. This definition is constructed so that both v and w are perpendicular to $v \times w$. That is, $(v \times w) \bullet v = 0$ and $(v \times w) \bullet w = 0$. The triple product $a \bullet (b \times c)$ can also be calculated by a determinant $\det[a|b|c] = a \bullet (b \times c)$. Geometrically, the triple product $a \bullet (b \times c)$ calculates the volume of a parallel-piped with sides a, b, c. It is not too hard to show $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$, $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$ and $\vec{A} \times (c\vec{B}) = (c\vec{A}) \times \vec{B} = c\vec{A} \times \vec{B}$. On the other hand, significant effort is required to show:

(i.) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \bullet \vec{C})\vec{B} - (\vec{A} \bullet \vec{B})\vec{C}$,

(ii.) **Jacobi Identity:** $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$,

(iii.) **cyclicity of triple product:** $\vec{A} \bullet (\vec{B} \times \vec{C}) = \vec{B} \bullet (\vec{C} \times \vec{A}) = \vec{C} \bullet (\vec{A} \times \vec{B})$,

(iv.) **Lagrange's identity:** $||\vec{A} \times \vec{B}||^2 = ||\vec{A}||^2\,||\vec{B}||^2 - \left[\ \vec{A} \bullet \vec{B}\ \right]^2$.

Since $\vec{A} \bullet \vec{B} = ||\vec{A}||\,||\vec{B}|| \cos\theta$ we see Lagrange's Identity implies $||\vec{A} \times \vec{B}|| = ||\vec{A}||\,||\vec{B}||\,|\sin\theta|$. See my *Multivariate Calculus Lecture Notes* from 2014 for details on how to prove these assertions as well as much added detail on the geometry of vectors in $\mathbb{R}^3$.

1.3 vector spaces

We say V is a **vector space** over $\mathbb{R}$ is a set V which contains elements called **vectors**. By definition, we are able to add vectors in V and scalar-multiply by real numbers and once more obtain vectors as a result. A set of vectors is said to be **linearly independent** if there is only trivial linear dependencies between the elements of the set. Computationally, $S \subseteq V$ is linearly independent[4] iff

$$c_1 v_1 + \cdots + c_k v_k = 0 \quad \Rightarrow \quad c_1 = 0, \ldots, c_k = 0 \tag{1.24}$$

[4]this definition works in infinite dimensions, for finite case it suffices to set $k = \#(S)$

for all finite subcollections $\{v_1, \dots, v_k\}$ in S. We also define[5] the **span** of a set of k-vectors

$$\text{span}\{v_1, \dots, v_k\} = \{c_1 v_1 + \cdots + c_k v_k \mid c_1, \dots, c_k \in \mathbb{R}\} \tag{1.25}$$

We say $W \subseteq V$ is a **subspace** of V if W is a vector space with respect to the addition and scalar multiplication of V suitably restricted to W. If W is a subspace of V then we denote this by $W \leq V$. The **subspace test** theorem stated if $W \neq \emptyset$ and W is closed under addition and scalar multiplication then $W \leq V$. Likewise, if $W = \text{span}(S)$ for some $S \subseteq V$ then we may immediately conclude $W \leq V$. Ultimately, the span is a subspace theorem rested on the fact that a linear combination of linear combinations is once more a linear combination.

The $\pm$ theorem is often a useful guide as we seek to modify a given set of vectors: If V be a vector space and S is a nonempty set of vectors in V then

1. If S is LI and $v \neq 0$ is such that $v \notin \mathit{span}(S)$ then $S \cup \{v\}$ is LI.

2. If $v \in S$ is a linear combination of other vectors in S then $\mathit{span}(S - \{v\}) = \mathit{span}(S)$.

The $\pm$ theorem gives us advice on how we may construct a *basis* for a given vector space; we remove redundant vectors as in (2.) and adjoin independent vectors as in (1.) until eventually we attain a LI set for which the span reaches the entire vector space.

If $\beta = \{v_1, \dots, v_n\} \subseteq V$ is a linearly independent set for which $\text{span}(\beta) = V$ then we say β is a **basis for** V. If there exists a finite basis for V then **all** other bases have the same number of elements. Hence, we define the **dimension** of V by $\dim(V) = n$ without ambiguity. Observe that $W \leq V$ implies $\dim(W) \leq \dim(V)$.

To each basis we pair a **coordinate system**. In particular, denote $\Phi_\beta : V \to \mathbb{R}^n$ to be the mapping:

$$\Phi_\beta(c_1 v_1 + \cdots + c_n v_n) = c_1 e_1 + \cdots + c_n e_n \tag{1.26}$$

where $\{e_1, \dots, e_n\}$ denotes the **standard basis** of $\mathbb{R}^n$; $(e_i)_j = \delta_{ij}$. In other words, Φ_β is defined by $\Phi_\beta(v_i) = e_i$ **extended linearly**. The coordinate map Φ_β establishes the isomorphism of V and $\mathbb{R}^n$. We also use the notation $\Phi_\beta(v) = [v]_\beta$ when convenient. We should appreciate that V and $\mathbb{R}^n$ have the same linear structure, the distinction is largely one of notation. That said, isomorphic vector spaces are far from being the same as point sets. Column vectors admit many special constructions which are not *directly* available for abstract vector spaces[6]

[5] ok for the purist, this is the intersection of all possible subspaces which contain $v_1, \dots, v_k$

[6] for example, for a set of vectors $S = \{v_1, \dots, v_k\}$ in $\mathbb{R}^n$ I can use the Gauss-Jordan method to efficiently calculate rref$[S]$ then quickly determine any linear-dependencies by the column-correspondence-property (CCP). If S is a subset of an abstract vector space V then I would first choose a basis β, then calculate $\Phi_\beta(S)$ from which we can use the Gauss-Jordan algorithm to find rref$[\Phi_\beta(S)]$, apply the CCP to read-off any dependencies in $\Phi_\beta(S)$ and finally use Φ_β^{-1} to transport the linear-dependencies back to the abstract vector space V.

1.3.1 examples of vector spaces, bases and coordinate maps

For our convenience of the class we set forth a standard list of examples in this subsection of the notes. For each example we describe the structure of the point-set, the definition of the vector addition and scalar multiplication, exhibit a natural choice of basis and finally the corresponding coordinate map. We seek to describe the real vector space structure here since we are ultimately interested in real analysis on these spaces or their subsets.

I. Vector space of real column vectors; $V = \mathbb{R}^n$

$V = \mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R},\ i = 1, \ldots, n\}$ has vector addition and scalar multiplication defined component-wise as follows: for each $j \in \mathbb{N}_n$ and $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$(x + y)_j = x_j + y_j \qquad (cx)_j = cx_j. \tag{1.27}$$

the standard basis is $\beta = \{e_1, \ldots, e_n\}$ where $(e_i)_j = \delta_{ij}$. In this case, the coordinate map is simply the identity mapping on $\mathbb{R}^n$; $\Phi_\beta(x) = x$. Unless otherwise instructed[7], we always assume this is the vector space structure of $\mathbb{R}^n$.

II. Vector space of real $m \times n$ matrices; $V = \mathbb{R}^{m\times n}$

If $A \in \mathbb{R}^{m\times n}$ we define vector addition and scalar multiplication by the standard addition and scalar multiplication defined component-wise; $(A + B)_{ij} = A_{ij} + B_{ij}$ and $(cA)_{ij} = cA_{ij}$. Observe, any matrix can be written as a sum of the unit-matrices $A = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} E_{ij}$ where $A_{ij} \in \mathbb{R}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ and the unit-matrix E_{ij} is zero in each component except the (i, j)-th component where we place 1; that is $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. We define the standard basis for $\mathbb{R}^{m\times n}$ as follows:

$$\beta = \{E_{11}, \ldots, E_{1n}, E_{21}, \ldots, E_{2n}, \ldots, E_{m1}, \ldots, E_{mn}\} \tag{1.28}$$

This gives us the coordinate map $\Phi_\beta : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ defined by simply stringing out the entries of the matrix row-by-row from first to last:

$$\Phi_\beta(A) = (A_{11}, \ldots, A_{1n}, A_{21}, \ldots, A_{2n}, \ldots, A_{m1}, \ldots, A_{mn}). \tag{1.29}$$

III. Vector space of complex column vectors; $V = \mathbb{C}^n$

$V = \mathbb{C}^n = \{(z_1, \ldots, z_n) \mid z_i \in \mathbb{C},\ i = 1, \ldots, n\}$ has vector addition and scalar multiplication defined component-wise as follows: for each $j \in \mathbb{N}_n$ and $z, w \in \mathbb{C}^n$ and $c \in \mathbb{C}$,

$$(z + w)_j = z_j + w_j \qquad (cz)_j = cz_j. \tag{1.30}$$

Notice, the restriction of c to real values gives a real scalar multiplication on $\mathbb{C}^n$. For each $z \in \mathbb{C}$ there exist $x, y \in \mathbb{R}$ for which $z = x + iy$. We denote $\mathfrak{Re}(z) = x$ and $\mathfrak{Im}(z) = y$ when convenient.

[7] later we study tangent vectors based at a point where the addition is based on the point of application, not the origin as is implicitly the case here

For each component z_j of z we have this decomposition into real and imaginary parts:

$$(z_1,\dots,z_n) = (x_1+iy_1,\dots,x_n+iy_n) = x_1e_1 + y_1(ie_1) + \cdots + x_ne_n + y_n(ie_n). \tag{1.31}$$

where $e_1 = (1,\dots,0)$ and $e_n = (0,\dots,1)$ as usual. Indeed, the calculation above reveals that the standard basis for $\mathbb{R}^n$ is almost what we need. We define the standard basis for $\mathbb{C}^n$ as follows:

$$\beta = \{e_1, ie_1, e_2, ie_2, \dots, e_n, ie_n\} \tag{1.32}$$

The corresponding coordinate map is given by simply stringing out the real and imaginary components of the complex-n-vector component-by-component from first to last:

$$\Phi_\beta(x_1+iy_1, x_2+iy_2, \dots, x_n+iy_n) = (x_1, y_1, x_2, y_2, \dots, x_n, y_n). \tag{1.33}$$

Notice $\Phi_\beta : \mathbb{C}^n \to \mathbb{R}^{2n}$ so $\dim(\mathbb{C}^n) = 2n$.

IV. VECTOR SPACE OF COMPLEX $m \times n$ MATRICES; $V = \mathbb{C}^{m\times n}$

If $A \in \mathbb{C}^{m\times n}$ we define vector addition and scalar multiplication by the standard addition and scalar multiplication defined component-wise; $(A+B)_{ij} = A_{ij} + B_{ij}$ and $(cA)_{ij} = cA_{ij}$. Observe, any matrix can be written as a sum of the unit-matrices $A = \sum_{i=1}^m \sum_{j=1}^n A_{ij}E_{ij}$ where $A_{ij} \in \mathbb{C}$ for $1 \le i \le m$ and $1 \le j \le n$. Let us denote $A_{ij} = \alpha_{ij} + i\beta_{ij}$ where $\mathfrak{Re}(A_{ij}) = \alpha_{ij}$ and $\mathfrak{Im}(A_{ij}) = \beta_{ij}$. For much the same reason as the last example, we define the standard basis for $\mathbb{C}^{m\times n}$ as follows:

$$\beta = \{E_{11}, iE_{11}, \dots, E_{1n}, iE_{1n}, \dots, E_{m1}, iE_{m1}, \dots, E_{mn}, iE_{mn}\} \tag{1.34}$$

This gives us the coordinate map $\Phi_\beta : \mathbb{C}^{m\times n} \to \mathbb{R}^{4mn}$ defined by simply stringing out the real and imaginary parts of the entries of the matrix row-by-row from first to last:

$$\Phi_\beta(A) = (\alpha_{11}, \beta_{11}, \dots, \alpha_{1n}, \beta_{1n}, \dots, \alpha_{m1}, \beta_{m1}, \dots, \alpha_{mn}, \beta_{mn}). \tag{1.35}$$

Of course, there are many other interesting vector spaces beyond this rather unimaginative list. Here is a token triple of examples.

Example 1.3.1. *Let* $V = \{A \in \mathbb{C}^{2\times 2} \mid A^T = A, \quad trace(A) = 0\ \}$. *Set* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *then* $A^T = A$ *yields* $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *thus* $b = c$ *and* a, d *are free. However,* $trace(A) = a + d = 0$ *hence* $d = -a$. *Note* $a, b \in \mathbb{C}$ *so we expand these with real variables* a_1, a_2, b_1, b_2 *to exhibit a typical element* $A \in V$ *has the form:*

$$A = \begin{bmatrix} a_1+ia_2 & b_1+ib_2 \\ b_1+ib_2 & -a_1-ia_2 \end{bmatrix} = a_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{v_1} + a_2 \underbrace{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}_{v_2} + b_1 \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{v_3} + b_2 \underbrace{\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}_{v_4}.$$

We may assign basis $\beta = \{v_1, v_2, v_3, v_4\}$ *for* V *with* $\Phi_\beta(A) = (a_1, a_2, b_1, b_2)$. *Notice the calculation above shows* $V \le \mathbb{C}^{2\times 2}$ *since it is clear* V *is formed by the span of* β. *I leave LI of* β *to the reader.*

Example 1.3.2. *Let W be the set of all $(x,y,z) \in \mathbb{R}^3$ for which $x+by+cz=0$ where $b,c \in \mathbb{R}$ are given constants. This is a plane with normal vector $(1,b,c)$. For $(x,y,z) \in W$ $x=-by-cz$ thus:*

$$(x,y,z) = (-by-cz,y,z) = y(-b,1,0)+z(-c,0,1)$$

The vectors $(-b,1,0)$ and $(-c,0,1)$ are orthogonal to $(1,b,c)$. Geometrically, we should recognize these as vectors which are tangent to the plane W. Let $\beta = \{(-b,1,0),\ (-c,0,1)\}$ and note $W = span(\beta)$. The coordinate map is given by

$$\Phi_\beta((-b,1,0)) = (1,0) \qquad \Phi_\beta((-c,0,1)) = (0,1)$$

extended linearly. In particular, we calculate:

$$\Phi_\beta(x,y,z) = \Phi_\beta\left(y(-b,1,0)+z(-c,0,1)\right) = y\Phi_\beta\left((-b,1,0)\right)+z\Phi_\beta\left((-c,0,1)\right) = (y,z).$$

Example 1.3.3. *Let $W = \{v_o+v_1t+v_2t^2+\cdots+v_nt^n \mid v_o,\ldots,v_n \in V\}$ where V is a vector space over $\mathbb{R}$. Addition of polynomials in W is done by adding coefficients of matching powers of t. Scalar multiplication is given by $c(v_o+v_1t+v_2t^2+\cdots+v_nt^n) = cv_o+cv_1t+\cdots+cv_nt^n$. We say W is the set of (possibly degenerate) polynomials of order n with coefficients from V. If $\beta_V = \{g_1,\ldots,g_k\}$ then a basis for W is given by*

$$\beta = \beta_V \cup \beta_V t \cup \cdots \beta_V t^n$$

where we define $\beta_V t^j = \{g_1t^j,\ldots,g_kt^j\}$ for $j=1,2,\ldots n$. Let us specialize this example to $W = \{v_o+v_1t+v_2t^2 \mid v_o,v_1,v_2 \in \mathbb{R}^{2\times 2}\}$ then

$$\beta = \{E_{ij}, E_{ij}t, E_{ij}t^2 \mid 1 \leq i,j \leq 2\}$$

Using the $11,12,21,22$ ordering,

$$\Phi_\beta(A+Bt+Ct^2) = (A_{11},A_{12},A_{21},A_{22},B_{11},B_{12},B_{21},B_{22},C_{11},C_{12},C_{21},C_{22})$$

We have $dim(W) = 12$. For n-th order polynomials with V-coefficients, $dim(W) = (n+1)dim(V)$.

There are many other examples of vector spaces, but, I'll stop here for brevity. Our goal here is not to be encylopedic. Rather, I want a set of examples which is robust enought for us to sharpen our mind on the theory of differentiation. I think we have more than enough.

1.4 linear transformations

Let us review the theory of linear transformations. Recall a **linear transformation** is a natural function of vector spaces. In particular $T: V \to W$ is a linear transformation iff T satisfies

$$(i.)\ \ T(x+y) = T(x)+T(y) \qquad \& \qquad (ii.)\ \ T(cx) = cT(x) \tag{1.36}$$

for all $x,y \in V$ and $c \in \mathbb{R}$. That is, T is both **additive**(i.) and **homogeneous**(ii.). If T is a linear transformation which is also injective and surjective then we say T is an **isomorphism** and we write $V \approx W$ by T. In other words, $V \approx W$ iff there exists a linear bijection $T: V \to W$. We also call an **injective** map an **injection** or a **one-to-one** linear transformation.

1.4.1 matrix of a linear transformation

If V has basis $\{v_1, \ldots, v_n\}$, the values of $T : V \to W$ on all of V are uniquely given by the values $T(v_i)$. Linearity implies:

$$T\left(\sum_i x_i v_i\right) = \sum_i x_i T(v_i). \tag{1.37}$$

If $\gamma = \{f_1, \ldots, f_m\}$ is a basis for W then we know $T(v_i) \in \text{span}(\gamma)$ implies there exist constants $A_{1i}, \ldots, A_{mi} \in \mathbb{R}$ for which $T(v_i) = A_{1i}f_1 + \cdots + A_{mi}f_m$. We define $[T]_{\beta,\gamma} = (A_{ij}) \in \mathbb{R}^{m\times n}$:

$$[T]_{\beta,\gamma} = [[T(v_1)]_\gamma | \cdots | [T(v_n)]_\gamma] \qquad \text{or, at the component level,} \qquad ([T]_{\beta,\gamma})_{ij} = (\Phi_\gamma(T(v_j)))_i \tag{1.38}$$

With respect to basis β for V and γ for W the linear function $T : V \to W$ is given by multiplication of $[T]_{\beta,\gamma}$ in the following sense:

$$[T(x)]_\gamma = [T]_{\beta,\gamma}[x]_\beta. \tag{1.39}$$

In the special case $T : \mathbb{R}^n \to \mathbb{R}^m$ we define the **standard matrix** $[T] \in \mathbb{R}^{m\times n}$ by $[T] = [T(e_1)| \cdots |T(e_n)]$. Observe, if $v = \sum_i v_i e_i$ then

$$T(v) = T\left(\sum_i v_i e_i\right) = \sum_i v_i T(e_i) = v_1 \text{col}_1[T] + \cdots + v_n \text{col}_n[T] = [T]v. \tag{1.40}$$

Let $A \in \mathbb{R}^{m\times n}$ then $L_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $L_A(v) = Av$ is the linear transformation **induced by left-multiplication of** A. For example, T is **induced** from the multiplication by $[T]$; $L_{[T]} = T$. For the case of column-vector transformations we often make use of the natural correspondence between L_A and A. Indeed, the fundamental theorem of linear algebra states that a function $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation iff $L = L_A$ for some $A \in \mathbb{R}^{m\times n}$.

1.4.2 change of coordinates

If we used other bases $\bar{\beta}$ for V and $\bar{\gamma}$ for W then the matrix $[T]_{\bar{\beta},\bar{\gamma}}$ is congruent to $[T]_{\beta,\gamma}$ in the following precise sense:

$$[T]_{\bar{\beta},\bar{\gamma}} = [\Phi_{\bar{\gamma}} \circ \Phi_\gamma^{-1}][T]_{\beta,\gamma}[\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}]. \tag{1.41}$$

Notice the matrix $[\Phi_{\bar{\gamma}} \circ \Phi_\gamma^{-1}] \in \mathbb{R}^{m\times m}$ whereas $[\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}] \in \mathbb{R}^{n\times n}$. In the special case $V = W$ with $\beta = \gamma$ the congruence above reduces to a **similarity transformation**; $[T]_{\bar{\beta},\bar{\beta}} = P^{-1}[T]_{\beta,\beta}P$ where $P = [\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}]$. Of course the actual calculation of P for an abstract vector space requires some calculation. That said, in the special case that $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ we have vastly simplified formulas born of concatenation: to begin the **basis matrix** for $\beta = \{v_1, \ldots, v_n\}$ is $[\beta] = [v_1| \cdots |v_n]$ and for $\gamma = \{f_1, \ldots f_m\}$ likewise define $[f_1| \cdots |f_m] \in \mathbb{R}^{m\times m}$. In fact,

$$\Phi_\beta(v) = [\beta]^{-1}v \qquad \& \qquad \Phi_\gamma(w) = [\gamma]^{-1}w \tag{1.42}$$

Recall once more the **standard matrix** of $T : \mathbb{R}^n \to \mathbb{R}^m$ is denoted $[T]$ and is the matrix of T for which $T(v) = [T]v$ for all $v \in \mathbb{R}^n$. Given this *standard* notation the previous equation is restated as:

$$[\Phi_\beta] = [\beta]^{-1} \qquad \& \qquad [\Phi_\gamma] = [\gamma]^{-1} \tag{1.43}$$

Now, one can argue that **matrix multiplication** is defined as it is simply to force $[T \circ S] = [T][S]$ for linear transformations on column vectors. Consider then, for $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ once more,

$$[\Phi_{\bar\gamma} \circ \Phi_\gamma^{-1}] = [\Phi_{\bar\gamma}][\Phi_\gamma^{-1}] = [\bar\gamma]^{-1}[\gamma] \qquad \& \qquad [\Phi_\beta \circ \Phi_{\bar\beta}^{-1}] = [\Phi_\beta][\Phi_{\bar\beta}^{-1}] = [\beta]^{-1}[\bar\beta] \tag{1.44}$$

Thus, for $T : \mathbb{R}^n \to \mathbb{R}^m$ we have the following:

$$[T]_{\bar\beta,\bar\gamma} = [\bar\gamma]^{-1}[\gamma][T]_{\beta,\gamma}[\beta]^{-1}[\bar\beta]. \tag{1.45}$$

Let us narrow the discussion even further. Suppose β, γ are the standard bases for $\mathbb{R}^n, \mathbb{R}^m$ respective then $[\beta] = [e_1|\cdots|e_n] = I_n$ and likewise $[\gamma] = I_m$ (the $m \times m$ identity matrix). Therefore, the formula above simplies further as, by convention $[T] = [T]_{\beta,\gamma}$ we derive:

$$[T]_{\bar\beta,\bar\gamma} = [\bar\gamma]^{-1}[T][\bar\beta]. \tag{1.46}$$

At last we reach the thing that most students are taught in introductory matrix algebra: for $T : \mathbb{R}^n \to \mathbb{R}^n$ if $\bar\beta$ is a non-standard basis then $P = [\bar\beta]^{-1}$ and

$$[T]_{\bar\beta\bar\beta} = P^{-1}[T]P. \tag{1.47}$$

Naturally, an important question to ask: *which choice of basis makes the formula for T most elegant.* This is the problem of **diagonalization** which ultimately leads us to study the **real Jordan form** which essentially gives you the most elegant formula which is possible for a given transformation on a finite dimensional vector space[8]. The structure of the Jordan form is in one-to-one correspondence with the **eigenvalues** and **generalized eigenspaces** of T.

We can also ask: *is there a nice choice of coordinates for a linear transformations $T : V \to W$ where $V \neq W$?*. More precisely, how can we choose basis β for V and γ for W such that $[T]_{\beta,\gamma}$ has a particularly simple form? Once answer is that we can choose bases such that

$$[T]_{\beta,\gamma} = \left[\begin{array}{c|c} I_p & 0 \\ \hline 0 & 0 \end{array}\right] = I_p \oplus 0_{n-p} \tag{1.48}$$

where $p = \text{rank}(T)$. The freedom to manipulate both β and γ allows us to force the matrix into this rather special form. Conversely, if there exist bases β, γ for which $[T]_{\beta,\gamma} = I_p \oplus 0_{n-p}$ then we can infer $\text{rank}(T) = p$ simply by inspection of the given matrix.

[8] this paragraphs need not be limited to the context $\mathbb{R}^n$, we can calculate the real Jordan form for linear transformations on abstract vector spaces.

Two important subspaces are defined for each linear transformation. The **kernel** of $T : V \to W$ is defined by $\text{Ker}(T) = T^{-1}\{0\} = \{v \in V \mid T(v) = 0\}$. The significance of the kernel is this: T is injective iff $\text{Ker}(T) = \{0\}$. On the other hand, we define the **image** or **range** of $T : V \to W$ by $\text{Range}(T) = T(V) = \{T(v) \mid v \in V\}$. If $\text{Range}(T) = T(V) = W$ then we have that T is a surjection. In the special case $T : \mathbb{R}^n \to \mathbb{R}^m$ the kernel and range are naturally connected to the **column space** and **null space** of the standard matrix:

$$\text{Ker}(T) = \text{Null}[T] \qquad \& \qquad \text{Range}(T) = \text{Col}([T]). \tag{1.49}$$

We can also calculate $\text{Ker}(T)$ and $\text{Range}(T)$ for a linear transformation of abstract vector spaces if we are willing to run our arguments through coordinate maps. If $[T]_{\beta,\gamma}$ is matrix of linear $T : V \to W$ with respect to bases β, γ of V, W respective then:

$$\text{Ker}(T) = \Phi_\beta^{-1}\left(\text{Null}[T]_{\beta,\gamma}\right) \qquad \& \qquad \text{Range}(T) = \Phi_\gamma^{-1}\left(\text{Col}[T]_{\beta,\gamma}\right). \tag{1.50}$$

The dimension of the kernel is known as the **nullity** of T; $\text{nullity}(T) = \dim(\text{Ker}(T))$. The dimension of the range is the **rank** of T; $\text{rank}(T) = \dim(\text{Range}(T))$. The **rank-nullity theorem** states

$$\dim(V) = \text{rank}(T) + \text{nullity}(T) \tag{1.51}$$

If $V \approx W$ then $\dim(V) = \dim(W)$ and the rank-nullity theorem specializes to

$$\dim(W) = \text{rank}(T) + \text{nullity}(T). \tag{1.52}$$

If T is injective then $\text{nullity}(T) = 0$ thus $\dim(W) = \text{rank}(T)$ hence T is surjective. Conversely, if T is surjective then $\dim(W) = \text{rank}(T)$ thus $\dim(W) = \dim(W) + \text{nullity}(T)$ and we find $\text{nullity}(T) = 0$ from which we conclude T is injective. In general, when $V \not\approx W$, we may have a surjection which is not an injection or vice-versa. [9] Notice, since $\text{Range}(T) \leq W$ it follows $\text{rank}(T) \leq \dim(W)$ and so $\text{rank}(T) = \dim(W)$ is the maximum possible rank for T. In other words, another characterization of a linear surjection is that it is a linear transformation with **full-rank**.

Remark 1.4.1.

We will continue to unfurl linear algebra throughout our study as it is needed.

[9] It is always possible to modify a given $T : V \to W$ to a related isomorphism. To obtain a surjection we simply replace W with the range(T). To obtain an injection we pass to the quotient vector space $V/\text{Ker}(T)$. Then $\tau(v + \text{Ker}(T)) = T(v)$ defines an isomorphism $\tau : V/\text{Ker}(T) \to T(V)$.

Chapter 2

topology and limits

Topology is the study of continuity and open sets in general. A **topology** on a given set of points is a collection of all sets which are defined to be **open**. Of course, the subject of topology is more interesting than just picking which sets are open, that's just the start of it. Topology also looks to find criteria with which to classify spaces as *topologically distinct*. Two topologies are equivalent if there exists a **homeomorphism** from one space to the other. A **homeomorphism** is bijection which preserves the topological structure. Just as an isomorphism was a bijection which preserved linear structure in the last Chapter. The abstract study of topology is a nontrivial task which we do not undertake here, I merely make these comments for some context.

Our study of topology is very introductory. We merely need topology as a langauge to describe sets in which are theorems and calculations typically hold. Moreover, the topologies we consider are all **metric** topologies. This means the definition of our open sets is based on some natural concept of distance. In the context of $\mathbb{R}^n$ it is simply Euclidean distance function. For a vector space with norm the distance is naturally induced from the norm. Typically, the theorems which interest us here can be shown true in the context of an abstract vector space with norm. Therefore, we give some proofs in that context. However, to be kind to the student we begin with a discussion of topology and limits in Euclidean space before we abstract to the arena of normed vector spaces. Continuity and limits in Euclidean spaces present new difficulties in two or more dimensions. Limits for finite dimensional vector spaces with a norm are not much different. In some sense, it's just the Euclidean case with some extra notational baggage. In particular, spaces of matrices provide an interesting class of examples which take us a bit beyond the context of Euclidean space.

Occasionally we need to quote a nontrivial topological result. For example, the fact that the real numbers are **complete** is an important base fact. However, discussing the justifcation of that fact is outside the scope (and interest) of this course. That said, we conclude this chapter by collecting a few basic theorems of a topological nature which we will not prove. I also recap all the interesting proofs from calculus I which we at times use in our study of advanced calculus. These are included for your edification and convenience, it is unlikely we devote much lecture time to the calculus I proofs.

2.1 elementary topology and limits

In this section we describe the *metric topology* for $\mathbb{R}^n$. In the study of functions of one real variable we often need to refer to open or closed intervals. The definition that follows generalizes those concepts to n-dimensions.

Definition 2.1.1.

An **open ball** of radius ϵ centered at $a \in \mathbb{R}^n$ is the subset all points in $\mathbb{R}^n$ which are less than ϵ units from a, we denote this open ball by

$$B_\epsilon(a) = \{x \in \mathbb{R}^n \mid ||x - a|| < \epsilon\}$$

The **closed ball** of radius ϵ centered at $a \in \mathbb{R}^n$ is likewise defined

$$\overline{B}_\epsilon(a) = \{x \in \mathbb{R}^n \mid ||x - a|| \leq \epsilon\}$$

Notice that in the $n = 1$ case we observe an open ball is an open interval: let $a \in \mathbb{R}$,

$$B_\epsilon(a) = \{x \in \mathbb{R} \mid ||x - a|| < \epsilon\} = \{x \in \mathbb{R} \mid |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

In the $n = 2$ case we observe that an open ball is an open disk: let $(a, b) \in \mathbb{R}^2$,

$$B_\epsilon((a, b)) = \left\{(x, y) \in \mathbb{R}^2 \mid ||\,(x, y) - (a, b)\,|| < \epsilon\right\} = \left\{(x, y) \in \mathbb{R}^2 \mid \sqrt{(x - a)^2 + (y - b)^2} < \epsilon\right\}$$

For $n = 3$ an open-ball is a sphere without the outer shell. In contrast, a closed ball in $n = 3$ is a solid sphere which includes the outer shell of the sphere.

Definition 2.1.2.

Let $D \subseteq \mathbb{R}^n$. We say $y \in D$ is an **interior point** of D iff there exists some open ball centered at y which is completely contained in D. We say $y \in \mathbb{R}^n$ is a **limit point** of D iff every open ball centered at y contains points in $D - \{y\}$. We say $y \in \mathbb{R}^n$ is a **boundary point** of D iff every open ball centered at y contains points not in D and other points which are in $D - \{y\}$. We say $y \in D$ is an **isolated point** of D if there exist open balls about y which do not contain other points in D. The set of all interior points of D is called the **interior of** D. Likewise the set of all boundary points for D is denoted ∂D. The **closure** of D is defined to be $\overline{D} = D \cup \{y \in \mathbb{R}^n \mid y \text{ a limit point}\}$

All the terms are aptly named. The term "limit point" is given because those points are the ones for which it is natural to define a limit. The picture below illustrates an interior point y_1, a boundary point y_2 and an isolated poin y_3. Also, the dotted-line around the hole indicates that edge is not part of the set. However, the closure of D would be formed by connecting those dots to give $\overline{D}$ a solid edge.

The dotted circle around y_1 is meant to illustrate an open ball which is centered on y_1 and it contained within D, the existence of this open ball is what proves y_1 is an interior point.

Definition 2.1.3.

Let $A \subseteq \mathbb{R}^n$ is an **open set** iff for each $x \in A$ there exists $\epsilon > 0$ such that $x \in B_\epsilon(x)$ and $B_\epsilon(x) \subseteq A$. Let $B \subseteq \mathbb{R}^n$ is an **closed set** iff its complement $\mathbb{R}^n - B = \{x \in \mathbb{R}^n \mid x \notin B\}$ is an open set.

Notice that $\mathbb{R} - [a, b] = (\infty, a) \cup (b, \infty)$. It is not hard to prove that open intervals are open hence we find that a closed interval is a closed set. Likewise it is not hard to prove that open balls are open sets and closed balls are closed sets. In fact, it can be shown that a closed set contains all its limit points, that is $A \subseteq \mathbb{R}^n$ is closed iff $A = \overline{A}$.

Limits of vector-valued functions of several real variables are defined below:

Definition 2.1.4.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping. We say that f has limit $b \in \mathbb{R}^m$ at limit point a of U iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $x \in \mathbb{R}^n$ with $0 < ||x - a|| < \delta$ implies $||f(x) - b|| < \epsilon$. In such a case we can denote the above by stating that

$$\lim_{x \to a} f(x) = b.$$

In calculus I the limit of a function is defined in terms of deleted open intervals centered about the limit point. We just defined the limit of a mapping in terms of deleted open balls centered at the limit point. The term "deleted" refers to the fact that we assume $0 < ||x - a||$ which means we do not consider $x = a$ in the limiting process. In other words, the limit of a mapping considers values close to the limit point but not necessarily the limit point itself. The case that the function is defined at the limit point is special, when the limit and the mapping agree then we say the mapping is continuous at that point.

Example 2.1.5. *Let $F : \mathbb{R}^n - \{a\} \to \mathbb{R}^n$ be defined by $F(x) = \frac{1}{||x-a||}(x - a)$. In this case, certainly a is a limit point of F but geometrically it is clear that $\lim_{x \to a} F(x)$ does not exist. Notice for $n = 1$, the discontinuity of F at a can be understood by seeing that left and right limits exist, but are not equal. On the other hand, $G(x) = \frac{||x-a||}{||x-a||}(x - a)$ clearly has $\lim_{x \to a} G(x) = 0$ and we could classify the discontinuity of G at $x = a$ as removeable. Clearly $\tilde{G}(x) = x - a$ is a continuous extension of G to all of $\mathbb{R}^n$*

Multivariate limits are much trickier than single-variate limits because there are infinitely many ways to approach a limit point. In the single-variate case we can only approach from the left $x \to a^-$ or from the right $x \to a^+$. However, even in $\mathbb{R}^2$ there are infinitely many lines on which we can approach a limit point. But, perhaps, even more insidiously, there are infinitely many parabolas, cubics, exponentials etc... which intersect the limit point. It turns out that the method of approach matters. It is possible for the function to limit to the same value by all linear paths and yet parabolic paths yield different values.

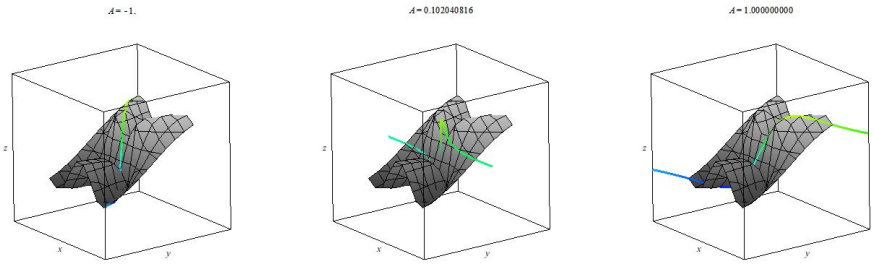
Example 2.1.6. *Suppose* $f(x,y) = \begin{cases} \frac{2x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$. *Notice that we can calculate the limit for* $(a,b) \neq (0,0)$ *with ease:*

$$\lim_{(x,y)\to(a,b)} = \frac{2a^2b}{a^4+b^2}.$$

However, if we consider the limit at $(0,0)$ *it is indeterminant since we have an expression of type* $0/0$. *Other calculation is required. Consider the path* $\vec{r}(t) = (t, mt)$ *then clearly this is continuous at* $t = 0$ *and* $\vec{r}(0) = (0,0)$*; in-fact, this is just the parametric equation of a line* $y = mx$. *Consider, for* $m \neq 0$,

$$\lim_{t\to 0} f(\vec{r}(t)) = \lim_{t\to 0} \frac{2mt^4}{t^4+m^2t^2} = \lim_{t\to 0} \frac{2mt^2}{t^2+m^2} = \frac{2m(0)}{0+m^2} = 0.$$

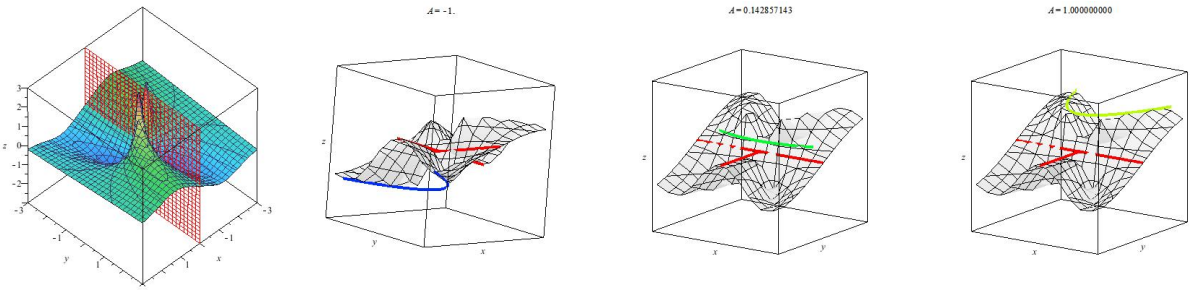
If $\vec{r}(t) = (t,0)$ *then for* $t \neq 0$ *we have* $f(\vec{r}(t)) = f(t,0) = 0$ *thus the limit of the function restricted to any linear path is just zero. The three pictures on the right illustrate how differing linear paths yield the same limits. The red lines are the* x, y *axes.*



What about parabolic paths? Those are easily constructed via $\vec{r}_2(t) = (t, kt^2)$ *again* $\vec{r}_2(0) = (0,0)$ *and* $\lim_{t\to 0} \vec{r}_2(t) = (0,0)$. *Calculate, for* $k \neq 0$,

$$\lim_{t\to 0} f(\vec{r}_2(t)) = \lim_{t\to 0} \frac{2kt^4}{t^4+k^2t^4} = \lim_{t\to 0} \frac{2k}{1+k^2} = \frac{2k}{1+k^2}.$$

Clearly if we choose differing values for k *we obtain different values for the limit hence the limit of* f *does not exist as* $(x,y) \to (0,0)$. *Here's the graph of this function, maybe you can see the problem at the origin. The red plane is vertical through the origin. The three pictures on the right illustrate how differing parabolic paths yield differing limits. The red lines are the* x, y *axes.*



See the limit of the blue path would be negative whereas the yellow path would give a positive limit.

Definition 2.1.7.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a function. If $a \in U$ is a limit point of f then we say that f is **continuous at** a iff

$$\lim_{x \to a} f(x) = f(a)$$

If $a \in U$ is an isolated point then we also say that f is continous at a. The mapping f is **continous on** S iff it is continous at each point in S. The **mapping f is continuous** iff it is continuous on its domain.

Notice that in the $m = n = 1$ case we recover the definition of continuous functions from calc. I. Another funny consequence of this definition is that sequences are by default continuous functions since each point in their domain is isolated.

Proposition 2.1.8.

Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a mapping with component functions $f_1, f_2, \ldots, f_m$ hence $f = (f_1, f_2, \ldots, f_m)$. If $a \in U$ is a limit point of f then

$$\lim_{x \to a} f(x) = b \qquad \Leftrightarrow \qquad \lim_{x \to a} f_j(x) = b_j \text{ for each } j = 1, 2, \ldots, m.$$

.

Proof: ($\Rightarrow$) Suppose $\lim_{x \to a} f(x) = b$. Then for each $\epsilon > 0$ choose $\delta > 0$ such that $0 < ||x - a|| < \delta$ implies $||f(x) - b|| < \epsilon$. This choice of δ suffices for our purposes as:

$$|f_j(x) - b_j| = \sqrt{(f_j(x) - b_j)^2} \leq \sqrt{\sum_{j=1}^{m} (f_j(x) - b_j)^2} = ||f(x) - b|| < \epsilon.$$

Hence we have shown that $\lim_{x \to a} f_j(x) = b_j$ for all $j = 1, 2, \ldots m$.

($\Leftarrow$) Suppose $\lim_{x \to a} f_j(x) = b_j$ for all $j = 1, 2, \ldots m$. Let $\epsilon > 0$. Note that $\epsilon/m > 0$ and therefore by the given limits we can choose $\delta_j > 0$ such that $0 < ||x - a|| < \delta$ implies $||f_j(x) - b_j|| < \sqrt{\epsilon/m}$. Choose $\delta = min\{\delta_1, \delta_2, \ldots, \delta_m\}$ clearly $\delta > 0$. Moreoever, notice $0 < ||x - a|| < \delta \leq \delta_j$ hence

requiring $0 < ||x - a|| < \delta$ automatically induces $0 < ||x - a|| < \delta_j$ for all j. Suppose that $x \in \mathbb{R}^n$ and $0 < ||x - a|| < \delta$ it follows that

$$||f(x) - b|| = ||\sum_{j=1}^{m}(f_j(x) - b_j)e_j|| = \sqrt{\sum_{j=1}^{m}|f_j(x) - b_j|^2} < \sum_{j=1}^{m}(\sqrt{\epsilon/m})^2 < \sum_{j=1}^{m}\epsilon/m = \epsilon.$$

Therefore, $\lim_{x\to a} f(x) = b$ and the proposition follows. □

The beauty of the previous proposition is that it means we can analyze the limit of a vector-valued function by analyzing the limits of the component functions. However, this does not remove the fundamental difficulty of analyzing the multivariate limits of the component functions. It just means we can tackel the problem one-component at time. This is a relief, it would be annoying if the range was as intertwined as the domain in this analysis.

Proposition 2.1.9.

Suppose that $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ is a vector-valued function with component functions $f_1, f_2, \ldots, f_m$. Let $a \in U$ be a limit point of f then f is continous at a iff f_j is continuous at a for $j = 1, 2, \ldots, m$. Moreover, f is continuous on S iff all the component functions of f are continuous on S. Finally, a vector-valued function f is continous iff all of its component functions are continuous. .

Proposition 2.1.10.

The projection functions are continuous. The identity mapping is continuous.

Proof: Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $x \in \mathbb{R}^n$ such that $0 < ||x - a|| < \delta$ then it follows that $||x-a|| < \epsilon$.. Therefore, $\lim_{x\to a} x = a$ which means that $\lim_{x\to a} Id(x) = Id(a)$ for all $a \in \mathbb{R}^n$. Hence Id is continuous on $\mathbb{R}^n$ which means Id is continuous. Since the projection functions are component functions of the identity mapping it follows that the projection functions are also continuous (using the previous proposition). □

Definition 2.1.11.

The **sum** and **product** are functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$s(x, y) = x + y \qquad p(x, y) = xy$$

Proposition 2.1.12.

The sum and product functions are continuous.

Preparing for the proof: Let the limit point be (a, b). Consider what we wish to show: given a point (x, y) such that $0 < ||(x, y) - (a, b)|| < \delta$ we wish to show that

$$|s(x, y) - (a + b)| < \epsilon \qquad \text{or for the product} \qquad |p(x, y) - (ab)| < \epsilon$$

follow for appropriate choices of δ. Think about the sum for a moment,

$$|s(x,y)-(a+b)| = |x+y-a-b| \leq |x-a|+|y-b|$$

I just used the triangle inequality for the absolute value of real numbers. We see that if we could somehow get control of $|x-a|$ and $|y-b|$ then we'd be getting closer to the prize. We have control of $0 < ||(x,y)-(a,b)|| < \delta$ notice this reduces to

$$||(x-a,y-b)|| < \delta \quad \Rightarrow \quad \sqrt{(x-a)^2+(y-b)^2} < \delta$$

it is clear that $(x-a)^2 < \delta^2$ since if it was otherwise the inequality above would be violated as adding a nonegative quantity $(y-b)^2$ only increases the radicand resulting in the squareroot to be larger than δ. Hence we may assume $(x-a)^2 < \delta^2$ and since $\delta > 0$ it follows $\boxed{|x-a| < \delta}$. Likewise, $\boxed{|y-b| < \delta}$. Thus $|s(x,y)-(a+b)| = |x+y-a-b| < |x-a|+|y-b| < 2\delta$. We see for the sum proof we can choose $\delta = \epsilon/2$ and it will work out nicely.

Proof: Let $\epsilon > 0$ and let $(a,b) \in \mathbb{R}^2$. Choose $\delta = \epsilon/2$ and suppose $(x,y) \in \mathbb{R}^2$ such that $||(x,y)-(a,b)|| < \delta$. Observe that

$$||(x,y)-(a,b)|| < \delta \;\Rightarrow\; ||(x-a,y-b)||^2 < \delta^2 \;\Rightarrow\; |x-a|^2+|y-b|^2 < \delta^2.$$

It follows $|x-a| < \delta$ and $|y-b| < \delta$. Thus

$$|s(x,y)-(a+b)| = |x+y-a-b| \leq |x-a|+|y-b| < \delta+\delta = 2\delta = \epsilon.$$

Therefore, $\lim_{(x,y)\to(a,b)} s(x,y) = a+b.$ and it follows that the sum function if continuous at (a,b). But, (a,b) is an arbitrary point thus s is continuous on $\mathbb{R}^2$ hence the sum function is continuous. $\square$.

Preparing for the proof of continuity of the product function: I'll continue to use the same notation as above. We need to study $|p(x,y)-(ab)| = |xy-ab| < \epsilon$. Consider that

$$|xy-ab| = |xy-ya+ya-ab| = |y(x-a)+a(y-b)| \leq |y||x-a|+|a||y-b|$$

We know that $|x-a| < \delta$ and $|y-b| < \delta$. There is one less obvious factor to bound in the expression. What should we do about $|y|$?. I leave it to the reader to show that:

$$\boxed{|y-b| < \delta \qquad \Rightarrow \qquad |y| < |b|+\delta}$$

Now put it all together and hopefully we'll be able to "solve" for ϵ.

$$|xy-ab| =\leq |y||x-a|+|a||y-b| < (|b|+\delta)\delta+|a|\delta = \delta^2+\delta(|a|+|b|) \text{ " = " } \epsilon$$

I put solve in quotes because we have considerably more freedom in our quest for finding δ. We could just as well[1] find δ which makes the " = " become an $<$. That said let's pursue equality,

$$\delta^2+\delta(|a|+|b|)-\epsilon = 0 \qquad \delta = \frac{-|a|-|b| \pm \sqrt{(|a|+|b|)^2+4\epsilon}}{2}$$

[1]Minh Nguyen showed me a much shorter argument involving well-known inequalities for the geometric and arithmetic means, this argument can be improved!

Since $\epsilon, |a|, |b| > 0$ it follows that $\sqrt{(|a|+|b|)^2+4\epsilon} < \sqrt{(|a|+|b|)^2} = |a|+|b|$ hence the $(+)$ solution to the quadratic equation yields a positive δ namely: $\delta = \frac{-|a|-|b|+\sqrt{(|a|+|b|)^2+4\epsilon}}{2}$.

Proof: Let $\epsilon > 0$ and let $(a,b) \in \mathbb{R}^2$. By the calculations that prepared for the proof we know that the following quantity is positive, hence choose

$$\delta = \frac{-|a|-|b|+\sqrt{(|a|+|b|)^2+4\epsilon}}{2} > 0.$$

Note that[2],

$$\begin{array}{rcll} |xy-ab| = |xy-ya+ya-ab| & = & |y(x-a)+a(y-b)| & \text{algebra} \\ & \leq & |y||x-a|+|a||y-b| & \text{triangle inequality} \\ & < & (|b|+\delta)\delta+|a|\delta & \text{by the boxed lemmas} \\ & = & \delta^2+\delta(|a|+|b|) & \text{algebra} \\ & = & \epsilon & \end{array}$$

where we know that last step follows due to the steps leading to the boxed equation in the proof preparation. Therefore, $\lim_{(x,y)\to(a,b)} p(x,y) = ab.$ and it follows that the product function if continuous at (a,b). But, (a,b) is an arbitrary point thus p is continuous on $\mathbb{R}^2$ hence the product function is continuous. $\square$.

Proposition 2.1.13.

Let $f : V \subseteq \mathbb{R}^p \to \mathbb{R}^m$ and $g : U \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be mappings. Suppose that $\lim_{x\to a} g(x) = b$ and suppose that f is continuous at b then

$$\lim_{x\to a}(f\circ g)(x) = f\big(\lim_{x\to a} g(x)\big).$$

The proof is on pages 46-47 of C.H. Edwards Advanced Calculus text. I will provide a proof in the setting of normed spaces later in this chapter. Notice that the proposition above immediately gives us the important result below:

Proposition 2.1.14.

Let f and g be mappings such that $f \circ g$ is well-defined. The composite function $f \circ g$ is continuous for points $a \in dom(f \circ g)$ such that the following two conditions hold:

1. g is continuous at a
2. f is continuous at $g(a)$.

[2] my notation is that when we stack inequalities the inequality in a particular line refers only to the immediate vertical successor.

I make use of the earlier proposition that a mapping is continuous iff its component functions are continuous throughout the examples that follow. For example, I know (Id, Id) is continuous since Id was previously proved continuous.

Example 2.1.15. *Note that if $f = p \circ (Id, Id)$ then $f(x) = \big(p \circ (Id, Id)\big)(x) = p\big((Id, Id)(x)\big) = p(x, x) = x^2$. Therefore, the quadratic function $f(x) = x^2$ is continuous on $\mathbb{R}$ as it is the composite of continuous functions.*

Example 2.1.16. *Note that if $f = p \circ (p \circ (Id, Id), Id)$ then $f(x) = p(x^2, x) = x^3$. Therefore, the cubic function $f(x) = x^3$ is continuous on $\mathbb{R}$ as it is the composite of continuous functions.*

Example 2.1.17. *The power function is inductively defined by $x^1 = x$ and $x^n = xx^{n-1}$ for all $n \in \mathbb{N}$. We can prove $f(x) = x^n$ is continous by induction on n. We proved the $n = 1$ case previously. Assume inductively that $f(x) = x^{n-1}$ is continuous. Notice that*

$$x^n = xx^{n-1} = xf(x) = p(x, f(x)) = (p \circ (Id, f))(x).$$

Therefore, using the induction hypothesis, we see that $g(x) = x^n$ is the composite of continuous functions thus it is continuous. We conclude that $f(x) = x^n$ is continuous for all $n \in \mathbb{N}$.

We can play similar games with the sum function to prove that sums of power functions are continuous. In your homework you will prove constant functions are continuous. Putting all of these things together gives us the well-known result that polynomials are continuous on $\mathbb{R}$.

Proposition 2.1.18.

Let a be a limit point of mappings $f, g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}$ and suppose $c \in \mathbb{R}$. If $\lim_{x\to a} f(x) = b_1 \in \mathbb{R}$ and $\lim_{x\to a} g(x) = b_2 \in \mathbb{R}$ then

1. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}(f(x)g(x)) = \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$.
3. $\lim_{x\to a}(cf(x)) = c \lim_{x\to a} f(x)$.

Moreover, if f, g are continuous then $f + g$, fg and cf are continuous.

Proof: Edwards proves (1.) carefully on pg. 48. I'll do (2.) here: we are given that If $\lim_{x\to a} f(x) = b_1 \in \mathbb{R}$ and $\lim_{x\to a} g(x) = b_2 \in \mathbb{R}$ thus by Proposition 2.1.8 we find $\lim_{x\to a}(f, g)(x) = (b_1, b_2)$. Consider then,

$$\begin{aligned}
\lim_{x\to a}(f(x)g(x)) &= \lim_{x\to a}\big(p(f, g)\big) && \text{defn. of product function}\\
&= p\big(\lim_{x\to a}(f, g)\big) && \text{since } p \text{ is continuous}\\
&= p(b_1, b_2) && \text{by Proposition 2.1.8.}\\
&= b_1 b_2 && \text{definition of product function}\\
&= \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big).
\end{aligned}$$

I leave it to the reader to show $\lim_{x\to a} c = c$ and hence item (3.) follows from (2.). □.

The proposition that follows does follow immediately from the proposition above, however I give a proof that again illustrates the idea we used in the examples. Reinterpreting a given function as a composite of more basic functions is a useful theoretical and calculational technique.

Proposition 2.1.19.

Assume $f, g : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}$ are continuous functions at $a \in U$ and suppose $c \in \mathbb{R}$.

1. $f + g$ is continuous at a.
2. fg is continuous at a
3. cf is continuous at a.

Moreover, if f, g are continuous then $f + g, fg$ and cf are continuous.

Proof: Observe that $(f + g)(x) = (s \circ (f, g))(x)$ and $(fg)(x) = (p \circ (f, g))(x)$. We're given that f, g are continuous at a and we know s, p are continuous on all of $\mathbb{R}^2$ thus the composite functions $s \circ (f, g)$ and $p \circ (f, g)$ are continuous at a and the proof of items (1.) and (2.) is complete. To prove (3.) I refer the reader to their homework where it was shown that $h(x) = c$ for all $x \in U$ is a continuous function. We then find (3.) follows from (2.) by setting $g = h$ (function multiplication commutes for real-valued functions). □.

We can use induction arguments to extend these results to arbitrarily many products and sums of power functions.To prove continuity of algebraic functions we'd need to do some more work with quotient and root functions. I'll stop here for the moment, perhaps I'll ask you to prove a few more fundamentals from calculus I. I haven't delved into the definition of exponential or log functions not to mention sine or cosine[3] We will assume that the basic functions of calculus are continuous on the interior of their respective domains. Basically if the formula for a function can be evaluated at the limit point then the function is continuous.

It's not hard to see that the comments above extend to functions of several variables and mappings. If the formula for a mapping is comprised of finite sums and products of power functions then we can prove such a mapping is continuous using the techniques developed in this section. If we have a mapping with a more complicated formula built from elementary functions then that mapping will be continuous provided its component functions have formulas which are sensibly calculated at the limit point. In other words, if you are willing to believe me that $\sin(x), \cos(x), e^x, \ln(x), \cosh(x), \sinh(x), \sqrt{x}, \frac{1}{x^n}, \ldots$ are continuous on the interior of their domains

[3]sine, cosine and exponentials are all nicely defined in terms of power series arguments, if time permits we may sketch the development of these basic functions when we discuss series calculation

then it's not hard to prove:

$$f(x,y,z) = \left(\sin(x) + e^x + \sqrt{\cosh(x^2) + \sqrt{y+e^x}},\ \cosh(xyz),\ xe^{\sqrt{x+\frac{1}{yz}})}\right)$$

is a continuous mapping at points where the radicands of the square root functions are nonnegative. It wouldn't be very fun to write explicitly but it is clear that this mapping is the Cartesian product of functions which are the sum, product and composite of continuous functions.

Definition 2.1.20.

A **polynomial in n-variables** has the form:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i_1,i_2,\ldots,i_k=0}^{\infty} c_{i_1,i_2,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_k}$$

where only finitely many coefficients $c_{i_1,i_2,\ldots,i_n} \neq 0$. We denote the set of multinomials in n-variables as $\mathbb{R}[x_1, x_2, \ldots, x_n]$.

Polynomials are $\mathbb{R}[x]$. Polynomials in two variables are $\mathbb{R}[x,y]$, for example,

$f(x,y)$	$= ax + by$	$deg(f) = 1$, linear function
$f(x,y)$	$= ax + by + c$	$deg(f) = 1$, affine function
$f(x,y)$	$= ax^2 + bxy + cy^2$	deg(f)=2, quadratic form
$f(x,y)$	$= ax^2 + bxy + cy^2 + dx + ey + g$	deg(f)=2

If all the terms in the polynomial have the same number of variables then it is said to be **homogeneous**. In the list above only the linear function and the quadratic form were homogeneous.

2.2 normed linear spaces

Definition 2.2.1.

Suppose V is a vector space. If $||\cdot|| : V \times V \to \mathbb{R}$ is a function such that for all $x, y \in V$ and $c \in \mathbb{R}$:

1. $||cx|| = |c|\,||x||$
2. $||x + y|| \leq ||x|| + ||y||$ (triangle inequality)
3. $||x|| \geq 0$
4. $||x|| = 0$ iff $x = 0$

then we say $(V, ||\cdot||)$ is a normed vector space. When there is no danger of ambiguity we also say that V is a normed vector space.

Notice that we did not assume V was finite-dimensional in the definition above. Our current focus is on finite-dimensional cases.

(1.) the standard **euclidean** norm on $\mathbb{R}^n$ is defined by $||v|| = \sqrt{v \bullet v}$.

(2.) the **taxicab** norm on $\mathbb{R}^n$ is defined by $||v||_1 = \sum_{j=1}^{n} |v_j|$.

(3.) the **sup** norm on $\mathbb{R}^n$ is defined by $||v||_\infty = \max\{|v_j| \mid j = 1, \ldots, n\}$

(4.) the p-norm on $\mathbb{R}^n$ is defined by $||v||_p = \left(\sum_{j=1}^{n} |v_j|^p\right)^{1/p}$ for $p \in \mathbb{N}$. Notice $||v|| = ||v||_2$ and the taxicabl is the $p = 1$ case and finally the sup-norm appears as $p \to \infty$.

If we identify points with vectors based at the origin then it is natural to think about a **circle** of radius 1 as the set of vectors (points) which have norm (distance) one. Focusing on $n = 2$,

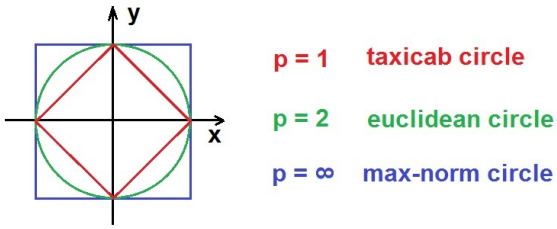
(1.) in the euclidean case a circle is the circle for $\mathbb{R}^2$ with $||v||_2$

(2.) if we use the taxicab norm on $\mathbb{R}^2$ then the circle is a diamond.

(3.) for $\mathbb{R}^2$ with the p-norm the circle is something like a blown-up circle.

(4.) as $p \to \infty$ the circle expands to a square.

In other words, to **square the circle** we need only study p-norms in the plane.



I'm not sure why philosophers insist on the impossibility of imagining this. Maybe they should have real analysis as part of the curriculum. In any event, we could play similar games for our other favorite examples, but primarily we just use the analog of the $p = 1, 2$ or ∞ norms in our application of norms.

(1.) the standard norm for $\mathbb{C}^n$ is given by $||v|| = ||\Phi_\beta(v)||$ where Φ_β is the standard coordinate map for $\mathbb{C}^n$ as defined in Equation 1.33.

(2.) the standard norm for $\mathbb{R}^{m\times n}$ is given by $||A|| = ||\Phi_\beta(A)||$ where Φ_β is the standard coordinate map for $\mathbb{R}^{m\times n}$ as defined in Equation 1.29.

(3.) the standard norm for $\mathbb{C}^{m\times n}$ is given by $||Z|| = ||\Phi_\beta(Z)||$ where Φ_β is the standard coordinate map for $\mathbb{C}^{m\times n}$ as defined in Equation 1.35.

In each case above there is some slick formula which hides the simple truth I described above; the length of matrices and complex vectors is simply the Euclidean length of the corresponding coordinate vectors.

$$||v||^2 = v^T\bar{v}, \qquad ||A||^2 = \text{trace}(A^TA), \qquad ||Z||^2 = \text{trace}(Z^\dagger Z)$$

where the complex vector $v = (v_1, \ldots, v_n)$ has conjugate vector $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_n)$ and the complex matrix Z has conjugates $\bar{Z}$ defined by $(\bar{Z})_{ij} = \bar{Z}_{ij}$ and $Z^\dagger = \bar{Z}^T$ is the **Hermitian conjugate**. Again, to be clear, there is not just one choice of norm for $\mathbb{C}^n, \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$. The set paired with the norm is what gives us the structure of a normed space. It is interesting that the choice does not actually matter to the limits we wish to calculate. We'll probably explore this claim explicitly in the assigned homework.

2.3 limits and continuity in normed linear spaces

In what follows here I give a brief overview of how we define open sets and limits in a normed vector space. The answer is simply to replace Euclidean vector length with the appropriate norm.

Definition 2.3.1. *Open sets and limit points for normed space V*

An open ball in $(V, ||\cdot||_V)$ is defined

$$B_\epsilon(x_o) = \{y \in V \mid ||y - x_o||_V < \epsilon\}.$$

We define the deleted open ball by removing the center from the open ball

$$B_\epsilon(x_o)_o = B_\epsilon(x_o) - \{x_o\} = \{y \in V \mid 0 < ||y - x_o||_V < \epsilon\}.$$

We say x_o is a limit point of a function f iff there exists a deleted open ball which is contained in the $dom(f)$. We say $U \subseteq V$ is an open set iff for each $u \in U$ there exists an open ball $B_\epsilon(u) \subseteq U$.

Limits and continuous functions are also defined in the same way as in $\mathbb{R}^n$.

Definition 2.3.2. *Limits and continuity in normed spaces.*

If $f : V \to W$ is a function from normed space $(V, ||\cdot||_V)$ to normed vector space $(W, ||\cdot||_W)$ then we say $\lim_{x\to x_o} f(x) = L$ iff for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in V$ subject to $0 < ||x - x_o||_V < \delta$ it follows $||f(x) - f(x_o)||_W < \epsilon$. If $\lim_{x\to x_o} f(x) = f(x_o)$ then we say that f is a continuous function at x_o.

Let $(V, ||\cdot||_V)$ be a normed vector space, a function from $\mathbb{N}$ to V is a called a **sequence**. Limits of sequences play an important role in analysis in normed linear spaces. The real analysis course makes great use of sequences to tackle questions which are more difficult with only $\epsilon-\delta$ arguments. In fact, we can reformulate limits in terms of sequences and subsequences. Perhaps one interesting feature of abstract topological spaces is the appearance of spaces in which sequential convergence is insufficient to capture the concept of limits. In general, one needs *nets* and *filters*. I digress. More important to our context, the criteria of **completeness**. Let us settle a few definitions to make the words meaningful.

Definition 2.3.3.

Suppose $\{a_n\}$ is a sequence then we say $\lim_{n\to\infty} a_n = L \in V$ iff for each $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $||a_n - L||_V < \epsilon$ for all $n \in \mathbb{N}$ with $n > M$. If $\lim_{n\to\infty} a_n = L \in V$ then we say $\{a_n\}$ is a **convergent sequence**.

We spent some effort attempting to understand the definition above and its application to the problem of infinite summations in calculus II. It is less likely you have thought much about the following:

Definition 2.3.4.

We say $\{a_n\}$ is a **Cauchy sequence** iff for each $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $||a_m - a_n||_V < \epsilon$ for all $m, n \in \mathbb{N}$ with $m, n > M$.

In other words, a sequence is Cauchy if the terms in the sequence get arbitarily close as we go sufficiently far out in the list. Many concepts we cover in calculus II are made clear with proofs built around the concept of a Cauchy sequence. The interesting thing about Cauchy is that for some spaces of numbers we can have a sequence which converges but is not Cauchy. For example, if you think about the rational numbers $\mathbb{Q}$ we can construct a sequence of truncated decimal expansions of π:

$$\{a_n\} = \{3, 3.1, 3.14, 3.141, 3.1415 \dots\}$$

note that $a_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and yet the $a_n \to \pi \notin \mathbb{Q}$. When spaces are missing their limit points they are in some sense incomplete.

Definition 2.3.5.

If every Cauchy sequence in a metric space converges to a point within the space then we say the metric space is **complete**. If a normed vector space V is complete then we say V is a **Banach space**.

A metric space need not be a vector space. In fact, we can take any open set of a normed vector space and construct a metric space. Metric spaces require less structure.

Fortunately all the main examples of this course are built on the real numbers which are complete, this induces completeness for $\mathbb{C}, \mathbb{R}^n$ and $\mathbb{R}^{m\times n}$. The proof that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ are Banach spaces follow from arguments similar to those given in the example below.

Example 2.3.6. Claim: *$\mathbb{R}$ complete implies $\mathbb{R}^2$ is complete.*

Proof: *suppose (x_n, y_n) is a Cauchy sequence in $\mathbb{R}^2$. Therefore, for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \in \mathbb{N}$ with $N < m < n$ implies $||(x_m, y_m) - (x_n, y_n)|| < \epsilon$. Consider that:*

$$||(x_m, y_m) - (x_n, y_n)|| = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2}$$

Therefore, as $|x_m - x_n| = \sqrt{(x_m - x_n)^2}$, it is clear that:

$$|x_m - x_n| \leq ||(x_m, y_m) - (x_n, y_n)||$$

But, this proves that $\{x_n\}$ is a Cauchy sequence of real numbers since for each $\epsilon > 0$ we can choose $N > 0$ such that $N < m < n$ implies $|x_m - x_n| < \epsilon$. The same holds true for the sequence $\{y_n\}$. By completeness of $\mathbb{R}$ we have $x_n \to x$ and $y_n \to y$ as $n \to \infty$. We propose that $(x_n, y_n) \to (x, y)$. Let $\epsilon > 0$ once more and choose $N_x > 0$ such that $n > N_x$ implies $|x_n - x| < \epsilon/2$ and $N_y > 0$ such that $n > N_y$ implies $|y_n - y| < \epsilon/2$. Let $N = max(N_x, N_y)$ and suppose $n > N$:

$$||(x_n, y_n) - (x, y)|| = ||x_n - x, 0) + (0, y_n - y)|| \leq |x_n - x| + |y_n - y| < \epsilon/2 + \epsilon/2 = \epsilon.$$

The key point here is that components of a Cauchy sequence form Cauchy sequences in $\mathbb{R}$. That will also be true for sets of matrices and complex numbers.

At this point it is appropriate to use the Cauchy sequence construction to prove the matrix exponential's convergence follows naturally from the single-variate exponential's convergence.

Moreover, this proof is almost identical to that which I gave to prove the limit of a sum was the sum of the limits. It is not an accident that the structure of sequential limits and continuum limits are so closely paired. However, I leave further analysis of that point to analysis. At the risk of killing that which is already dead[4]

Proposition 2.3.7.

Let V, W be normed vector spaces. Let a be a limit point of mappings $f, g : U \subseteq V \to W$ and suppose $c \in \mathbb{R}$. If $\lim_{x\to a} f(x) = b_1 \in W$ and $\lim_{x\to a} g(x) = b_2 \in W$ then

1. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}(cf(x)) = c \lim_{x\to a} f(x)$.

Moreover, if f, g are continuous then $f + g$ and cf are continuous.

[4] turns out this is quite interesting television and cinema for what it's worth

Proof: Let $\epsilon > 0$ and suppose $\lim_{x\to a} f(x) = b_1 \in W$ and $\lim_{x\to a} g(x) = b_2 \in W$. Choose $\delta_1, \delta_2 > 0$ such that $0 < ||x-a|| < \delta_1$ implies $||f(x)-b_1|| < \epsilon/2$ and $0 < ||x-a|| < \delta_2$ implies $||g(x)-b_2|| < \epsilon/2$. Choose $\delta = min(\delta_1, \delta_2)$ and suppose $0 < ||x-a|| < \delta \leq \delta_1, \delta_2$ hence

$$||(f+g)(x) - (b_1+b_2)|| = ||f(x) - b_1 + g(x) - b_2|| \leq ||f(x) - b_1|| + ||g(x) - b_2|| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Item (2.) follows. To prove (2.) note that if $c = 0$ the result is clearly true so suppose $c \neq 0$. Suppose $\epsilon > 0$ and choose $\delta > 0$ such that $||f(x) - b_1|| < \epsilon/|c|$. Note that if $0 < ||x-a|| < \delta$ then

$$||(cf)(x) - cb_1|| = ||c(f(x) - b_1)|| = |c|||f(x) - b_1|| < |c|\epsilon/|c| = \epsilon.$$

The claims about continuity follow immediately from the limit properties $\square$.

Perhaps you recognize these arguments from calculus I. The logic used to prove the basic limit theorems on $\mathbb{R}$ is essentially identical.

Proposition 2.3.8.

Suppose V_1, V_2, V_3 are normed vector spaces with norms $||\cdot||_1, ||\cdot||_2, ||\cdot||_3$ respective. Let $f : dom(f) \subseteq V_2 \to V_3$ and $g : dom(g) \subseteq V_1 \to V_2$ be mappings. Suppose that $\lim_{x\to x_o} g(x) = y_o$ and suppose that f is continuous at y_o then

$$\lim_{x\to x_o} (f \circ g)(x) = f\big(\lim_{x\to x_o} g(x)\big).$$

Proof: Let $\epsilon > 0$ and choose $\beta > 0$ such that $0 < ||y-b||_2 < \beta$ implies $||f(y) - f(y_o)||_3 < \epsilon$. We can choose such a β since Since f is continuous at y_o thus it follows that $\lim_{y\to y_o} f(y) = f(y_o)$. Next choose $\delta > 0$ such that $0 < ||x - x_o||_1 < \delta$ implies $||g(x) - y_o||_2 < \beta$. We can choose such a δ because we are given that $\lim_{x\to x_o} g(x) = y_o$. Suppose $0 < ||x - x_o||_1 < \delta$ and let $y = g(x)$ note $||g(x) - y_o||_2 < \beta$ yields $||y - y_o||_2 < \beta$ and consequently $||f(y) - f(y_o)||_3 < \epsilon$. Therefore, $0 < ||x-x_o||_1 < \delta$ implies $||f(g(x))-f(y_o)||_3 < \epsilon$. It follows that $\lim_{x\to x_o}(f(g(x)) = f(\lim_{x\to x_o} g(x))$. $\square$

The squeeze theorem relies heavily on the order properties of $\mathbb{R}$. Generally a normed vector space has no natural ordering. For example, is $1 > i$ or is $1 < i$ in $\mathbb{C}$? That said, we can state a squeeze theorem for functions whose domain reside in a normed vector space. This is a generalization of what we learned in calculus I. That said, the proof offered below is very similar to the typical proof which is not given in calculus I[5]

[5]this is lifted word for word from my calculus I notes, however here the meaning of open ball is considerably more general and the linearity of the limit which is referenced is the one proven earlier in this section

Proposition 2.3.9. *squeeze theorem.*

Suppose $f : dom(f) \subseteq V \to \mathbb{R}$, $g : dom(g) \subseteq V \to \mathbb{R}$, $h : dom(h) \subseteq V \to \mathbb{R}$ where V is a normed vector space with norm $||\cdot||$. Let $f(x) \leq g(x) \leq h(x)$ for all x on some $\delta > 0$ ball of[6] $a \in V$ then we find that the limits at x_o follow the same ordering,

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x) \leq \lim_{x\to a} h(x).$$

Moreover, if $\lim_{x\to x_o} f(x) = \lim_{x\to x_o} h(x) = L \in \mathbb{R}$ then $\lim_{x\to x_o} f(x) = L$.

Proof: Suppose $f(x) \leq g(x)$ for all $x \in B_{\delta_1}(a)_o$ for some $\delta_1 > 0$ and also suppose $\lim_{x\to a} f(x) = L_f \in \mathbb{R}$ and $\lim_{x\to a} g(x) = L_g \in \mathbb{R}$. We wish to prove that $L_f \leq L_g$. Suppose otherwise towards a contradiction. That is, suppose $L_f > L_g$. Note that $\lim_{x\to a}[g(x) - f(x)] = L_g - L_f$ by the linearity of the limit. It follows that for $\epsilon = \frac{1}{2}(L_f - L_g) > 0$ there exists $\delta_2 > 0$ such that $x \in B_{\delta_2}(a)_o$ implies $|g(x) - f(x) - (L_g - L_f)| < \epsilon = \frac{1}{2}(L_f - L_g)$. Expanding this inequality we have

$$-\frac{1}{2}(L_f - L_g) < g(x) - f(x) - (L_g - L_f) < \frac{1}{2}(L_f - L_g)$$

adding $L_g - L_f$ yields,

$$-\frac{3}{2}(L_f - L_g) < g(x) - f(x) < -\frac{1}{2}(L_f - L_g) < 0.$$

Thus, $f(x) > g(x)$ for all $x \in B_{\delta_2}(a)_o$. But, $f(x) \leq g(x)$ for all $x \in B_{\delta_1}(a)_o$ so we find a contradiction for each $x \in B_\delta(a)$ where $\delta = min(\delta_1, \delta_2)$. Hence $L_f \leq L_g$. The same proof can be applied to g and h thus the first part of the theorem follows.

Next, we suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \in \mathbb{R}$ and $f(x) \leq g(x) \leq h(x)$ for all $x \in B_{\delta_1}(a)$ for some $\delta_1 > 0$. We seek to show that $\lim_{x\to a} f(x) = L$. Let $\epsilon > 0$ and choose $\delta_2 > 0$ such that $|f(x) - L| < \epsilon$ and $|h(x) - L| < \epsilon$ for all $x \in B_\delta(a)_o$. We are free to choose such a $\delta_2 > 0$ because the limits of f and h are given at $x = a$. Choose $\delta = min(\delta_1, \delta_2)$ and note that if $x \in B_\delta(a)_o$ then

$$f(x) \leq g(x) \leq h(x)$$

hence,

$$f(x) - L \leq g(x) - L \leq h(x) - L$$

but $|f(x) - L| < \epsilon$ and $|h(x) - L| < \epsilon$ imply $-\epsilon < f(x) - L$ and $h(x) - L < \epsilon$ thus

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon.$$

Therefore, for each $\epsilon > 0$ there exists $\delta > 0$ such that $x \in B_\delta(a)_o$ implies $|g(x) - L| < \epsilon$ so $\lim_{x\to a} g(x) = L$. □

Our typical use of the theorem above applies to equations of norms from a normed vector space. The norm takes us from V to $\mathbb{R}$ so the theorem above is essential to analyze interesting limits. We shall make use of it in the next chapter.

Proposition 2.3.10. *norm is continuous with respect to itself.*

Suppose V has norm $||\cdot||$ then $f : V \to \mathbb{R}$ defined by $f(x) = ||x||$ is continuous.

Proof: Suppose $a \in V$ and let $\epsilon > 0$. Choose $\delta = \epsilon$ and consider $x \in V$ such that $0 < ||x - a|| < \delta$. Observe $||x|| = ||x - a + a|| \leq ||x - a|| + ||a|| = \delta + ||a||$ and hence

$$|f(x) - f(a)| = |||x|| - ||a||| < |\delta + ||a|| - ||a||| = |\delta| = \epsilon.$$

Thus $f(x) \to f(a)$ as $x \to a$ and as $a \in V$ was arbitrary the proposition follows $\Box$.

Perhaps the most interesting feature of an abstract vector space is the loss of a canonical basis in general. One might worry that this ambiguity spoils the result of Proposition 2.1.8, however, this is not the case. Existence of limits for a set of component functions with respect to a particular basis implies existence for all others[7]

Theorem 2.3.11.

Let V, W be normed vector spaces and suppose W has basis $\beta = \{w_j\}_{j=1}^m$. Let $a \in V$ then

$$\lim_{x \to a} F(x) = B = \sum_{j=1}^{m} B_j w_j \qquad \Leftrightarrow \qquad \lim_{x \to a} F_j(x) = B_j \text{ for all } j = 1, 2, \ldots m.$$

Proof: Throughout this proof we should keep in mind the basis vector $w_j \neq 0$ implies $||w_j|| \neq 0$ hence we may form quotients by the length of any basis vector if the need arises.

Suppose $\lim_{x \to a} F(x) = B = \sum_{j=1}^m B_j w_j$. Also, let $F_j : V \to \mathbb{R}$ denote the j-th component function of F with respect to basis β; that is suppose $F = \sum_{j=1}^m F_j w_j$. Consider that by the triangle inequality:

$$\big|\big|(F_j(x) - B_j)w_j\big|\big| \leq ||(F_1(x) - B_1)w_1 + \cdots + (F_m(x) - B_m)w_m|| = ||F(x) - B||$$

Therefore, using the above in the last step, observe:

$$F_j(x) - B_j = \frac{||w_j||}{||w_j||}|F_j(x) - B_j| \leq \frac{1}{||w_j||}\big|\big|(F_j(x) - B_j)w_j\big|\big| \leq \frac{1}{||w_j||}||F(x) - B||.$$

Therefore, $|F_j(x) - B_j| \leq \frac{1}{||w_j||}||F(x) - B||$. Let $\epsilon > 0$ and choose $\delta > 0$ such that $||F(x) - B|| < \epsilon||w_j||$. If $x \in V$ such that $0 < ||x - a|| < \delta$ then note,

$$|F_j(x) - B_j| = \frac{1}{||w_j||} \leq ||F(x) - B|| < \epsilon.$$

[7] component functions with respect to different bases are related by multiplication by a constant nonsingular matrix, so this result is not too surprising.

Thus $\lim_{x\to a} F_j(x) = B_j$ as $x \to a$ for all $j \in \mathbb{N}_n$.
Conversely, suppose $\lim_{x\to a} F_j(x) = B_j$ as $x \to a$ for all $j \in \mathbb{N}_n$. Let $\epsilon > 0$ and choose $\delta_j > 0$ such that $0 < ||x-a|| < \delta_j$ implies $||F_j(x) - B_j|| < \frac{\epsilon}{||w_j||m}$. We are free to choose such δ_j by the given limits as clearly $\frac{\epsilon}{||w_j||m} > 0$ for each j. Choose $\delta = min(\delta_j \mid j \in \mathbb{N}_m\}$ and suppose $x \in V$ such that $0 < ||x-a|| < \delta$. Using properties $||x+y|| \leq ||x|| + ||y||$ and $||cx|| = |c|||x||$ multiple times yield:

$$||F(x) - B|| = ||\sum_{j=1}^{m}(F_j(x) - B_j)w_j|| \leq \sum_{j=1}^{m}|F_j(x) - B_j|||w_j|| < \sum_{j=1}^{m}\frac{\epsilon}{||w_j||m}||w_j|| = \sum_{j=1}^{m}\frac{\epsilon}{m} = \epsilon.$$

Therefore, $\lim_{x\to a} F(x) = B$ and this completes the proof $\square$.

At this point it is appropriate to show $\mathrm{Gl}(n,\mathbb{R})$ is an open set of matrices and other such examples.

2.4 intuitive theorems of calculus

In this section we review some of the important theorems of calculus which justify analytical arguments via derivatives and continuity. Most of these you should have seen in Calculus I, but perhaps the proof was not given. I give the proofs here for your mathematical edification. I doubt we'll cover them in lecture. That said, the end of this section presents the Extreme Value Theorem's generalization to multivariate, real-valued functions and we will probably discuss that. One interesting thing to think about, which theorems in the next subsection generalize to functions whose domain is a Banach space and range if $\mathbb{R}$?

2.4.1 theorems for single-variable calculus

Proposition 2.4.1.

Let f be continuous at c such that $f(c) \neq 0$ then there exists $\delta > 0$ such that either $f(x) > 0$ or $f(x) < 0$ for all $x \in (c-\delta, c+\delta)$.

Proof: we are given that $\lim_{x\to c} f(x) = f(a) \neq 0$.

1.) Assume that $f(a) > 0$. Choose $\epsilon = \frac{f(a)}{2}$ and use existence of the limit $\lim_{x\to c} f(x) = f(a)$ to select $\delta > 0$ such that $0 < |x-c| < \delta$ implies $|f(x)-f(a)| < \frac{f(a)}{2}$ hence $-\frac{f(a)}{2} < f(x)-f(a) < \frac{f(a)}{2}$. Adding $f(a)$ across the inequality yields $0 < \frac{f(a)}{2} < f(x) < \frac{3f(a)}{2}$.

2.) If $f(a) < 0$ then we can choose $\epsilon = -\frac{f(a)}{2} > 0$ and select $\delta > 0$ such that $0 < |x-c| < \delta$ implies $|f(x) - f(a)| < -\frac{f(a)}{2}$ hence $\frac{f(a)}{2} < f(x) - f(a) < -\frac{f(a)}{2}$. It follows that $\frac{3f(a)}{2} < f(x) < \frac{f(a)}{2} < 0$.

The proposition follows. $\square$

Bolzano understood there was a gap in the arguments of the founders of calculus. Often, theorems like those stated in this section would merely be claimed without proof. The work of Bolzano and

others like him ultimately gave rise to the careful rigorous study of the real numbers and more generally the study of *real analysis* [8]

Proposition 2.4.1 is clearly extended to sets which have boundary points. If we know a function is continuous on $[a, b)$ and $f(a) \neq 0$ then we can find $\delta > 0$ such that $f([a, a + \delta)) > 0$. (*This is needed in the proof below in the special case that* $c = a$ *and a similar comment applies to* $c = b$.)

Theorem 2.4.2. *Bolzano's theorem*

Let f be continuous on $[a.b]$ such that $f(a)f(b) < 0$ then there exists $c \in (a, b)$ such that $f(c) = 0$.

Proof: suppose $f(a) < f(b)$ then $f(a)f(b) < 0$ implies $f(a) < 0$ and $f(b) > 0$. We can use axiom A11 for the heart of this proof. Our goal is to find a nonempty subset $S \subseteq \mathbb{R}$ which has an upper bound. Axiom A11 will then provides the existence of the least upper bound. We should like to construct a set which has the property desired in this theorem. Define $S = \{x \in [a, b] \mid f(x) < 0\}$. Notice that $a \in S$ since $f(a) < 0$ thus $S \neq \emptyset$. Moreover, it is clear that $x \leq b$ for all $x \in S$ thus S is bounded above. Axiom A11 states that there exists a least upper bound $c \in S$. To say c is the least upper bound means that any other upperbound of S is larger than c.

We now seek to show that $f(c) = 0$. Consider that there exist three possibilities:

1. if $f(c) < 0$ then the continuous function f has $f(c) \neq 0$ so by prop. 2.4.1 there exists $\delta > 0$ such that $x \in (c - \delta, c + \delta) \cap [a, b]$ implies $f(x) < 0$. However, this implies there is a value $x \in [c, c + \delta)$ such that $f(x) < 0$ and $x > c$ which means x is in S and is larger than the upper bound c. Therefore, c is not an upper bound of S. Obviously this is a contradiction therefore $f(c) \not< 0$.

2. if $f(c) > 0$ then the continuous function f has $f(c) \neq 0$ so by prop. 2.4.1 there exists $\delta > 0$ such that $x \in (c - \delta, c + \delta) \cap [a, b]$ implies $f(x) > 0$. However, this implies that all values $x \in (c - \delta, c]$ have $f(x) > 0$ and thus $x \notin S$ which means $x = c - \delta/2 < c$ is an upper bound of S which is smaller than the least upper bound c. Therefore, c is not the least upper bound of S. Obviously this is a contradiction therefore $f(c) \not> 0$.

3. if $f(c) = 0$ then no contradiction is found. The theorem follows. □

My proof here essentially follows Apostol's argument, however I suspect this argument in one form or another can be found in many serious calculus texts. With Bolzano's theorem settled we can prove the IVT without much difficulty.

[8] the Bolzano-Weierstrauss theorem is one of the central theorems of real analysis, in 1817 Bolzano used it to prove the IVT. It states every bounded sequence contains a convergent subsequence. Sequences can also be used to formulate limits and continuity. Sequential convergence is dealt with properly in Math 431 at LU.

Proposition 2.4.3. **I***ntermediate* **V***alue* **T***heorem* .

Suppose that f is continuous on an interval $[a,b]$ with $f(a) \neq f(b)$ and let N be a number such that N is between $f(a)$ and $f(b)$ then there exists $c \in (a,b)$ such that $f(c) = N$.

Proof: let N be as described above and define $g(x) = f(x) - N$. Note that g is clearly continuous. Suppose that $f(a) < f(b)$ then we must have $f(a) < N < f(b)$ which gives $f(a) - N \leq 0 \leq f(b) - N$ hence $g(a) < 0 < g(b)$. Applying Bolzano's theorem to g gives $c \in (a,b)$ such that $g(c) = 0$. But, $g(c) = f(c) - N = 0$ therefore $f(c) = N$. If $f(a) > f(b)$ then a similar argument applies. □.

Proposition 2.4.4. *Extreme value theorem.*

Suppose that f is a function which is continuous on $[a,b]$ then f attains its absolute maximum $f(c)$ on $[a,b]$ and its absolute minimum $f(d)$ on $[a,b]$ for some $c, d \in [a,b]$.

It's easy to see why the requirement of continuity is essential. If the function had a vertical asymptote on $[a,b]$ then the function gets arbitrarily large or negative so there is no biggest or most negative value the function takes on the closed interval. Of course, if we had a vertical asymptote then the function is not continuous at the asymptote. The proof of this theorem is technical and beyond the scope of this course. See Apostol pages 150-151 for a nice proof.

Definition 2.4.5. *critical numbers.*

We say $c \in \mathbb{R}$ is a **critical number** of a function f if either $f'(c) = 0$ or $f'(c)$ does not exist. If $c \in dom(f)$ is a critical number then $(c, f(c))$ is a **critical point** of f.

Notice that a critical number need not be in the domain of a given function. For example, $f(x) = 1/x$ has $f'(x) = -1/x^2$ and thus $c = 0$ is a critical numbers as $f'(0)$ does not exist in $\mathbb{R}$. Clearly $0 \notin dom(f)$ either. It is usually the case that a vertical asymptote of the function will likewise be a vertical asymptote of the derivative function.

Proposition 2.4.6. *Fermat's theorem.*

If f has a local extreme value of $f(c)$ and $f'(c)$ exists then $f'(c) = 0$.

Proof: suppose $f(c)$ is a local maximum. Then there exists $\delta_1 > 0$ such that $f(c+h) \leq f(c)$ for all $h \in B_{\delta_1}(0)$. Furthermore, since $f'(c) \in \mathbb{R}$ we have $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = f'(c) \in \mathbb{R}$. If $h > 0$ and $h \in B_{\delta_1}(0)$ then $f(c+h) - f(c) \leq 0$ hence,

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

Using the squeeze theorem we find $f'(c) = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq \lim_{h \to 0}(0) = 0$. Likewise, if $h < 0$ and $h \in B_{\delta_1}(0)$ then $f(c+h) - f(c) \leq 0$ hence,

$$\frac{f(c+h) - f(c)}{h} \geq 0$$

Using the squeeze theorem we find $f'(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq \lim_{h\to 0}(0) = 0$. Consequently, $f'(c) \leq 0$ and $f'(c) \geq 0$ therefore $f'(c) = 0$. The proof in the case that $f(c)$ is a local minimum is similar. □

Remember, if $f'(c)$ does not exist then c is a critical point by definition. Therefore, if $f(c)$ is a local extrema then c must be a critical point for one of two general reasons:

1. $f'(c)$ exists so Fermat's theorem proves $f'(c) = 0$ so c is a critical point.
2. $f'(c)$ does not exist so by definition c is a critical point.

Sometimes Fermat's Theorem is simply stated as "local extrema happen at critical points".

The converse of this Theorem is not true. We can have a critical number c such that $f(c)$ is **not** a local maximum or minimum. For example, $f(x) = x^3$ has critical number $c = 0$ yet $f(0) = 0$ which is neither a local max. nor min. value of $f(x) = x^3$. It turns out that $(0, 0)$ is actually an *inflection point* as we'll discuss soon. Another example of a critical point which yields something funny is a constant function; if $g(x) = k$ then $g'(x) = 0$ for each and every $x \in dom(g)$. Technically, $y = k$ is both the minimum and maximum value of g. Constant functions are a sort of exceptional case in this game we are playing.

Proposition 2.4.7. *Rolle's theorem.*

Suppose that f is a function such that

1. f is continuous on $[a, b]$,
2. f is differentiable on (a, b),
3. $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof: If $f(x) = k$ for all $x \in [a, b]$ then every point is a critical point and the theorem is trivially satisfied. Suppose f is not constant, apply the Extreme Value Theorem to show there exists $c, d \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$ and $f(d) \leq f(x)$ for all $x \in [a, b]$. Since $f(x_o) \neq f(a)$ for at least one $x_o \in (a, b)$ it follows that $f(x_o) > f(a)$ or $f(x_o) < f(a)$. If $x_o \in (a, b)$ and $f(x_o) > f(a)$ then $f(a)$ is not the absolute maximum therefore we deduce $c \in (a, b)$ is the absolute maximum. Likewise, if $x_o \in (a, b)$ and $f(x_o) < f(a)$ then $f(a)$ is not the absolute minimum therefore we deduce $d \in (a, b)$ is the absolute maximum. In all cases there is an absolute extremum in the open set (a, b) hence there exists a critical point in the interior of the set. Moreover, since f is differentiable on (a, b) it follows that either $f'(c) = 0$ or $f'(d) = 0$ and Rolle's theorem follows. □

Proposition 2.4.8. *Mean Value Theorem (MVT).*

Suppose that f is a function such that

1. f is continuous on $[a,b]$,
2. f is differentiable on (a,b),

Then there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. That is, there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

Proof:(essentially borrowed from Stewart's Calculus, this proof is common to a host of calculus texts). The equation of the secant line to $y = f(x)$ on the interval $[a,b]$ is $y = s(x)$ where $s(x)$ is defined via the standard formula for a line going from $(a, f(a)$ to $(b, f(b))$

$$s(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

The Mean Value Theorem proposes that there is some point on the interval $[a,b]$ such that the slope of the tangent line is equal to the slope of the secant line $y = s(x)$. Consider a new function defined to be the difference of the secant line and the given function, call it h:

$$h(x) = f(x) - s(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Observe that $h(a) = h(b) = 0$ and h is clearly continuous on $[a,b]$ because f is continuous and besides that the function is constructed from a sum of a polynomial with f. Additionally, it is clear that h is differentiable on (a,b) since polynomials are differentiable everywhere and f was assumed to be differentiable on (a,b). Thus Rolle's Theorem applies to h so there exists a $c \in (a,b)$ such that $h'(c) = 0$ which yields

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \quad \Longrightarrow \quad \boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}. \qquad \square$$

Proposition 2.4.9. *sign of the derivative function f' indicates strict increase or decrease of f.*

Suppose that f is a function and J is a connected subset of $dom(f)$

1. if $f'(x) > 0$ for all $x \in J$ then f is strictly increasing on J
2. if $f'(x) < 0$ for all $x \in J$ then f is strictly decreasing on J.

Proof: suppose $f'(x) > 0$ for all $x \in J$. Let $[a,b] \subseteq J$ and note f is continuous on $[a,b]$ since it is given to be differentiable on a superset of $[a,b]$. The MVT applied to f with respect to $[a,b]$ implies there exists $c \in [a,b]$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Notice that $f(b)-f(a) = (b-a)f'(c)$ but $b-a > 0$ and $f'(c) > 0$ hence $f(b)-f(a) > 0$. Therefore, for each pair $a, b \in J$ with $a < b$ we find $f(a) < f(b)$ which means f is strictly increasing on J. Likewise, if $f'(c) < 0$ then almost the same argument applies to show $a < b$ implies $f(a) > f(b)$. □

2.4.2 theorems for multivariate calculus

To begin, we need to define[9] compact sets in $\mathbb{R}^n$.

Definition 2.4.10.

We say C is a **bounded** subset of $\mathbb{R}^n$ if there exists an open ball which contains C. A subset $C \subseteq \mathbb{R}^n$ is **compact** iff it is both closed and bounded.

In the one-dimensional case, clearly closed intervals $[a, b]$ are compact sets. In higher dimensions it's not hard to see that spheres, cubes, disks, ellipses, ellipsoids and more generally finite shapes with solid edges are compact. In particular, if there is some maximum distance between points then it is clear we can place a slightly larger ball around the set so it's bounded. Also, if the edges are solid then the set should be closed.

Definition 2.4.11.

Let $S \subset \mathbb{R}$. We say $u \in \mathbb{R}$ is an **upper bound** of S if $u \geq s$ for all $s \in S$. We say $l \in \mathbb{R}$ is an **lower bound** of S if $l \leq s$ for all $s \in S$. Define the **least upper bound** to be $sup(S) \in \mathbb{R}$ for which $sup(S) \leq u$ for all upper bounds u of S. Similarly, define the **greatest lower bound** to be $inf(S) \in \mathbb{R}$ for which $inf(S) \geq l$ for all lower bounds l of S.

The **completeness** of the real numbers can be expressed by the statement that every set which is bounded above has a suprememum $sup(S)$. This is a clever way[10] to capture the idea that every Cauchy sequence of real numbers converges to a real number.

Finally we reach the punchline of this section. The following theorem generalizes the **Extreme Value Theorem** for $\mathbb{R}$ to the n-dimensional case.

Theorem 2.4.12.

Let C be a compact subset of $\mathbb{R}^n$ and suppose $f : C \to \mathbb{R}$ is a continuous function. Then $sup(f(C)), inf(f(C)) \in \mathbb{R}$. The values of f attain a minimum and maximum over C.

Even in the one-dimensional case the proof of this theorem is non-trivial. Take a look at Apostol pages 150-151. Note that pages 49-55 of C.H. Edwards' *Advanced Calculus of Several Variables* discuss compactness in greater depth and list several theorems which are related to the Extreme Value Theorem given here. I think my comments here suffice for our purposes.

[9] the real definition of compactness says that C is compact if every open cover of C admits a finite subcover then this definition is a theorem which can be proved in the context of $\mathbb{R}^n$. See the Heine Borel theorem and related discussion in a good real analysis text

[10] some mathematicians construct the real numbers by simply adjoining these limit points to the rational numbers.

Chapter 3

differentiation

Our goal in this chapter is to describe differentiation for functions to and from normed linear spaces. It turns out this is actually quite simple given the background of the preceding chapter. The differential at a point is a linear transformation which best approximates the change in a function at a particular point. We can quantify "best" by a limiting process which is naturally defined in view of the fact there is a norm on the spaces we consider.

The most important example is of course the case $f : \mathbb{R}^n \to \mathbb{R}^m$. In this context it is natural to write the differential as a matrix multiplication. The matrix of the differential is what Edwards calls the *derivative*. Partial derivatives are also defined in terms of directional derivatives. The directional derivative is sometimes defined where the differential fails to exist. We will discuss how the criteria of continuous differentiability allows us to build the differential from the directional derivatives. We'll see how the Cauchy-Riemann equations of complex analysis are really just an algebraic result if we already have the theorem for continuously differentiability. We will see how this general concept of differentiation recovers all the derivatives you've seen previously in calculus and much more.

On the other hand, I postpone implicit differentiation for a future chapter where we have the existence theorems for implicit and inverse functions. I also postpone discussion of the geometry of the differential. In short, existence of the differential and the tangent space are essentially two sides of the same problem. In fact, the approach of this chapter is radically different than my first set of notes on advanced calculus. In those notes I followed Edwards a bit more and built up to the definition of the differential on the basis of the directional derivative and geometry. I don't think students appreciate geometry or directional differentiation well enough to make that approach successful. Consquently, I begin with the unjustified definition of the derivative and then spend the rest of the chapter working out precise implications and examples that flow from the defintition. I essentially ignore the question of motivating the defintiion here. If you want motivation, think backward with this chapter or perhaps read Edwards or my old notes.

3.1 the Frechet differential

The definition[1] below says that $\triangle F = F(a+h) - F(a) \approx dF_a(h)$ when h is close to zero.

Definition 3.1.1.

Let $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ be normed vector spaces. Suppose that U is open and $F : U \subseteq V \to W$ is a function the we say that F is **differentiable** at $a \in U$ iff there exists a linear mapping $L : V \to W$ such that

$$\lim_{h\to 0}\left[\frac{F(a+h) - F(a) - L(h)}{||h||_V} \right] = 0.$$

In such a case we call the linear mapping L the **differential at** a and we denote $L = dF_a$. In the case $V = \mathbb{R}^m$ and $W = \mathbb{R}^n$ are given the standard euclidean norms, the matrix of the differential is called the **derivative of** F **at** a and we denote $[dF_a] = F'(a) \in \mathbb{R}^{m\times n}$ which means that $dF_a(v) = F'(a)v$ for all $v \in \mathbb{R}^n$.

Notice this definition gives an equation which implicitly defines dF_a. For the moment the only way we have to calculate dF_a is educated guessing. We simply use brute-force calculation to suggest a guess for L which forces the Frechet quotient to vanish. In the next section we'll discover a systematic calculational method for functions on euclidean spaces. The purpose of this section is to understand the definition of the differential and to connect it to some previous concepts of basic calculus. Efficient calculational schemes are discussed later in this chapter.

Example 3.1.2. *Suppose $T : V \to W$ is a linear transformation of normed vector spaces V and W. I propose $L = T$. In other words, I think we can show the best linear approximation to the change in a linear function is simply the function itself. Clearly L is linear since T is linear. Consider the difference quotient:*

$$\frac{T(a+h) - T(a) - L(h)}{||h||_V} = \frac{T(a) + T(h) - T(a) - T(h)}{||h||_V} = \frac{0}{||h||_V}.$$

Note $h \neq 0$ implies $||h||_V \neq 0$ by the definition of the norm. Hence the limit of the difference quotient vanishes since it is identically zero for every nonzero value of h. We conclude that $dT_a = T$.

Example 3.1.3. *Let $T : V \to W$ where V and W are normed vector spaces and define $T(v) = w_o$ for all $v \in V$. I claim the differential is the zero transformation. Linearity of $L(v) = 0$ is trivially verified. Consider the difference quotient:*

$$\frac{T(a+h) - T(a) - L(h)}{||h||_V} = \frac{w_o - w_o - 0}{||h||_V} = \frac{0}{||h||_V}.$$

Using the arguments to the preceding example, we find $dT_a = 0$.

[1] Some authors might put a norm in the numerator of the quotient. That is an equivalent condition since a function $g : V \to W$ has $\lim_{h\to 0} g(h) = 0$ iff $\lim_{h\to 0} ||g(h)||_W = 0$

Typically the difference quotient is not identically zero. The pair of examples above are very special cases. Now that we've seen a sample pair of abstract examples now we turn to the question of how this general definition recovers the concept of differentiation we studied in introductory calculus.

Example 3.1.4. *Suppose $f : dom(f) \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at x. It follows that there exists a linear function $df_x : \mathbb{R} \to \mathbb{R}$ such that*[2]

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - df_x(h)}{|h|} = 0.$$

Note that

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - df_x(h)}{|h|} = 0 \quad \Leftrightarrow \quad \lim_{h\to 0^{\pm}} \frac{f(x+h) - f(x) - df_x(h)}{|h|} = 0.$$

In the left limit $h \to 0^-$ we have $h < 0$ hence $|h| = -h$. On the other hand, in the right limit $h \to 0^+$ we have $h > 0$ hence $|h| = h$. Thus, differentiability suggests that $\lim_{h\to 0^{\pm}} \frac{f(x+h)-f(x)-df_x(h)}{\pm h} = 0$. But we can pull the minus out of the left limit to obtain $\lim_{h\to 0^-} \frac{f(x+h)-f(x)-df_x(h)}{h} = 0$. Therefore, after an algebra step, we find:

$$\lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h} - \frac{df_x(h)}{h}\right] = 0.$$

Linearity of $df_x : \mathbb{R} \to \mathbb{R}$ implies there exists $m \in \mathbb{R}^{1\times 1} = \mathbb{R}$ such that $df_x(h) = mh$. Observe that

$$\lim_{h\to 0} \frac{df_x(h)}{h} = \lim_{h\to 0} \frac{mh}{h} = m.$$

It is a simple exercise to show that if $\lim(A - B) = 0$ and $\lim(B)$ exists then $\lim(A)$ exists and $\lim(A) = \lim(B)$. Identify $A = \frac{f(x+h)-f(x)}{h}$ and $B = \frac{df_x(h)}{h}$. Therefore,

$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, we find the 1×1 matrix m of the differential is precisely $f'(x)$ as we defined it via a difference quotient in first semester calculus. In summary, we find $\boxed{df_x(h) = f'(x)h}$. In other words, if a function is differentiable in the sense we defined at the beginning of this chapter then it is differentiable in the terminology we used in calculus I. Moreover, the derivative at x is precisely the matrix of the differential.

Remark 3.1.5.

Incidentally, I should mention that df_x is the differential of f at the point x. The differential of f would be the mapping $x \mapsto df_x$. Technically, the differential df is a function from $\mathbb{R}$ to the set of linear transformations on $\mathbb{R}$. You can contrast this view with that of first semester calculus. There we say the mapping $x \mapsto f'(x)$ defines the derivative f' as a function from $\mathbb{R}$ to $\mathbb{R}$. This simplification in perspective is only possible because calculus in one-dimension is so special. More on this later. This distinction is especially important to understand if you begin to look at questions of higher derivatives.

[2] unless we state otherwise, $\mathbb{R}^n$ is assumed to have the euclidean norm, in this case $||x||_{\mathbb{R}} = \sqrt{x^2} = |x|$.

Example 3.1.6. *Suppose $F : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $F(x,y) = (xy,\ x^2,\ x+3y)$ for all $(x,y) \in \mathbb{R}^2$. Consider the difference function $\triangle F$ at (x,y):*

$$\triangle F = F((x,y)+(h,k)) - F(x,y) = F(x+h, y+k) - F(x,y)$$

Calculate,

$$\triangle F = \big((x+h)(y+k),\ (x+h)^2,\ x+h+3(y+k)\big) - \big(xy,\ x^2,\ x+3y\big)$$

Simplify by cancelling terms which cancel with $F(x,y)$:

$$\triangle F = \big(xk+hy+hk,\ 2xh+h^2,\ h+3k)\big)$$

Identify the linear part of $\triangle F$ as a good candidate for the differential. I claim that:

$$L(h,k) = \big(xk+hy,\ 2xh,\ h+3k\big).$$

is the differential for f at (x,y). Observe first that we can write

$$L(h,k) = \begin{bmatrix} y & x \\ 2x & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}.$$

therefore $L : \mathbb{R}^2 \to \mathbb{R}^3$ is manifestly linear. Use the algebra above to simplify the difference quotient below:

$$\lim_{(h,k)\to(0,0)} \left[\frac{\triangle F - L(h,k)}{||(h,k)||} \right] = \lim_{(h,k)\to(0,0)} \left[\frac{(hk, h^2, 0)}{||(h,k)||} \right]$$

Note $||(h,k)|| = \sqrt{h^2+k^2}$ therefore we fact the task of showing that $\frac{1}{\sqrt{h^2+k^2}}(hk,\ h^2,\ 0) \to (0,0,0)$ as $(h,k) \to (0,0)$. Notice that:

$$||(hk,\ h^2,\ 0)|| = |h|\sqrt{h^2+k^2}$$

Therefore, as $(h,k) \to 0$ we find

$$\left|\left| \frac{1}{\sqrt{h^2+k^2}}(hk,\ h^2,\ 0) \right|\right| = |h| \to 0.$$

However, if $||v|| \to 0$ it follows $v \to 0$ so we derive the desired limit. Therefore,

$$df_{(x,y)}(h,k) = \begin{bmatrix} y & x \\ 2x & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}.$$

Computation of less trivial multivariate limits is an art we'd like to avoid if possible. It turns out that we can actually avoid these calculations by computing partial derivatives. However, we still need a certain multivariate limit to exist for the partial derivative functions so in some sense it's unavoidable. The limits are there whether we like to calculate them or not.

Calculation of the differential simplifies considerably when the domain is one-dimensional. We already worked out the case of $f : \mathbb{R} \to \mathbb{R}$ in Example 3.1.4 and the following pair of examples work out the concrete case of $F : \mathbb{R} \to \mathbb{C}$ and then the general case $F : \mathbb{R} \to V$ for an arbitrary finite dimensional normed linear space V.

Example 3.1.7. *Suppose $F(t) = U(t)+iV(t)$ for all $t \in dom(f)$ and both U and V are differentiable functions on $dom(F)$. By the arguments given in Example 3.1.4 it suffices to find $L : \mathbb{R} \to \mathbb{C}$ such that*

$$\lim_{h\to 0}\left[\frac{F(t+h) - F(t) - L(h)}{h}\right] = 0.$$

I propose that on the basis of analogy to Example 3.1.4 we ought to have $dF_t(h) = (U'(t)+iV'(t))h$. Let $L(h) = (U'(t) + iV'(t))h$. Observe that, using properties of $\mathbb{C}$:

$$\begin{aligned} L(h_1 + ch_2) &= (U'(t) + iV'(t))(h_1 + ch_2) \\ &= (U'(t) + iV'(t))h_1 + c(U'(t) + iV'(t))h_2 \\ &= L(h_1) + cL(h_2). \end{aligned}$$

for all $h_1, h_2 \in \mathbb{R}$ and $c \in \mathbb{R}$. Hence $L : \mathbb{R} \to \mathbb{C}$ is linear. Moreover,

$$\begin{aligned} \tfrac{F(t+h)-F(t)-L(h)}{h} &= \tfrac{1}{h}\Big(U(t+h) + iV(t+h) - U(t) + iV(t) - (U'(t) + iV'(t))h\Big) \\ &= \tfrac{1}{h}\Big(U(t+h) - U(t) - U'(t)h\Big) + i\tfrac{1}{h}\Big(V(t+h) - V(t) - V'(t)h\Big) \end{aligned}$$

Consider the problem of calculating $\lim_{h\to 0} \frac{F(t+h)-F(t)-L(h)}{h}$. We observe that a complex function converges to zero iff the real and imaginary parts of the function separately converge to zero (this is covered by Theorem 2.3.11). By differentiability of U and V we find again using Example 3.1.4

$$\lim_{h\to 0}\frac{1}{h}\Big(U(t+h) - U(t) - U'(t)h\Big) = 0 \qquad \lim_{h\to 0}\frac{1}{h}\Big(V(t+h) - V(t) - V'(t)h\Big) = 0.$$

Therefore, $dF_t(h) = (U'(t) + iV'(t))h$. Note that the quantity $U'(t) + iV'(t)$ is not a real matrix in this case. To write the derivative in terms of a real matrix multiplication we need to construct some further notation which makes use of the isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$. Actually, it's pretty easy if you agree that $a + ib = (a, b)$ then $dF_t(h) = (U'(t), V'(t))h$ so the matrix of the differential is $(U'(t), V'(t)) \in \mathbb{R}^{1\times 2}$ which makes since as $F : \mathbb{C} \approx \mathbb{R}^2 \to \mathbb{R}$.

Example 3.1.8. *Suppose V is a normed vector space with basis $\beta = \{f_1, f_2, \ldots, f_n\}$. Futhermore, let $G : I \subseteq \mathbb{R} \to V$ be defined by*

$$G(t) = \sum_{i=1}^{n} G_i(t) f_i$$

where $G_i : I \to \mathbb{R}$ is differentiable on I for $i = 1, 2, \ldots, n$. Recall Theorem 2.3.11 revealed that $T = \sum_{j=1}^{n} T_j f_j : \mathbb{R} \to V$ then $\lim_{t\to 0} T(t) = \sum_{j=1}^{n} l_j f_j$ iff $\lim_{t\to 0} T_j(t) = l_j$ for all $j = 1, 2, \ldots, n$. In words, the limit of a vector-valued function can be parsed into a vector of limits. With this in mind consider (again we can trade $|h|$ for h as we explained in-depth in Example 3.1.4) the difference quotient $\lim_{h\to 0}\left[\frac{G(t+h)-G(t)-h\sum_{i=1}^{n}\frac{dG_i}{dt}f_i}{h}\right]$, factoring out the basis yields:

$$\lim_{h\to 0}\left[\frac{\sum_{i=1}^{n}[G_i(t+h) - G_i(t) - h\frac{dG_i}{dt}]f_i}{h}\right] = \sum_{i=1}^{n}\left[\lim_{h\to 0}\frac{G_i(t+h) - G_i(t) - h\frac{dG_i}{dt}}{h}\right] f_i = 0$$

where the zero above follows from the supposed differentiability of each component function. It follows that:

$$\boxed{dG_t(h) = h\sum_{i=1}^{n}\frac{dG_i}{dt}f_i}$$

The example above encompasses a number of cases at once:

1. $V = \mathbb{R}$, functions on $\mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$
2. $V = \mathbb{R}^n$, space curves in $\mathbb{R}$, $\vec{r} : \mathbb{R} \to \mathbb{R}^n$
3. $V = \mathbb{C}$, complex-valued functions of a real variable, $f = u + iv : \mathbb{R} \to \mathbb{C}$
4. $V = \mathbb{R}^{m\times n}$, matrix-valued functions of a real variable, $F : \mathbb{R} \to \mathbb{R}^{m\times n}$.

In short, when we differentiate a function which has a real domain then we can define the derivative of such a function by component-wise differentiation. It gets more interesting when the domain has several independent variables as Examples 3.1.6 and 3.1.9 illustrate.

Example 3.1.9. *Suppose $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is defined by $F(X) = X^2$. Notice*

$$\triangle F = F(X + H) - F(X) = (X + H)(X + H) - X^2 = XH + HX + H^2$$

I propose that F is differentiable at X and $L(H) = XH + HX$. Let's check linearity,

$$L(H_1 + cH_2) = X(H_1 + cH_2) + (H_1 + cH_2)X = XH_1 + H_1X + c(XH_2 + H_2X)$$

Hence $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is a linear transformation. By construction of L the linear terms in the numerator cancel leaving just the quadratic term,

$$\lim_{H\to 0}\frac{F(X + H) - F(X) - L(H)}{||H||} = \lim_{H\to 0}\frac{H^2}{||H||}.$$

It suffices to show that $\lim_{H\to 0} \frac{||H^2||}{||H||} = 0$ *since* $\lim(||g||) = 0$ *iff* $\lim(g) = 0$ *in a normed vector space. Fortunately the normed vector space* $\mathbb{R}^{n\times n}$ *is actually a* **Banach algebra**. *A vector space with a multiplication operation is called an algebra. In the current context the multiplication is simply matrix multiplication. A Banach algebra is a normed vector space with a multiplication that satisfies* $||XY|| \leq ||X||\,||Y||$. *Thanks to this inequality*[3] *we can calculate our limit via the squeeze theorem. Observe* $0 \leq \frac{||H^2||}{||H||} \leq ||H||$. *As* $H \to 0$ *it follows* $||H|| \to 0$ *hence* $\lim_{H\to 0} \frac{||H^2||}{||H||} = 0$. *We find* $dF_X(H) = XH + HX$.

Generally constructing the matrix for a function $f : V \to W$ where $V, W \neq \mathbb{R}$ involves a fair number of relatively ad-hoc conventions because the constructions necessarily involving choosing coordinates. The situation is similar in linear algebra. Writing abstract linear transformations in terms of matrix multiplication takes a little thinking. If you look back you'll notice that I did not bother to try to write a matrix for the differential in Examples 3.1.2 or 3.1.3.

Remark 3.1.10.

I have deliberately defined the derivative in slightly more generality than we need for this course. It's probably not much trouble to continue to develop the theory of differentiation for a normed vector space, however I will for the most part stop here modulo an example here or there.

If you understand many of the theorems that follow from here on out for $\mathbb{R}^n$ then it is a simple matter to transfer arguments to the setting of a Banach space by using an appropriate isomorphism. Traditionally this type of course only covers continuous differentiability, inverse and implicit function theorems in the context of mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$.

For the reader interested in generalizing these results to the context of an abstract normed vector space feel free to discuss it with me sometime. Also, if you want to read a master on these topics you could look at the text by Shlomo Sternberg on Advanced Calculus. He develops many things for normed spaces. Or, take a look at Dieudonne's Modern Analysis which pays special attention to reaping infinite dimensional results from our finite-dimensional arguments. Both of those texts would be good to read to follow-up my course with something deeper.

[3] it does take a bit of effort to prove this inequality holds for the matrix norm, I omit it since it would be distracting here

3.2 partial derivatives and the Jacobian matrix

In the preceding section we calculated the differential at a point via educated guessing. We should like to find better method to derive differentials. It turns out that we can systematically calculate the differential from partial derivatives of the component functions. However, certain topological conditions are required for us to properly paste together the partial derivatives of the component functions. The story is given near the end of the Chapter in Section 3.7. There we describe how the criteria of continuous differentiability achieves this goal. The difference between what we do here and calculus III is that the derivative is no longer a number or a vector, instead as we shall see soon, it is a matrix. Even this is a convenience, the better view is that the differential is primary. Thus the better concept for the derivative is that of linear transformations. Dieudonne said it best: this is the introduction to his chapter on differentiation in *Modern Analysis* Chapter VIII.

> The subject matter of this Chapter is nothing else but the elementary theorems of Calculus, which however are presented in a way which will probably be new to most students. That presentation, which throughout adheres strictly to our general "geometric" outlook on Analysis, aims at keeping as close as possible to the fundamental idea of Calculus, namely the "local" approximation of functions by *linear* functions. **In the classical teaching of Calculus, the idea is immediately obscured by the accidental fact that, on a one-dimensional vector space, there is a one-to-one correspondence between *linear* forms and numbers, and therefore the derivative at a point is defined as a *number* instead of a *linear form*. This slavish subservience to the shibboleth[4] of numerical interpretation at any cost becomes much worse when dealing with functions of several variables...**

Dieudonne's then spends the next half page continuing this thought with explicit examples of how this custom of our calculus presentation injures the conceptual generalization. If you want to see differentiation written for mathematicians, that is the place to look. He proves many results for infinite dimensions because, well, why not?

3.2.1 directional derivatives

The directional derivative of a mapping F at a point $a \in dom(F)$ along v is defined to be the derivative of the curve $\gamma(t) = F(a + tv)$. In other words, the directional derivative gives you the instantaneous vector-rate of change in the mapping F at the point a along v.

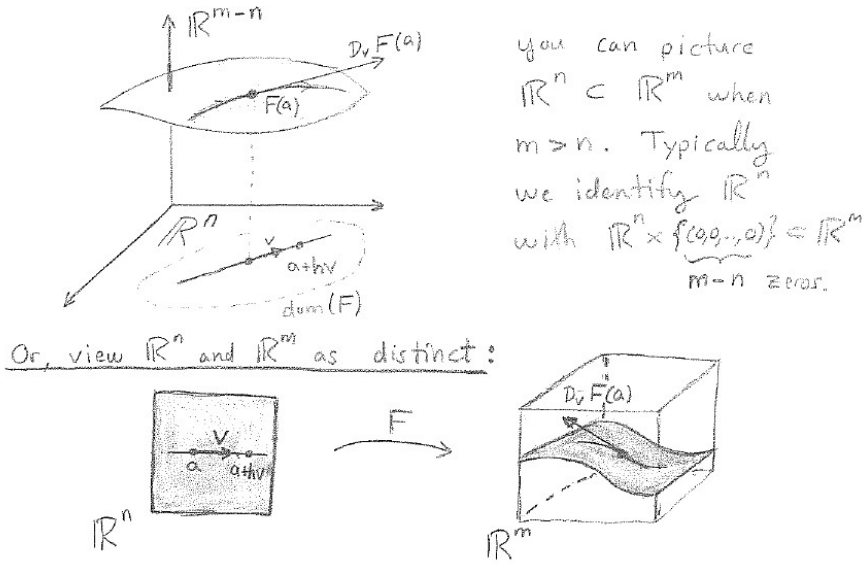
[4] from wikipedia: is a word, sound, or custom that a person unfamiliar with its significance may not pronounce or perform correctly relative to those who are familiar with it. It is used to identify foreigners or those who do not belong to a particular class or group of people. It also refers to features of language, and particularly to a word or phrase whose pronunciation identifies a speaker as belonging to a particular group.

Definition 3.2.1.

Let $F : dom(F) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and suppose the limit below exists for $a \in dom(F)$ and $v \in \mathbb{R}^n$ then we define the **directional derivative of F at a along v** to be $D_vF(a) \in \mathbb{R}^m$ where

$$D_vF(a) = \lim_{h \to 0} \frac{F(a+hv) - F(a)}{h}$$

One great contrast we should pause to note is that the definition of the directional derivative is explicit whereas the definition of the differential was implicit. The picture below might help motivate the definition we just offered.



In the case that $m = 1$ then $F : dom(F) \subseteq \mathbb{R}^n \to \mathbb{R}$ and the directional derivative gives the instantaneous rate of change of the function F at the point a along v. You probably insisted that $||v|| = 1$ in calculus III but we make no such demand here. We define the directional derivative for mappings and vectors of non-unit length.

Proposition 3.2.2.

If $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in U$ then the directional derivative $D_vF(a)$ exists for each $v \in \mathbb{R}^n$ and $D_vF(a) = dF_a(v)$.

Proof: Suppose $a \in U$ such that dF_a is well-defined then we are given that

$$\lim_{h \to 0} \frac{F(a+h) - F(a) - dF_a(h)}{||h||} = 0.$$

This is a limit in $\mathbb{R}^n$, when it exists it follows that the limits that approach the origin along particular paths also exist and are zero. In particular we can consider the path $t \mapsto tv$ for $v \neq 0$ and $t > 0$, we find

$$\lim_{tv \to 0,\ t>0} \frac{F(a+tv) - F(a) - dF_a(tv)}{||tv||} = \frac{1}{||v||} \lim_{t \to 0^+} \frac{F(a+tv) - F(a) - tdF_a(v)}{|t|} = 0.$$

Hence, as $|t| = t$ for $t > 0$ we find

$$\lim_{t \to 0^+} \frac{F(a+tv) - F(a)}{t} = \lim_{t \to 0} \frac{tdF_a(v)}{t} = dF_a(v).$$

Likewise we can consider the path $t \mapsto tv$ for $v \neq 0$ and $t < 0$

$$\lim_{tv \to 0,\ t<0} \frac{F(a+tv) - F(a) - dF_a(tv)}{||tv||} = \frac{1}{||v||} \lim_{t \to 0^-} \frac{F(a+tv) - F(a) - tdF_a(v)}{|t|} = 0.$$

Note $|t| = -t$ thus the limit above yields

$$\lim_{t \to 0^-} \frac{F(a+tv) - F(a)}{-t} = \lim_{t \to 0^-} \frac{tdF_a(v)}{-t} \quad \Rightarrow \quad \lim_{t \to 0^-} \frac{F(a+tv) - F(a)}{t} = dF_a(v).$$

Therefore,

$$\lim_{t \to 0} \frac{F(a+tv) - F(a)}{t} = dF_a(v)$$

and we conclude that $D_v F(a) = dF_a(v)$ for all $v \in \mathbb{R}^n$ since the $v = 0$ case follows trivially. □

Let's think about the problem we face. We want to find a nice formula for the differential. We now know that if it exists then the directional derivatives allow us to calculate the values of the differential in particular directions. The natural thing to do is to calculate the standard matrix for the differential using the preceding proposition. Recall that if $L : \mathbb{R}^n \to \mathbb{R}^m$ then the standard matrix was simply

$$[L] = [L(e_1)|L(e_2)|\cdots|L(e_n)]$$

and thus the action of L is expressed nicely as a matrix multiplication; $L(v) = [L]v$. Similarly, $df_a : \mathbb{R}^n \to \mathbb{R}^m$ is linear transformation and thus $df_a(v) = [df_a]v$ where

$$[df_a] = [df_a(e_1)|df_a(e_2)|\cdots|df_a(e_n)].$$

Moreover, by the preceding proposition we can calculate $df_a(e_j) = D_{e_j} f(a)$ for $j = 1, 2, \ldots, n$. Clearly the directional derivatives in the coordinate directions are of great importance. For this reason we make the following definition:

Definition 3.2.3. *Partial derivatives are directional derivatives in coordinate directions.*

Suppose that $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a mapping the we say that F is **has partial derivative** $\frac{\partial F}{\partial x_i}(a)$ at $a \in U$ iff the directional derivative in the e_i direction exists at a. In this case we denote,

$$\frac{\partial F}{\partial x_i}(a) = D_{e_i}F(a).$$

Also, the notation $D_{e_i}F(a) = D_iF(a)$ or $\partial_i F = \frac{\partial F}{\partial x_i}$ is convenient. We construct the **partial derivative function** $\partial_i F : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$ as the function defined pointwise for each $v \in V$ where $\partial_i F(v)$ exists.

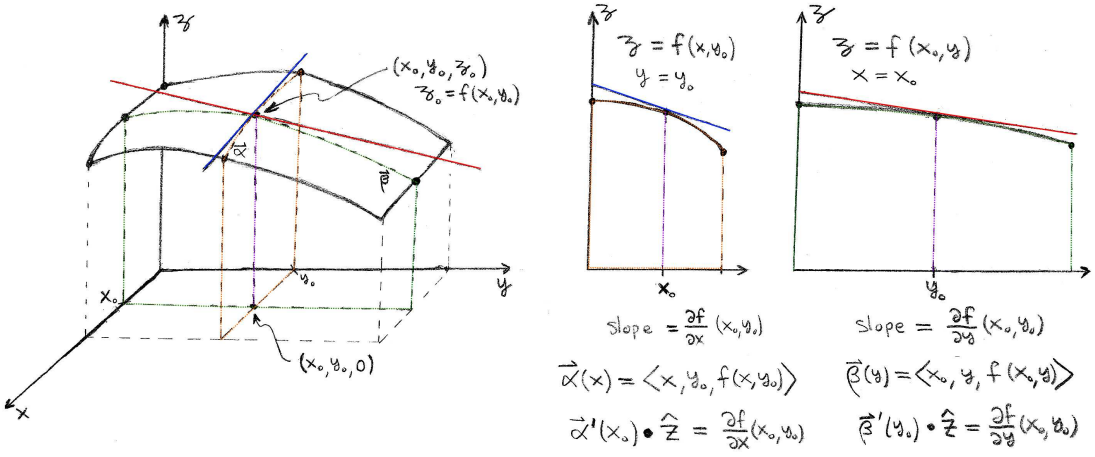
Let's expand this definition a bit. Note that if $F = (F_1, F_2, \ldots, F_m)$ then

$$D_{e_i}F(a) = \lim_{h\to 0} \frac{F(a + he_i) - F(a)}{h} \quad \Rightarrow \quad [D_{e_i}F(a)] \cdot e_j = \lim_{h\to 0} \frac{F_j(a + he_i) - F_j(a)}{h}$$

for each $j = 1, 2, \ldots m$. But then the limit of the component function F_j is precisely the directional derivative at a along e_i hence we find the result

$$\frac{\partial F}{\partial x_i} \cdot e_j = \frac{\partial F_j}{\partial x_i} \qquad \text{in other words,} \qquad \boxed{\partial_i F = (\partial_i F_1, \partial_i F_2, \ldots, \partial_i F_m).}$$

In the particular case $f : \mathbb{R}^2 \to \mathbb{R}$ the partial derivatives with respect to x and y at (x_o, y_o) are related to the graph $z = f(x, y)$ as illustrated below:



Similar pictures can be imagined for partial derivatives of more variables, even for vector-valued maps, but direct visualization is not possible (at least for me).

The proposition below shows how the differential of a m-vector-valued function of n-real variables is connected to a matrix of partial derivatives.

Proposition 3.2.4.

If $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in U$ then the differential dF_a has derivative matrix $F'(a)$ and it has components which are expressed in terms of partial derivatives of the component functions:

$$[dF_a]_{ij} = \partial_j F_i$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Perhaps it is helpful to expand the derivative matrix explicitly for future reference:

$$F'(a) = \begin{bmatrix} \partial_1 F_1(a) & \partial_2 F_1(a) & \cdots & \partial_n F_1(a) \\ \partial_1 F_2(a) & \partial_2 F_2(a) & \cdots & \partial_n F_2(a) \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 F_m(a) & \partial_2 F_m(a) & \cdots & \partial_n F_m(a) \end{bmatrix}$$

Let's write the operation of the differential for a differentiable mapping at some point $a \in \mathbb{R}$ in terms of the explicit matrix multiplication by $F'(a)$. Let $v = (v_1, v_2, \ldots v_n) \in \mathbb{R}^n$,

$$dF_a(v) = F'(a)v = \begin{bmatrix} \partial_1 F_1(a) & \partial_2 F_1(a) & \cdots & \partial_n F_1(a) \\ \partial_1 F_2(a) & \partial_2 F_2(a) & \cdots & \partial_n F_2(a) \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 F_m(a) & \partial_2 F_m(a) & \cdots & \partial_n F_m(a) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

You may recall the notation from calculus III at this point, omitting the a-dependence,

$$\nabla F_j = grad(F_j) = \begin{bmatrix} \partial_1 F_j, & \partial_2 F_j, & \cdots, & \partial_n F_j \end{bmatrix}^T$$

So if the derivative exists we can write it in terms of a stack of gradient vectors of the component functions: (I used a transpose to write the stack side-ways),

$$F' = \begin{bmatrix} \nabla F_1 | \nabla F_2 | \cdots | \nabla F_m \end{bmatrix}^T$$

Finally, just to collect everything together,

$$F' = \begin{bmatrix} \partial_1 F_1 & \partial_2 F_1 & \cdots & \partial_n F_1 \\ \partial_1 F_2 & \partial_2 F_2 & \cdots & \partial_n F_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 F_m & \partial_2 F_m & \cdots & \partial_n F_m \end{bmatrix} = \begin{bmatrix} \partial_1 F & | & \partial_2 F & | \cdots | & \partial_n F \end{bmatrix} = \begin{bmatrix} (\nabla F_1)^T \\ \hline (\nabla F_2)^T \\ \hline \vdots \\ \hline (\nabla F_m)^T \end{bmatrix}$$

Example 3.2.5. *Recall that in Example 3.1.6 we showed that $F : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $F(x, y) = (xy,\ x^2,\ x + 3y)$ for all $(x, y) \in \mathbb{R}^2$ was differentiable. In fact we calculated that*

$$dF_{(x,y)}(h, k) = \begin{bmatrix} y & x \\ 2x & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}.$$

If you recall from calculus III the mechanics of partial differentiation it's simple to see that

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(xy,\ x^2,\ x+3y) = (y,\ 2x, 1) = \begin{bmatrix} y \\ 2x \\ 1 \end{bmatrix}$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(xy,\ x^2,\ x+3y) = (x,\ 0, 3) = \begin{bmatrix} x \\ 0 \\ 3 \end{bmatrix}$$

Thus $[dF] = [\partial_x F | \partial_y F]$ *(as we expect given the derivations in this section!)*

Directional derivatives and partial derivatives are of secondary importance in this course. They are merely the substructure of what is truly of interest: the differential. That said, it is useful to know how to construct directional derivatives via partial derivative formulas. In fact, in careless calculus texts it sometimes presented as the definition.

Proposition 3.2.6.

If $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in U$ then the directional derivative $D_v F(a)$ can be expressed as a sum of partial derivative maps for each $v =< v_1, v_2, \ldots, v_n > \in \mathbb{R}^n$:

$$D_v F(a) = \sum_{j=1}^{n} v_j \partial_j F(a)$$

Proof: since F is differentiable at a the differential dF_a exists and $D_v F(a) = dF_a(v)$ for all $v \in \mathbb{R}^n$. Use linearity of the differential to calculate that

$$D_v F(a) = dF_a(v_1 e_1 + \cdots + v_n e_n) = v_1 dF_a(e_1) + \cdots + v_n dF_a(e_n).$$

Note $dF_a(e_j) = D_{e_j} F(a) = \partial_j F(a)$ and the prop. follows. □

Example 3.2.7. *Suppose* $f : \mathbb{R}^3 \to \mathbb{R}$ *then* $\nabla f = [\partial_x f, \partial_y f, \partial_z f]^T$ *and we can write the directional derivative in terms of*

$$D_v f = [\partial_x f, \partial_y f, \partial_z f]^T v = \nabla f \cdot v$$

if we insist that $||v|| = 1$ *then we recover the standard directional derivative we discuss in calculus III. Naturally the* $||\nabla f(a)||$ *yields the maximum value for the directional derivative at* a *if we limit the inputs to vectors of unit-length. If we did not limit the vectors to unit length then the directional derivative at* a *can become arbitrarily large as* $D_v f(a)$ *is proportional to the magnitude of* v*. Since our primary motivation in calculus III was describing rates of change along certain directions for some multivariate function it made sense to specialize the directional derivative to vectors of unit-length. The definition used in these notes better serves the theoretical discussion.*[5]

In Section 3.4 we give many explicit examples.

[5] If you read my calculus III notes you'll find a derivation of how the directional derivative in Stewart's calculus arises from the general definition of the derivative as a linear mapping. Look up page 305g.

3.3 derivatives of sum and scalar product

Suppose $F_1 : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $F_2 : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Furthermore, suppose both of these are differentiable at $a \in \mathbb{R}^n$. It follows that $(dF_1)_a = L_1$ and $(dF_2)_a = L_2$ are linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$ which approximate the change in F_1 and F_2 near a, in particular,

$$\lim_{h\to 0} \frac{F_1(a+h) - F_1(a) - L_1(h)}{||h||} = 0 \qquad \lim_{h\to 0} \frac{F_2(a+h) - F_2(a) - L_2(h)}{||h||} = 0$$

To prove that $H = F_1 + F_2$ is differentiable at $a \in \mathbb{R}^n$ we need to find a differential at a for H. Naturally, we expect $dH_a = d(F_1 + F_2)_a = (dF_1)_a + (dF_2)_a$. Let $L = (dF_1)_a + (dF_2)_a$ and consider,

$$\begin{aligned}
\lim_{h\to 0} \tfrac{H(a+h)-H(a)-L(h)}{||h||} &= \lim_{h\to 0} \tfrac{F_1(a+h)+F_2(a+h)-F_1(a)-F_2(a)-L_1(h)-L_2(h)}{||h||} \\
&= \lim_{h\to 0} \tfrac{F_1(a+h)-F_1(a)-L_1(h)}{||h||} + \lim_{h\to 0} \tfrac{F_2(a+h)-F_2(a)-L_2(h)}{||h||} \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

Note that breaking up the limit was legal because we knew the subsequent limits existed and were zero by the assumption of differentiability of F_1 and F_2 at a. Finally, since $L = L_1 + L_2$ we know L is a linear transformation since the sum of linear transformations is a linear transformation. Moreover, the matrix of L is the sum of the matrices for L_1 and L_2. Let $c \in \mathbb{R}$ and suppose $G = cF_1$ then we can also show that $dG_a = d(cF_1)_a = c(dF_1)_a$, the calculation is very similar except we just pull the constant c out of the limit. I'll let you write it out. Collecting our observations:

Proposition 3.3.1.

Suppose $F_1 : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $F_2 : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are differentiable at $a \in U$ then $F_1 + F_2$ is differentiable at a and

$$d(F_1 + F_2)_a = (dF_1)_a + (dF_2)_a \quad \text{or} \quad (F_1 + F_2)'(a) = F_1'(a) + F_2'(a)$$

Likewise, if $c \in \mathbb{R}$ then

$$d(cF_1)_a = c(dF_1)_a \quad \text{or} \quad (cF_1)'(a) = c(F_1'(a))$$

These results suggest that the differential of a function is a new object which has a vector space structure. Of course, from a calculational view, these also say that the Jacobian matrix of a sum or scalar product is simply the sum or scalar product of the Jacobian matrices.

3.4 a gallery of explicit derivatives

Our goal here is simply to exhibit the Jacobian matrix and partial derivatives for a few mappings. At the base of all these calculations is the observation that partial differentiation is just ordinary differentiation where we treat all the independent variable not being differentiated as constants. The criteria of independence is important. We'll study the case where variables are not independent in a later section (see implicit differentiation).

Example 3.4.1. *Let $f(t) = (t, t^2, t^3)$ then $f'(t) = (1, 2t, 3t^2)$. In this case we have*

$$f'(t) = [df_t] = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}$$

Example 3.4.2. *Let $f(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$ be a mapping from $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. I'll denote the coordinates in the domain by $(x_1, x_2, x_3, y_1, y_2, y_3)$ thus $f(\vec{x}, \vec{y}) = x_1y_1 + x_2y_2 + x_3y_3$. Calculate,*

$$[df_{(\vec{x},\vec{y})}] = \nabla f(\vec{x}, \vec{y})^T = [y_1, y_2, y_3, x_1, x_2, x_3]$$

Example 3.4.3. *Let $f(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$ be a mapping from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. I'll denote the coordinates in the domain by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ thus $f(\vec{x}, \vec{y}) = \sum_{i=1}^n x_iy_i$. Calculate,*

$$\frac{\partial}{x_j}\left[\sum_{i=1}^n x_iy_i\right] = \sum_{i=1}^n \frac{\partial x_i}{x_j} y_i = \sum_{i=1}^n \delta_{ij}y_i = y_j$$

Likewise,

$$\frac{\partial}{y_j}\left[\sum_{i=1}^n x_iy_i\right] = \sum_{i=1}^n x_i \frac{\partial y_i}{y_j} = \sum_{i=1}^n x_i\delta_{ij} = x_j$$

Therefore, noting that $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_n} f, \partial_{y_1} f, \ldots, \partial_{y_n} f)$,

$$[df_{(\vec{x},\vec{y})}]^T = (\nabla f)(\vec{x}, \vec{y}) = \vec{y} \times \vec{x} = (y_1, \ldots, y_n, x_1, \ldots, x_n)$$

Example 3.4.4. *Suppose $F(x, y, z) = (xyz, y, z)$ we calculate,*

$$\frac{\partial F}{\partial x} = (yz, 0, 0) \qquad \frac{\partial F}{\partial y} = (xz, 1, 0) \qquad \frac{\partial F}{\partial z} = (xy, 0, 1)$$

Remember these are actually column vectors in my sneaky notation; $(v_1, \ldots, v_n) = [v_1, \ldots, v_n]^T$. This means the **derivative** *or* **Jacobian matrix** *of F at (x, y, z) is*

$$F'(x, y, z) = [dF_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example 3.4.5. *Suppose $F(x,y,z) = (x^2+z^2, yz)$ we calculate,*

$$\frac{\partial F}{\partial x} = (2x, 0) \qquad \frac{\partial F}{\partial y} = (0, z) \qquad \frac{\partial F}{\partial z} = (2z, y)$$

The derivative is a 2×3 matrix in this example,

$$F'(x,y,z) = [dF_{(x,y,z)}] = \begin{bmatrix} 2x & 0 & 2z \\ 0 & z & y \end{bmatrix}$$

Example 3.4.6. *Suppose $F(x,y) = (x^2+y^2, xy, x+y)$ we calculate,*

$$\frac{\partial F}{\partial x} = (2x, y, 1) \qquad \frac{\partial F}{\partial y} = (2y, x, 1)$$

The derivative is a 3×2 matrix in this example,

$$F'(x,y) = [dF_{(x,y)}] = \begin{bmatrix} 2x & 2y \\ y & x \\ 1 & 1 \end{bmatrix}$$

Example 3.4.7. *Suppose $P(x,v,m) = (P_o, P_1) = (\frac{1}{2}mv^2 + \frac{1}{2}kx^2, mv)$ for some constant k. Let's calculate the derivative via gradients this time,*

$$\nabla P_o = (\partial P_o/\partial x, \partial P_o/\partial v, \partial P_o/\partial m) = (kx, mv, \tfrac{1}{2}v^2)$$

$$\nabla P_1 = (\partial P_1/\partial x, \partial P_1/\partial v, \partial P_1/\partial m) = (0, m, v)$$

Therefore,

$$P'(x,v,m) = \begin{bmatrix} kx & mv & \frac{1}{2}v^2 \\ 0 & m & v \end{bmatrix}$$

Example 3.4.8. *Let $F(r,\theta) = (r\cos\theta, r\sin\theta)$. We calculate,*

$$\partial_r F = (\cos\theta, \sin\theta) \qquad \text{and} \qquad \partial_\theta F = (-r\sin\theta, r\cos\theta)$$

Hence,

$$F'(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Example 3.4.9. *Let $G(x,y) = (\sqrt{x^2+y^2}, \tan^{-1}(y/x))$. We calculate,*

$$\partial_x G = \left(\tfrac{x}{\sqrt{x^2+y^2}}, \tfrac{-y}{x^2+y^2}\right) \qquad \text{and} \qquad \partial_y G = \left(\tfrac{y}{\sqrt{x^2+y^2}}, \tfrac{x}{x^2+y^2}\right)$$

Hence,

$$G'(x,y) = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} = \begin{bmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{bmatrix} \qquad (\text{ using } r = \sqrt{x^2+y^2} \text{ })$$

Example 3.4.10. *Let $F(x,y) = (x, y, \sqrt{R^2 - x^2 - y^2})$ for a constant R. We calculate,*

$$\nabla\sqrt{R^2 - x^2 - y^2} = \left(\frac{-x}{\sqrt{R^2-x^2-y^2}},\ \frac{-y}{\sqrt{R^2-x^2-y^2}} \right)$$

Also, $\nabla x = (1,0)$ and $\nabla y = (0,1)$ thus

$$F'(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{R^2-x^2-y^2}} & \frac{-y}{\sqrt{R^2-x^2-y^2}} \end{bmatrix}$$

Example 3.4.11. *Let $F(x,y,z) = (x, y, z, \sqrt{R^2 - x^2 - y^2 - z^2})$ for a constant R. We calculate,*

$$\nabla\sqrt{R^2 - x^2 - y^2 - z^2} = \left(\frac{-x}{\sqrt{R^2-x^2-y^2-z^2}},\ \frac{-y}{\sqrt{R^2-x^2-y^2-z^2}},\ \frac{-z}{\sqrt{R^2-x^2-y^2-z^2}} \right)$$

Also, $\nabla x = (1,0,0)$, $\nabla y = (0,1,0)$ and $\nabla z = (0,0,1)$ thus

$$F'(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-x}{\sqrt{R^2-x^2-y^2-z^2}} & \frac{-y}{\sqrt{R^2-x^2-y^2-z^2}} & \frac{-z}{\sqrt{R^2-x^2-y^2-z^2}} \end{bmatrix}$$

Example 3.4.12. *Let $f(x,y,z) = (x+y, y+z, x+z, xyz)$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ yz & xz & xy \end{bmatrix}$$

Example 3.4.13. *Let $f(x,y,z) = xyz$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \end{bmatrix}$$

Example 3.4.14. *Let $f(x,y,z) = (xyz, 1-x-y)$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \\ -1 & -1 & 0 \end{bmatrix}$$

Example 3.4.15. *Let $f : \mathbb{R}^3 \times \mathbb{R}^3$ be defined by $f(x) = x \times v$ for a fixed vector $v \neq 0$. We denote $x = (x_1, x_2, x_3)$ and calculate,*

$$\frac{\partial}{\partial x_a}(x \times v) = \frac{\partial}{\partial x_a}\Big(\sum_{i,j,k} \epsilon_{ijk} x_i v_j e_k \Big) = \sum_{i,j,k} \epsilon_{ijk} \frac{\partial x_i}{\partial x_a} v_j e_k = \sum_{i,j,k} \epsilon_{ijk}\delta_{ia} v_j e_k = \sum_{j,k} \epsilon_{ajk} v_j e_k$$

It follows,

$$\frac{\partial}{\partial x_1}(x \times v) = \sum_{j,k} \epsilon_{1jk} v_j e_k = v_2 e_3 - v_3 e_2 = (0, -v_3, v_2)$$

$$\frac{\partial}{\partial x_2}(x \times v) = \sum_{j,k} \epsilon_{2jk} v_j e_k = v_3 e_1 - v_1 e_3 = (v_3, 0, -v_1)$$

$$\frac{\partial}{\partial x_3}(x \times v) = \sum_{j,k} \epsilon_{3jk} v_j e_k = v_1 e_2 - v_2 e_1 = (-v_2, v_1, 0)$$

Thus the Jacobian is simply,

$$[df_{(x,y)}] = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & -v_1 \\ v_2 & v_1 & 0 \end{bmatrix}$$

In fact, $df_p(h) = f(h) = h \times v$ for each $p \in \mathbb{R}^3$. The given mapping is linear so the differential of the mapping is precisely the mapping itself (we could short-cut much of this calculation and simply quote Example 3.1.2 where we proved $dT = T$ for linear T).

Example 3.4.16. *Let $f(x, y) = (x, y, 1 - x - y)$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Example 3.4.17. *Let $X(u, v) = (x, y, z)$ where x, y, z denote functions of u, v and I prefer to omit the explicit depedendence to reduce clutter in the equations to follow.*

$$\frac{\partial X}{\partial u} = X_u = (x_u, y_u, z_u) \quad \text{and} \quad \frac{\partial X}{\partial v} = X_v = (x_v, y_v, z_v)$$

Then the Jacobian is the 3×2 matrix

$$\left[dX_{(u,v)}\right] = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$$

Remark 3.4.18.

I return to these examples in the next chapter and we'll explore the geometric content of these formulas as they support the application of certain theorems. More on that later, for the remainder of this chapter we continue to focus on properties of differentiation.

3.5 chain rule

The proof in Edwards is on 77-78. I'll give a heuristic proof here which captures the essence of the argument. The simplicity of this rule continues to amaze me.

Proposition 3.5.1.

If $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at a and $G : V \subseteq \mathbb{R}^p \to \mathbb{R}^m$ is differentiable at $F(a) \in V$ then $G \circ F$ is differentiable at a and

$$d(G \circ F)_a = (dG)_{F(a)} \circ dF_a \qquad \text{or, in matrix notation,} \qquad (G \circ F)'(a) = G'(F(a))F'(a)$$

Proof Sketch: (please forgive my lazy formatting, if only the summer was a year)

$$F(a+h) \approx F(a) + dF_a(h) = F(a) + F'(a)h$$
$$G(\beta+k) \approx G(\beta) + dG_\beta(k) = G(\beta) + G'(\beta)k$$

Consider then,

$$\begin{aligned}(G \circ F)(a+h) &= G(F(a+h)) \\ &\approx G(F(a) + F'(a)h) \quad : \text{ let } \beta = F(a) \text{ and } k = F'(a)h, \text{ note } h \text{ small} \Rightarrow k \text{ small} \\ &= G(\beta + k) \\ &\approx G(\beta) + G'(\beta)k \quad \leftarrow \text{hence this reasonable} \\ &= G(F(a)) + G'(F(a))F'(a)h\end{aligned}$$

Thus $\Delta(G \circ F) \approx G'(F(a))F'(a)h \Rightarrow \underline{d(G \circ F) = dG \circ dF}$.

In calculus III you may have learned how to calculate partial derivatives in terms of tree-diagrams and intermediate variable etc... We now have a way of understanding those rules and all the other chain rules in terms of one over-arching calculation: matrix multiplication of the constituent Jacobians in the composite function. Of course once we have this rule for the composite of two functions we can generalize to n-functions by a simple induction argument. For example, for three suitably defined mappings F, G, H,

$$(F \circ G \circ H)'(a) = F'(G(H(a)))G'(H(a))H'(a)$$

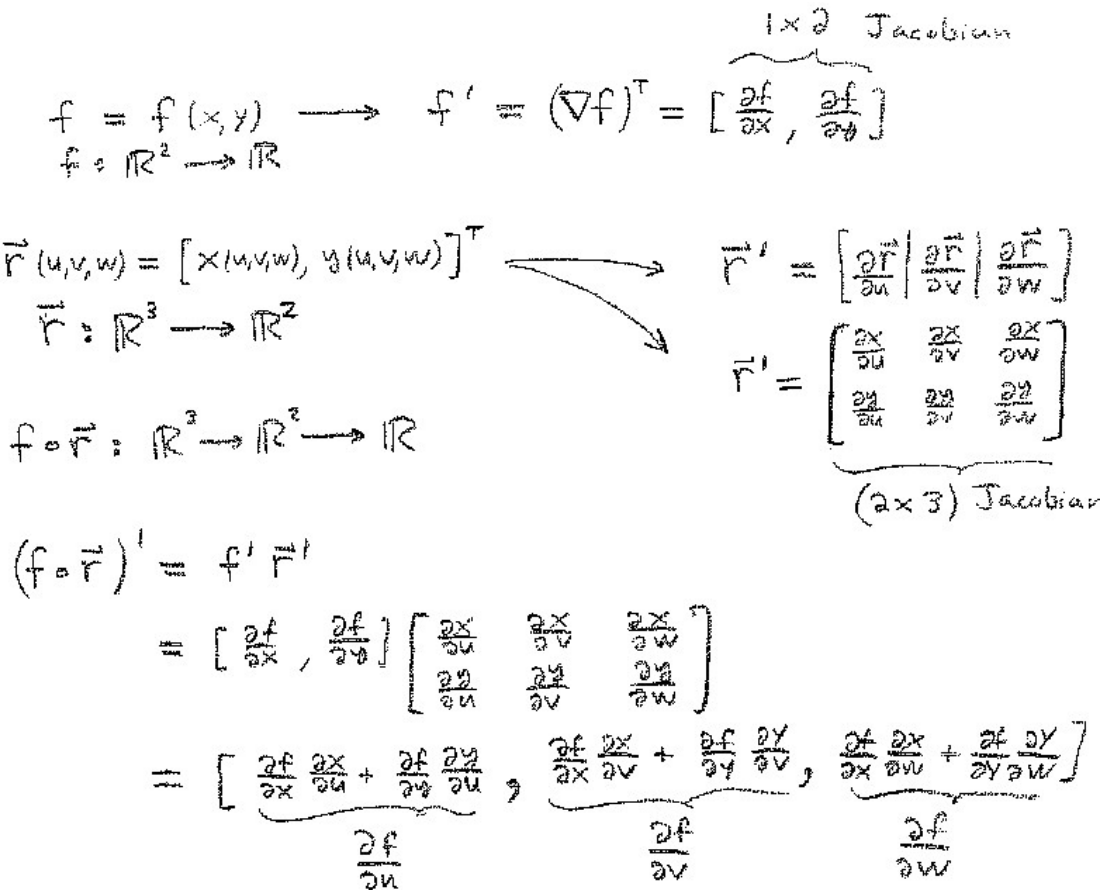
Example 3.5.2. .

Recall, $x = x(u,v,w)$, $y = y(u,v,w)$, $f = f(x,y)$ then,

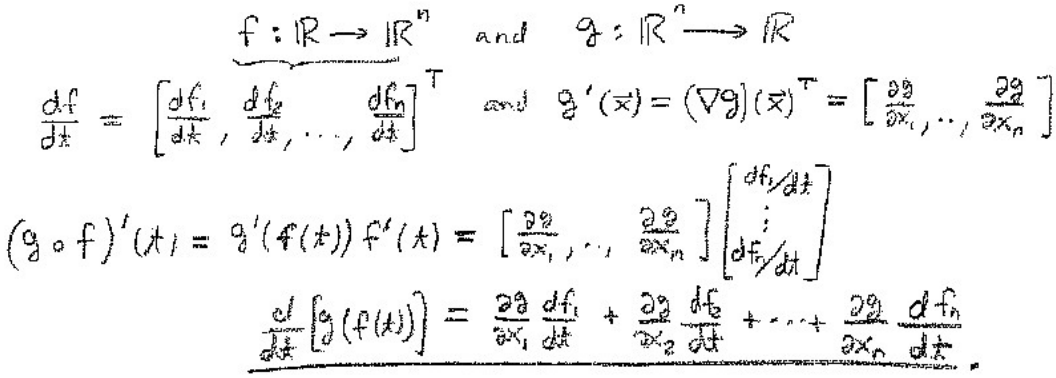
$$\frac{\partial f}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial f}{\partial y}$$
$$\frac{\partial f}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial f}{\partial y}$$
$$\frac{\partial f}{\partial w} = \frac{\partial x}{\partial w}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial w}\frac{\partial f}{\partial y}$$

Note we derive these via matrix multiplication in example below

Example 3.5.3. .



Example 3.5.4. .



Example 3.5.5. .

Let $f(x,y) = x^2y^2$ and $g(t) = (t, t^2)$

We have $f: \mathbb{R}^2 \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}^2$ note,

$f'(x,y) = [2xy^2, 2x^2y]$ and $g'(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$

Note $f \circ g : \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}$ has

$$(f \circ g)'(t) = f'(g(t))g'(t) = f'(t,t^2)g'(t) = \begin{bmatrix} 2t^5, 2t^4 \end{bmatrix} \begin{bmatrix} 1 \\ 2t \end{bmatrix} = 6t^5,$$

Note that $(f \circ g)(t) = f(t,t^2) = t^2t^4 = t^6$ so this result is not surprising!

Example 3.5.6. .

$$T(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$$

$$T' = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

If $w = f(x,y)$ then $w = g(r,\theta) = f(T(r,\theta))$ — f rewritten in polars.

$$\left[\frac{\partial g}{\partial r}, \frac{\partial g}{\partial \theta} \right] = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right] \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = \left[\cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}, -r\sin\theta \frac{\partial f}{\partial x} + r\cos\theta \frac{\partial f}{\partial y} \right]$$

With the proper understanding we have derived,

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$$

You can invert these, $r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}(y/x)$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial \theta} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

You can use these to change coordinates. For example

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right)$$
$$+ \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right)$$

$$= \cos^2\theta \frac{\partial^2 f}{\partial r^2} - \cos\theta\sin\theta \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial f}{\partial \theta}\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial f}{\partial r}\right] + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial f}{\partial \theta}\right]$$
$$+ \sin^2\theta \frac{\partial^2 f}{\partial r^2} + \sin\theta\cos\theta \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial f}{\partial \theta}\right] + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial f}{\partial r}\right] + \frac{\cos\theta}{r^2}\frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial f}{\partial \theta}\right]$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial \theta \partial r} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 f}{\partial \theta \partial r} + \cdots$$
$$\cdots + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial \theta} - \frac{\cos\theta\sin\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

$$\therefore \quad \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}}$$

3.6 common product rules

What sort of product can we expect to find among mappings? Remember two mappings have vector outputs and there is no way to multiply vectors in general. Of course, in the case we have two mappings that have equal-dimensional outputs we could take their dot-product. There is a product rule for that case: if $\vec{A}, \vec{B} : \mathbb{R}^n \to \mathbb{R}^m$ then

$$\boxed{\partial_j(\vec{A} \cdot \vec{B}) = (\partial_j \vec{A}) \cdot \vec{B}) + \vec{A} \cdot (\partial_j \vec{B})}$$

Or in the special case of $m = 3$ we could even take their cross-product and there is another product rule in that case:

$$\boxed{\partial_j(\vec{A} \times \vec{B}) = (\partial_j \vec{A}) \times \vec{B} + \vec{A} \times (\partial_j \vec{B})}$$

What other case can we "multiply" vectors? One very important case is $\mathbb{R}^2 = \mathbb{C}$ where is is customary to use the notation $(x, y) = x + iy$ and $f = u + iv$. If our range is complex numbers then we again have a product rule: if $f : \mathbb{R}^n \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{C}$ then

$$\boxed{\partial_j(fg) = (\partial_j f)g + f(\partial_j g)}$$

I have relegated the proof of most of these product rules to the end of this section. One other object worth differentiating is a matrix-valued function of $\mathbb{R}^n$. If we **define** the partial derivative of a matrix to be the matrix of partial derivatives then partial differentiation will respect the sum and product of matrices (we may return to this in depth if need be later on):

$$\boxed{\partial_j(A + B) = \partial_j B + \partial_j B} \qquad \boxed{\partial_j(AB) = (\partial_j A)B + A(\partial_j B)}$$

Moral of this story? If you have a pair mappings whose ranges allow some sort of product then it is entirely likely that there is a corresponding product rule [6].

3.6.1 scalar-vector product rule

There is one product rule which we can state for arbitrary mappings, note that we can always sensibly multiply a mapping by a function. Suppose then that $G : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ are differentiable at $a \in U$. It follows that there exist linear transformations $L_G : \mathbb{R}^n \to \mathbb{R}^m$ and $L_f : \mathbb{R}^n \to \mathbb{R}$ where

$$\lim_{h \to 0} \frac{G(a+h) - G(a) - L_G(h)}{||h||} = 0 \qquad \lim_{h \to 0} \frac{f(a+h) - f(a) - L_f(h)}{h} = 0$$

Since $G(a + h) \approx G(a) + L_G(h)$ and $f(a + h) \approx f(a) + L_f(h)$ we expect

$$\begin{aligned} fG(a+h) &\approx (f(a) + L_f(h))(G(a) + L_G(h)) \\ &\approx (fG)(a) + \underbrace{G(a)L_f(h) + f(a)L_G(h)}_{\text{linear in } h} + \underbrace{L_f(h)L_G(h)}_{2^{nd} \text{ order in } h} \end{aligned}$$

[6] In my research I consider functions on supernumbers, these also can be multiplied. Naturally there is a product rule for super functions, the catch is that super numbers z, w do not necessarily commute. However, if they're homogeneneous $zw = (-1)^{\epsilon_w \epsilon_z} wz$. Because of this the super product rule is $\partial_M(fg) = (\partial_M f)g + (-1)^{\epsilon_f \epsilon_M} f(\partial_M g)$

Thus we propose: $L(h) = G(a)L_f(h) + f(a)L_G(h)$ is the best linear approximation of fG.

$$\begin{aligned}
&\lim_{h\to 0}\frac{(fG)(a+h)-(fG)(a)-L(h)}{||h||} = \\
&\quad= \lim_{h\to 0}\frac{f(a+h)G(a+h)-f(a)G(a)-G(a)L_f(h)-f(a)L_G(h)}{||h||} \\
&\quad= \lim_{h\to 0}\frac{f(a+h)G(a+h)-f(a)G(a)-G(a)L_f(h)-f(a)L_G(h)}{||h||}+ \\
&\qquad+\lim_{h\to 0}\frac{f(a)G(a+h)-G(a+h)f(a)}{||h||} \\
&\qquad+\lim_{h\to 0}\frac{f(a+h)G(a)-G(a)f(a+h)}{||h||} \\
&\qquad+\lim_{h\to 0}\frac{f(a)G(a)-G(a)f(a)}{||h||} \\
&\quad= \lim_{h\to 0}\left[f(a)\frac{G(a+h)-G(a)-L_G(h)}{||h||} + \frac{f(a+h)-f(a)-L_f(h)}{||h||}G(a)+\right. \\
&\qquad\qquad\left.+\Big(f(a+h)-f(a)\Big)\frac{G(a+h)-G(a)}{||h||}\right] \\
&\quad= f(a)\left[\lim_{h\to 0}\frac{G(a+h)-G(a)-L_G(h)}{||h||}\right]+\left[\lim_{h\to 0}\frac{f(a+h)-f(a)-L_f(h)}{||h||}\right]G(a) \\
&\quad= 0
\end{aligned}$$

Where we have made use of the differentiability and the consequent continuity of both f and G at a. Furthermore, note

$$\begin{aligned}
L(h+ck) &= G(a)L_f(h+ck)+f(a)L_G(h+ck) \\
&= G(a)(L_f(h)+cL_f(k))+f(a)(L_G(h)+cL_G(k)) \\
&= G(a)L_f(h)+f(a)(L_G(h)+c(G(a)L_f(k)+f(a)L_G(k)) \\
&= L(h)+cL(k)
\end{aligned}$$

for all $h, k \in \mathbb{R}^n$ and $c \in \mathbb{R}$ hence $L = G(a)L_f + f(a)L_G$ is a linear transformation. We have proved (most of) the following proposition:

Proposition 3.6.1.

If $G : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ are differentiable at $a \in U$ then fG is differentiable at a and

$$d(fG)_a = (df)_a G(a) + f(a)dG_a \qquad (fG)'(a) = f'(a)G(a) + f(a)G'(a)$$

The argument above covers the ordinary product rule and a host of other less common rules. Note again that $G(a)$ and $G'(a)$ are vectors.

3.6.2 calculus of paths in $\mathbb{R}^3$

A **path** is a mapping from $\mathbb{R}$ to $\mathbb{R}^m$. We use such mappings to model position, velocity and acceleration of particles in the case $m = 3$. Some of these things were proved in previous sections of this chapter but I intend for this section to be self-contained so that you can read it without digging through the rest of this chapter.

Proposition 3.6.2.

If $F, G : U \subseteq \mathbb{R} \to \mathbb{R}^m$ are differentiable vector-valued functions and $\phi : U \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable real-valued function then for each $t \in U$,

1. $(F + G)'(t) = F'(t) + G'(t)$.
2. $(cF)'(t) = cF'(t)$.
3. $(\phi F)'(t) = \phi'(t)F(t) + \phi(t)F'(t)$.
4. $(F \cdot G)'(t) = F'(t) \cdot G(t) + F(t) \cdot G'(t)$.
5. provided $m = 3$, $(F \times G)'(t) = F'(t) \times G(t) + F(t) \times G'(t)$.
6. provided $\phi(U) \subset dom(F')$, $(F \circ \phi)'(t) = \phi'(t)F(\phi(t))$.

We have to insist that $m = 3$ for the statement with cross-products since we only have a standard cross-product in $\mathbb{R}^3$. We prepare for the proof of the proposition with a useful lemma. Notice this lemma tells us how to actually calculate the derivative of paths in examples. The derivative of component functions is nothing more than calculus I and one of our goals is to reduce things to those sort of calculations whenever possible.

Lemma 3.6.3.

If $F : U \subseteq \mathbb{R} \to \mathbb{R}^m$ is differentiable vector-valued function then for all $t \in U$,

$$F'(t) = (F_1'(t), F_2'(t), \ldots, F_m'(t))$$

We are given that the following vector limit exists and is equal to $F'(t)$,

$$F'(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h}$$

then by Proposition 2.1.8 the limit of a vector is related to the limits of its components as follows:

$$F'(t) \cdot e_j = \lim_{h \to 0} \frac{F_j(t+h) - F_j(t)}{h}.$$

Thus $(F'(t))_j = F_j'(t)$ and the lemma follows[7]. $\triangledown$

[7] this notation I first saw in a text by Marsden, it means the proof is partially completed but you should read on to finish the proof

Proof of proposition: We use the notation $F = \sum F_j e_j = (F_1, \ldots, F_m)$ and $G = \sum_i G_i e_i = (G_1, \ldots, G_m)$ throughout the proofs below. The $\sum$ is understood to range over $1, 2, \ldots m$. Begin with (1.),

$$\begin{aligned}
[(F+G)']_j &= \tfrac{d}{dt}[(F+G)_j] && \text{using the lemma} \\
&= \tfrac{d}{dt}[F_j + G_j] && \text{using def. } (F+G)_j = F_j + G_j \\
&= \tfrac{d}{dt}[F_j] + \tfrac{d}{dt}[G_j] && \text{by calculus I, } (f+g)' = f' + g'. \\
&= [F' + G']_j && \text{def. of vector addition for } F' \text{ and } G'
\end{aligned}$$

Hence $(F \times G)' = F' \times G + F \times G'$.The proofs of 2,3,5 and 6 are similar. I'll prove (5.),

$$\begin{aligned}
[(F \times G)']_k &= \tfrac{d}{dt}[(F \times G)_k] && \text{using the lemma} \\
&= \tfrac{d}{dt}[\sum \epsilon_{ijk} F_i G_j] && \text{using def. } F \times G \\
&= \sum \epsilon_{ijk} \tfrac{d}{dt}[F_i G_j] && \text{repeatedly using, } (f+g)' = f' + g' \\
&= \sum \epsilon_{ijk} [\tfrac{dF_i}{dt} G_j + F_i \tfrac{dG_j}{dt}] && \text{repeatedly using, } (fg)' = f'g + fg' \\
&= \sum \epsilon_{ijk} \tfrac{dF_i}{dt} G_j \sum \epsilon_{ijk} F_i \tfrac{dG_j}{dt}] && \text{property of finite sum } \textstyle\sum \\
&= (\tfrac{dF}{dt} \times G)_k + (F \times \tfrac{dG}{dt})_k) && \text{def. of cross product} \\
&= \left(\tfrac{dF}{dt} \times G + F \times \tfrac{dG}{dt} \right)_k && \text{def. of vector addition}
\end{aligned}$$

Notice that the calculus step really just involves calculus I applied to the components. The ordinary product rule was the crucial factor to prove the product rule for cross-products. We'll see the same for the dot product of mappings. Prove (4.)

$$\begin{aligned}
(F \cdot G)'(t) &= \tfrac{d}{dt}[\sum F_k G_k] && \text{using def. } F \cdot G \\
&= \sum \tfrac{d}{dt}[F_k G_k] && \text{repeatedly using, } (f+g)' = f' + g' \\
&= \sum [\tfrac{dF_k}{dt} G_k + F_k \tfrac{dG_k}{dt}] && \text{repeatedly using, } (fg)' = f'g + fg' \\
&= \tfrac{dF}{dt} \cdot G + F \cdot \tfrac{dG}{dt}. && \text{def. of dot product}
\end{aligned}$$

The proof of (3.) follows from applying the product rule to each component of $\phi(t)F(t)$. The proof of (2.) follow from (3.) in the case that $\phi(t) = c$ so $\phi'(t) = 0$. Finally the proof of (6.) follows from applying the chain-rule to each component. □

3.6.3 calculus of matrix-valued functions of a real variable

Definition 3.6.4.

A matrix-valued function of a real variable is a function from $I \subseteq \mathbb{R}$ to $\mathbb{R}^{m\times n}$. Suppose $A : I \subseteq \mathbb{R} \to \mathbb{R}^{m\times n}$ is such that $A_{ij} : I \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable for each i, j then we define

$$\frac{dA}{dt} = \left[\frac{dA_{ij}}{dt}\right]$$

which can also be denoted $(A')_{ij} = A'_{ij}$. We likewise define $\int A\,dt = [\int A_{ij}\,dt]$ for A with integrable components. Definite integrals and higher derivatives are also defined component-wise.

Example 3.6.5. *Suppose* $A(t) = \begin{bmatrix} 2t & 3t^2 \\ 4t^3 & 5t^4 \end{bmatrix}$. *I'll calculate a few items just to illustrate the definition above. calculate; to differentiate a matrix we differentiate each component one at a time:*

$$A'(t) = \begin{bmatrix} 2 & 6t \\ 12t^2 & 20t^3 \end{bmatrix} \qquad A''(t) = \begin{bmatrix} 0 & 6 \\ 24t & 60t^2 \end{bmatrix} \qquad A'(0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Integrate by integrating each component:

$$\int A(t)\,dt = \begin{bmatrix} t^2 + c_1 & t^3 + c_2 \\ t^4 + c_3 & t^5 + c_4 \end{bmatrix} \qquad \int_0^2 A(t)\,dt = \begin{bmatrix} t^2\big|_0^2 & t^3\big|_0^2 \\ t^4\big|_0^2 & t^5\big|_0^2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 16 & 32 \end{bmatrix}$$

Proposition 3.6.6.

Suppose A, B are matrix-valued functions of a real variable, f is a function of a real variable, c is a constant, and C is a constant matrix then

1. $(AB)' = A'B + AB'$ (product rule for matrices)
2. $(AC)' = A'C$
3. $(CA)' = CA'$
4. $(fA)' = f'A + fA'$
5. $(cA)' = cA'$
6. $(A + B)' = A' + B'$

where each of the functions is evaluated at the same time t and I assume that the functions and matrices are differentiable at that value of t and of course the matrices A, B, C are such that the multiplications are well-defined.

Proof: Suppose $A(t) \in \mathbb{R}^{m\times n}$ and $B(t) \in \mathbb{R}^{n\times p}$ consider,

$$\begin{aligned}
(AB)'_{ij} &= \tfrac{d}{dt}((AB)_{ij}) && \text{defn. derivative of matrix}\\
&= \tfrac{d}{dt}(\textstyle\sum_k A_{ik}B_{kj}) && \text{defn. of matrix multiplication}\\
&= \textstyle\sum_k \tfrac{d}{dt}(A_{ik}B_{kj}) && \text{linearity of derivative}\\
&= \textstyle\sum_k \left[\tfrac{dA_{ik}}{dt}B_{kj} + A_{ik}\tfrac{dB_{kj}}{dt}\right] && \text{ordinary product rules}\\
&= \textstyle\sum_k \tfrac{dA_{ik}}{dt}B_{kj} + \sum_k A_{ik}\tfrac{dB_{kj}}{dt} && \text{algebra}\\
&= (A'B)_{ij} + (AB')_{ij} && \text{defn. of matrix multiplication}\\
&= (A'B + AB')_{ij} && \text{defn. matrix addition}
\end{aligned}$$

this proves (1.) as i, j were arbitrary in the calculation above. The proof of (2.) and (3.) follow quickly from (1.) since C constant means $C' = 0$. Proof of (4.) is similar to (1.):

$$\begin{aligned}
(fA)'_{ij} &= \tfrac{d}{dt}((fA)_{ij}) && \text{defn. derivative of matrix}\\
&= \tfrac{d}{dt}(fA_{ij}) && \text{defn. of scalar multiplication}\\
&= \tfrac{df}{dt}A_{ij} + f\tfrac{dA_{ij}}{dt} && \text{ordinary product rule}\\
&= (\tfrac{df}{dt}A + f\tfrac{dA}{dt})_{ij} && \text{defn. matrix addition}\\
&= (\tfrac{df}{dt}A + f\tfrac{dA}{dt})_{ij} && \text{defn. scalar multiplication.}
\end{aligned}$$

The proof of (5.) follows from taking $f(t) = c$ which has $f' = 0$. I leave the proof of (6.) as an exercise for the reader. $\square$.

To summarize: the calculus of matrices is the same as the calculus of functions with the small qualifier that we must respect the rules of matrix algebra. The noncommutativity of matrix multiplication is the main distinguishing feature.

3.6.4 calculus of complex-valued functions of a real variable

Differentiation of functions from $\mathbb{R}$ to $\mathbb{C}$ is defined by splitting a given function into its real and imaginary parts then we just differentiate with respect to the real variable one component at a time. For example:

$$\begin{aligned}
\frac{d}{dt}(e^{2t}\cos(t) + ie^{2t}\sin(t)) &= \frac{d}{dt}(e^{2t}\cos(t)) + i\frac{d}{dt}(e^{2t}\sin(t))\\
&= (2e^{2t}\cos(t) - e^{2t}\sin(t)) + i(2e^{2t}\sin(t) + e^{2t}\cos(t))\\
&= e^{2t}(2+i)(\cos(t) + i\sin(t))\\
&= (2+i)e^{(2+i)t}
\end{aligned} \tag{3.1}$$

where I have made use of the identity[8] $e^{x+iy} = e^x(\cos(y) + i\sin(y))$. We just saw that $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$ which seems obvious enough until you appreciate that we just proved it for $\lambda = 2 + i$.

[8] or definition, depending on how you choose to set-up the complex exponential, I take this as the definition in calculus II

3.7 continuous differentiability

We have noted that differentiablility on some set U implies all sorts of nice formulas in terms of the partial derivatives. Curiously the converse is not quite so simple. It is possible for the partial derivatives to exist on some set and yet the mapping may fail to be differentiable. We need an extra topological condition on the partial derivatives if we are to avoid certain pathological[9] examples.

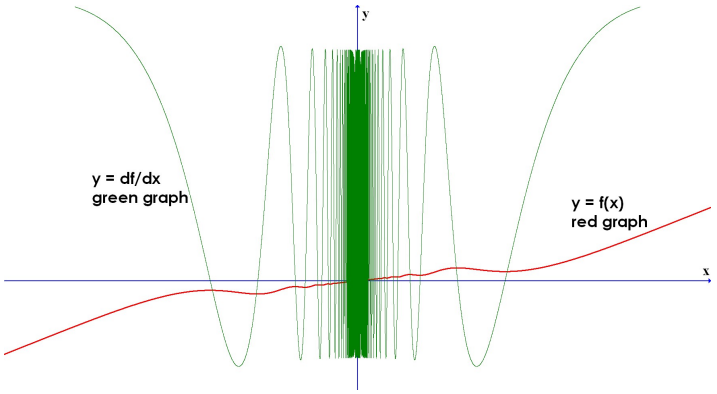
Example 3.7.1. *I found this example in Hubbard's advanced calculus text(see Ex. 1.9.4, pg. 123). It is a source of endless odd examples, notation and bizarre quotes. Let* $f(x) = 0$ *and*

$$f(x) = \frac{x}{2} + x^2 \sin \frac{1}{x}$$

for all $x \neq 0$*. I can be shown that the derivative* $f'(0) = 1/2$*. Moreover, we can show that* $f'(x)$ *exists for all* $x \neq 0$*, we can calculate:*

$$f'(x) = \frac{1}{2} + 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

Notice that $dom(f') = \mathbb{R}$*. Note then that the tangent line at* $(0, 0)$ *is* $y = x/2$*.*



You might be tempted to say then that this function is increasing at a rate of 1/2 for x *near zero. But this claim would be false since you can see that* $f'(x)$ *oscillates wildly without end near zero. We have a tangent line at* $(0, 0)$ *with positive slope for a function which is not increasing at* $(0, 0)$ *(recall that increasing is a concept we must define in a open interval to be careful). This sort of thing cannot happen if the derivative is continuous near the point in question.*

The one-dimensional case is really quite special, even though we had discontinuity of the derivative we still had a well-defined tangent line to the point. However, many interesting theorems in calculus of one-variable require the function to be continuously differentiable near the point of interest. For example, to apply the 2nd-derivative test we need to find a point where the first derivative is zero and the second derivative exists. We cannot hope to compute $f''(x_o)$ unless f' is continuous at x_o. The next example is *sick*.

[9] "pathological" as in, "your clothes are so pathological, where'd you get them?"

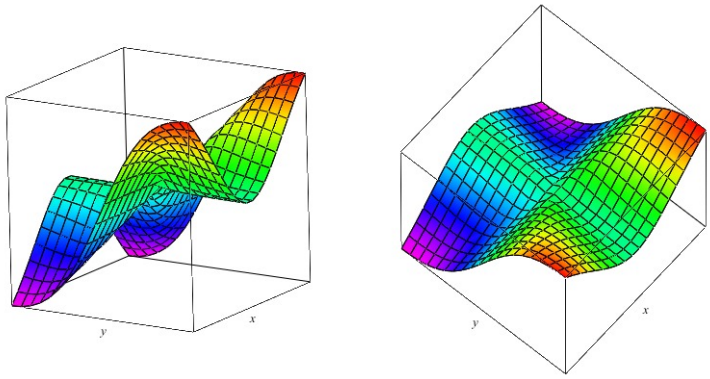
Example 3.7.2. *Let us define* $f(0,0) = 0$ *and*

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}$$

for all $(x,y) \neq (0,0)$ *in* $\mathbb{R}^2$. *It can be shown that* f *is continuous at* $(0,0)$. *Moreover, since* $f(x,0) = f(0,y) = 0$ *for all* x *and all* y *it follows that* f *vanishes identically along the coordinate axis. Thus the rate of change in the* e_1 *or* e_2 *directions is zero. We can calculate that*

$$\frac{\partial f}{\partial x} = \frac{2xy^3}{(x^2+y^2)^2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}$$

If you examine the plot of $z = f(x,y)$ *you can see why the tangent plane does not exist at* $(0,0,0)$.



Notice the sides of the box in the picture are parallel to the x *and* y *axes so the path considered below would fall on a diagonal slice of these boxes*[10]. *Consider the path to the origin* $t \mapsto (t,t)$ *gives* $f_x(t,t) = 2t^4/(t^2+t^2)^2 = 1/2$ *hence* $f_x(x,y) \to 1/2$ *along the path* $t \mapsto (t,t)$, *but* $f_x(0,0) = 0$ *hence the partial derivative* f_x *is not continuous at* $(0,0)$. *In this example, the discontinuity of the partial derivatives makes the tangent plane fail to exist.*

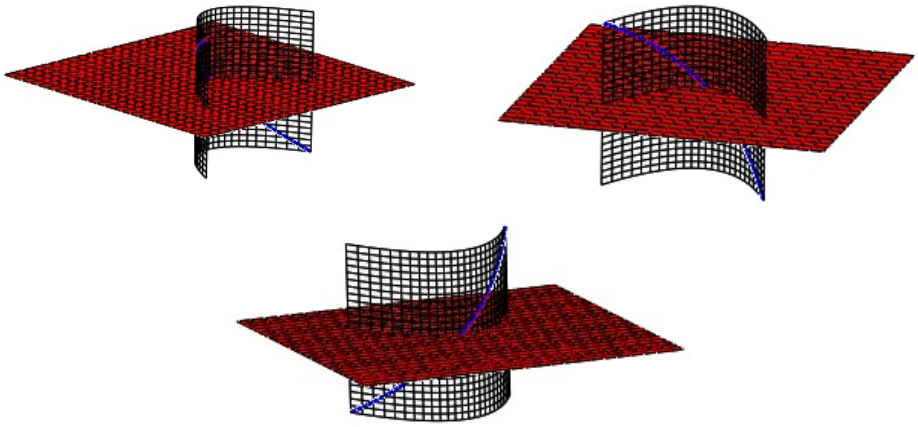
One might be tempted to suppose that if a function is continuous at a given point and if all the possible directional derivatives exist then differentiability should follow. It turns out this is not sufficient since continuity of the function does not imply some continuity along the partial derivatives. For example:

Example 3.7.3. *Let us define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by* $f(x,y) = 0$ *for* $y \neq x^2$ *and* $f(x,x^2) = x$. *I invite the reader to verify that this function is continuous at the origin. Moreover, consider the directional derivatives at* $(0,0)$. *We calculate, if* $v = \langle a, b \rangle$

$$D_v f(0,0) = \lim_{h \to 0} \frac{f(0+hv) - f(0)}{h} = \lim_{h \to 0} \frac{f(ah, bh)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

[10] the argument to follow stands alone, you don't need to understand the picture to understand the math here, but it's nice if you do

To see why $f(ah, bh) = 0$, consider the intersection of $\vec{r}(h) = (ha, hb)$ and $y = x^2$ the intersection is found at $hb = (ha)^2$ hence, noting $h = 0$ is not of interest in the limit, $b = ha^2$. If $a = 0$ then clearly (ah, bh) only falls on $y = x^2$ at $(0, 0)$. If $a \neq 0$ then the solution $h = b/a^2$ gives $f(ha, hb) = ha$ a nontrivial value. However, as $h \to 0$ we eventually reach values close enough to $(0, 0)$ that $f(ah, bh) = 0$. Hence we find **all** *directional derivatives exist and are zero at $(0, 0)$. Let's examine the graph of this example to see how this happened. The pictures below graph the xy-plane as red and the nontrivial values of f as a blue curve. The union of these forms the graph $z = f(x, y)$.*



Clearly, f is continuous at $(0, 0)$ as I invited you to prove. Moreover, clearly $z = f(x, y)$ cannot be well-approximated by a tangent plane at $(0, 0, 0)$. If we capture the xy-plane then we lose the blue curve of the graph. On the other hand, if we use a tilted plane then we lose the xy-plane part of the graph.

The moral of the story in the last two examples is simply that derivatives at a point, or even all directional derivatives at a point do not necessarily tell you much about the function near the point. This much is clear: something else is required if the differential is to have meaning which extends beyond one point in a nice way. Therefore, we consider the following:

Definition 3.7.4.

A mapping $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is **continuously differentiable** at $a \in U$ iff the partial derivative mappings $D_j F$ exist on an open set containing a and are continuous at a.

Technically, the term **continuously differentiable** would seem to indicate that the mapping $x \to dF_x$ is a continuous mapping at $x = a$. If dF_x is the **differential** at x then surely continuous differentiability ought to indicate that the linear transformations dF_x (thinking of x as varying) are glued together in some continuous fashion. That is correct, however, it is beyond our techniques at the present to discuss continuity of an *operator-valued* function. Further techniques must be developed to properly address the continuity in question. That said, once we do those things[11]

[11] see C.H. Edwards pages 172-180 and Proposition 2.4 in particular if you are so impatient as to not wait for me

then we'll find that this much more abstract idea of continuity indicates that the partial derivatives of the Jacobian are continuous at $x = a$. Therefore, the definition above is not at odds with the natural definition[12]

The import of the theorem below is that we can build the tangent plane from the Jacobian matrix provided the partial derivatives exist near the point of tangency and are continuous at the point of tangency. This is a very nice result because the concept of the linear mapping is quite abstract but partial differentiation of a given mapping is often easy. The proof that follows here is found in many texts, in particular see C.H. Edwards *Advanced Calculus of Several Variables* on pages 72-73.

Theorem 3.7.5.

If $F : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at a then F is differentiable at a

Proof: Consider $a+h$ sufficiently close to a that all the partial derivatives of F exist. Furthermore, consider going from a to $a+h$ by traversing a hyper-parallel-piped travelling n-perpendicular paths:

$$\underbrace{a}_{p_o} \to \underbrace{a + h_1 e_1}_{p_1} \to \underbrace{a + h_1 e_1 + h_2 e_2}_{p_2} \to \cdots \underbrace{a + h_1 e_1 + \cdots + h_n e_n}_{p_n} = a + h.$$

Let us denote $p_j = a + b_j$ where clearly b_j ranges from $b_o = 0$ to $b_n = h$ and $b_j = \sum_{i=1}^{j} h_i e_i$. Notice that the difference between p_j and p_{j-1} is given by:

$$p_j - p_{j-1} = a + \sum_{i=1}^{j} h_i e_i - a - \sum_{i=1}^{j-1} h_i e_i = h_j e_j$$

Consider then the following identity,

$$F(a + h) - F(a) = F(p_n) - F(p_{n-1}) + F(p_{n-1}) - F(p_{n-2}) + \cdots + F(p_1) - F(p_o)$$

This is to say the change in F from $p_o = a$ to $p_n = a + h$ can be expressed as a sum of the changes along the n-steps. Furthermore, if we consider the difference $F(p_j) - F(p_{j-1})$ you can see that only the j-th component of the argument of F changes. Since the j-th partial derivative exists on the interval for h_j considered by construction we can apply the mean value theorem to locate c_j such that:

$$h_j \partial_j F(p_{j-1} + c_j e_j) = F(p_j) - F(p_{j-1})$$

Therefore, using the mean value theorem for each interval, we select $c_1, \ldots c_n$ with:

$$F(a + h) - F(a) = \sum_{j=1}^{n} h_j \partial_j F(p_{j-1} + c_j e_j)$$

[12] Obviously, the reason this is given as the definition of continuously differentiable here and in most textbooks is to avoid the discussion about continuity of operators.

It follows we should propose L to satisfy the definition of Frechet differentation as follows:

$$L(h) = \sum_{j=1}^{n} h_j \partial_j F(a)$$

It is clear that L is linear (in fact, perhaps you recognize this as $L(h) = (\nabla F)(a) \bullet h$). Let us prepare to study the Frechet quotient,

$$\begin{aligned} F(a+h) - F(a) - L(h) &= \sum_{j=1}^{n} h_j \partial_j F(p_{j-1} + c_j e_j) - \sum_{j=1}^{n} h_j \partial_j F(a) \\ &= \sum_{j=1}^{n} h_j \underbrace{\left[\partial_j F(p_{j-1} + c_j e_j) - \partial_j F(a)\right]}_{g_j(h)} \end{aligned}$$

Observe that $p_{j-1} + c_j e_j \to a$ as $h \to 0$. Thus, $g_j(h) \to 0$ by the continuity of the partial derivatives at $x = a$. Finally, consider the Frechet quotient:

$$\lim_{h \to 0} \frac{F(a+h) - F(a) - L(h)}{||h||} = \lim_{h \to 0} \frac{\sum_j h_j g_j(h)}{||h||} = \lim_{h \to 0} \sum_j \frac{h_j}{||h||} g_j(h)$$

Notice $|h_j| \leq ||h||$ hence $\left| \frac{h_j}{||h||} \right| \leq 1$ and

$$0 \leq \left| \frac{h_j}{||h||} g_j(h) \right| \leq |g_j(h)|$$

Apply the squeeze theorem to deduce each term in the sum $\star$ limits to zero. Consquently, $L(h)$ satisfies the Frechet quotient and we have shown that F is differentiable at $x = a$ and the differential is expressed in terms of partial derivatives as expected; $dF_x(h) = \sum_{j=1}^{n} h_j \partial_j F(a)$ $\square$.

Given the result above it is a simple matter to extend the proof to $F : \mathbb{R}^n \to \mathbb{R}^m$.

Theorem 3.7.6.

If $F : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable at a then F is differentiable at a

Proof: If F is continuously differentiable at a then clearly each component function $F^j : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable at a. Thus, by Theorem 3.7.5 we have F^j differentiable at a hence

$$\lim_{h \to 0} \frac{F^j(a+h) - F^j(a) - dF_a^j(h)}{||h||} = 0 \quad \text{for all } j \in \mathbb{N}_m \quad \Rightarrow \quad \lim_{h \to 0} \frac{F(a+h) - F(a) - dF_a(h)}{||h||} = 0$$

by Proposition 2.1.8. This proves F is differentiable at a $\square$.

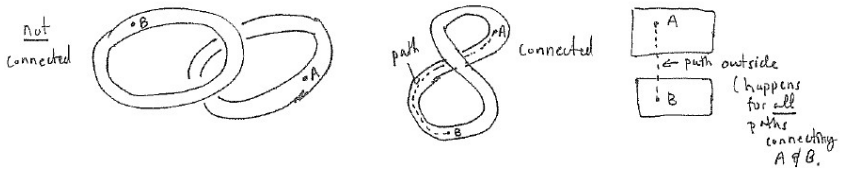
3.8 on why partial derivatives commute

The goal of this section is to prove the partial derivatives commute for nice functions. Of course some of the results we discuss on the way to that goal are interesting in their own right as well.

Definition 3.8.1.

We say $U \subseteq \mathbb{R}^n$ is **path connected** iff any two points in U can be connected by a path which is contained within the set.

For example, $\mathbb{R}^n$ is connected since given any two points $a, b \in \mathbb{R}^n$ we can construct the path $\phi(t) = a + t(b - a)$ from a to b and naturally the path is within the set. You can easily verify that open and closed balls are also path connected. Even a donut is path connected. However, a pair donuts is not path connected unless it's one of those artsy figure-8 deals.



Proposition 3.8.2.

If U is a connected open subset of $\mathbb{R}^n$ then a differentiable mapping $F : U \to \mathbb{R}^m$ is constant iff $F'(u) = 0$ for all $u \in U$.

Let $A, B \in U$ and connect these points with a smooth path $\gamma : [a, b] \longrightarrow U$.
(perhaps we should insist path connected $\Rightarrow$ smooth paths exist inside U...)
Consider, $F(\gamma(t)) = g(t)$. If F is constant then $g(t)$ is constant so
hence $F'(\gamma(t))\,\gamma'(t) = 0$ but γ smooth $\Rightarrow \gamma'(t) \neq 0$ hence
$F'(\gamma(a)) = F'(a) = 0 \Rightarrow F'(a) = 0 \quad \forall a \in U$. Conversely
if $F'(u) = 0 \quad \forall u \in U \Rightarrow \frac{d}{dt}[g(t)] = \underbrace{F'(\gamma(t))}_{\text{zero}}\,\gamma'(t) = 0$
Thus $g(a) = g(b) \Rightarrow F(\gamma(a)) = F(\gamma(b))$
$\Rightarrow F(A) = F(B) \quad \forall A, B \in U \therefore F$ constant on U.

Proposition 3.8.3.

If U is a connected open subset of $\mathbb{R}^n$ and the differentiable mappings $F, G : U \to \mathbb{R}$ such that $F'(x) = G'(x)$ for all $x \in U$ then there exists a constant vector $c \in \mathbb{R}^m$ such that $F(x) = G(x) + c$ for all $x \in U$.

Construct $H(x) = F(x) - G(x)$ and note $H'(x) = F'(x) - G'(x) = 0 \quad \forall x \in U$.
By the previous prop. $H(x) = C \quad \forall x \in U$ hence
$F(x) = G(x) + C \quad \forall x \in U$.

There is no mean value theorem for mappings since counter-examples exist. For example, Exercise 1.12 on page 63 shows the mean value theorem fails for the helix. In particular, you can find average velocity vector over a particular time interval such that the velocity vector never matches the average velocity over that time period. Fortunately, if we restrict our attention to mappings with one-dimensional codomains we still have a nice theorem:

Proposition 3.8.4. *(Mean Value Theorem)*

Suppose that $f : U \to \mathbb{R}$ is a differentiable function and U is an open set. Furthermore, suppose U contains the line segment from a to b in U;

$$L_{a,b} = \{a + t(b-a) \mid t \in [0,1]\} \subset U.$$

It follows that there exists some point $c \in L_{a,b}$ such that $f(b) - f(a) = f'(c)(b-a)$.

The proof follows from the construction of a function on $\mathbb{R}$ to which the elementary mean value Thm is applied. Let

$$g(t) = f\big(a + t(b-a)\big) \quad \text{for} \quad 0 \le t \le 1.$$

Or if you prefer, construct $\varphi(t) = a + t(b-a)$ which parametrizes the line segment from a to b. Clearly $\varphi'(t) = b - a$ and by the chain-rule,

$$g'(t) = f'\big(\varphi(t)\big)\varphi'(t)$$

Note $g : [0,1] \xrightarrow{\varphi} U \xrightarrow{f} \mathbb{R}$ is differentiable on $[0,1]$ thus the MVT gives $c_0 \in [0,1]$ such that $g(1) - g(0) = g'(c_0)$. Thus,

$$f(b) - f(a) = g(1) - g(0) = g'(c_0) = f'(\varphi(c_0))\varphi'(c_0) = f'(c) \cdot (b-a).$$

with $c = \varphi(c_0)$.

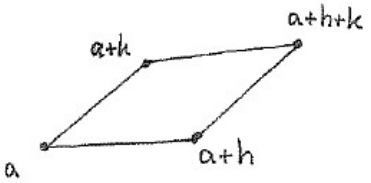
Definition 3.8.5. *(higher derivatives)*

We define nested directional derivatives in the natural way:

$$D_k D_h f(x) = D_k(D_h f(x)) = \lim_{t \to 0} \frac{D_h f(x + tk) - D_h f(x)}{t}$$

Furthermore, the **second difference** is defined by

$$\Delta^2 f_a(h,k) = f(a + h + k) - f(a + h) - f(a + k) + f(a)$$



note by MVT
$f(a+h) - f(a) = f'(a + \theta h)h$
$f(a+h+k) - f(a+k) = f'(a + k + \beta h)h$

$$\Delta^2 f_a(h,k) = \Delta f_a(h) - \Delta f_{a+k}(h)$$

This is Lemma 3.5 on page 86 of Edwards.

Proposition 3.8.6.

Suppose U us an open set and $f : U \to \mathbb{R}$ which is differentiable on U with likewise differentiable directional derivative function on U. Suppose that $a, a+h, a+k, a+h+k$ are all in U then there exist $\alpha, \beta \in (0,1)$ such that $\Delta^2 f_a(h,k) = D_k D_h f(a + \alpha h + \beta k)$.

The proof is rather neat. The α and β stem from two applications of the MVT, once for the function then once for its directional derivative.

Let $g(x) = f(x+k) - f(x)$ then $dg_x = df_{x+k} - df_x$.
Furthermore, using $\Delta^2 f_a(h,k) = f(a+h+k) - f(a+h) - f(a+k) + f(a)$ note,

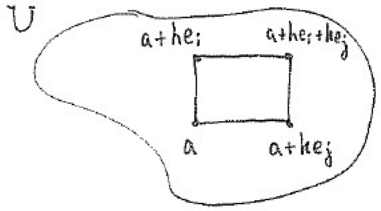
following Edwards pg. 87.

$$\begin{aligned}
\Delta^2 f_a(h,k) &= g(a+h) - g(a) \\
&= g'(a+\alpha h)\, h && \text{: by MVT } \exists\, \alpha \in (0,1). \\
&= (D_h g)(a+\alpha h) && \text{: diff} \Rightarrow \text{directional derivative exist.} \\
&= dg_{a+\alpha h}(h) \\
&= df_{a+\alpha h+k}(h) - df_{a+\alpha h}(h) \\
&= D_h f(a+\alpha h+k) - D_h f(a+\alpha h) \\
&= (D_h f)'(a+\alpha h+\beta k)(h) && \text{for some } \beta \in (0,1) \text{ by MVT again.} \\
&= (D_k D_h f)(a+\alpha h+\beta k) && \text{: unraveling notation.}
\end{aligned}$$

Proposition 3.8.7.

Let U be an open subset of $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is a function with continuous first and second partial derivatives on U then for all $i, j = 1, 2, \ldots, n$ we have $D_i D_j f = D_j D_i f$ on U;

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$



$$\Delta^2 f_a(he_i, ke_j) = D_{ke_j} D_{he_i} f(a + \alpha_1 he_i + \beta_1 ke_j)$$

$$\Delta^2 f_a(ke_j, he_i) = D_{he_i} D_{ke_j} f(a + \alpha_2 ke_j + \beta_2 he_i)$$

Then $\Delta^2 f_a(he_i, ke_j) = \Delta^2 f_a(ke_j, he_i)$

$$\Rightarrow D_j D_i f(a) = D_i D_j f(a)$$

Pull out h, k using homogeneity of $D_{cv} f = c D_v f$ and take limit $(h,k) \to 0$ to drop the h, k inside the $Df(\cdots)$ terms. This is possible thanks to the continuity of the partial derivatives near a.

3.9 complex analysis in a nutshell

Differentiation with respect to a real variable can be reduced to the slogan that we differentiate componentwise. Differentiation with respect to a complex variable requires additional structure. They key distinguishing ingredient is complex linearity:

Definition 3.9.1.

If we have some function $T : \mathbb{C} \to \mathbb{C}$ such that

(1.) $T(v+w) = T(v) + T(w)$ for all $v, w \in \mathbb{C}$ (2.) $T(cv) = cT(v)$ for all $c, v \in \mathbb{C}$

then we would say that T is **complex-linear**.

Condition (1.) is **additivity** whereas condition (2.) is **homogeneity**. Note that complex linearity implies real linearity however the converse is not true.

Example 3.9.2. *Suppose $T(z) = \bar{z}$ where if $z = x+iy$ for $x, y \in \mathbb{R}$ then $\bar{z} = x-iy$ is the* **complex conjugate** *of z. Consider for $c = a + ib$ where $a, b \in \mathbb{R}$,*

$$\begin{aligned} T(cz) &= T((a+ib)(x+iy)) \\ &= T(ax - by + i(ay + bx)) \\ &= ax - by - i(ay + bx) \\ &= (a - ib)(x - iy) \\ &= \bar{c}\, T(z) \end{aligned}$$

hence this map is not complex linear. On the other hand, if we study mutiplication by just $a \in \mathbb{R}$,

$$T(az) = T(a(x+iy)) = T(ax + iay) = ax - iay = a(x - iy) = aT(x+iy) = aT(z)$$

thus T is homogeneous with respect to real-number multiplication and it is also additive hence T is real linear.

Suppose that L is a linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. It is known from linear algebra that there exists a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $L(v) = Av$ for all $v \in \mathbb{R}^2$. In this section we use the notation $a + ib = (a, b)$ and

$$\boxed{(a,b) * (c,d) = (a+ib)(c+id) = ac - bd + i(ad + bc) = (ac - bd,\ ad + bc).}$$

This construction is due to Gauss in the early nineteenth century, the idea is to use two component vectors to *construct* complex numbers. There are other ways to construct complex numbers[13]. Notice that $L(x+iy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (ax + by, cx + dy) = ax + by + i(cx + dy)$ defines a real linear mapping on $\mathbb{C}$ for any choice of the real constants a, b, c, d. In contrast, complex linearity puts strict conditions on these constants:

[13] the same is true for real numbers, you can construct them in more than one way, however all constructions agree on the basic properties and as such it is the properties of real or complex numbers which truly defined them. That said, we choose Gauss' representation for convenience.

Theorem 3.9.3.

The linear mapping $L(v) = Av$ is complex linear iff the matrix A has the special form below:

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

To be clear, we mean to identify $\mathbb{R}^2$ with $\mathbb{C}$ as before. Thus the condition of complex homogeneity reads $L((a,b) * (x,y)) = (a,b) * L(x,y)$

Proof: assume L is complex linear. Define the matrix of L as before:

$$L(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This yields,

$$L(x+iy) = ax + by + i(cx + dy)$$

We can gain conditions on the matrix by examining the special points $1 = (1,0)$ and $i = (0,1)$

$$L(1,0) = (a,c) \qquad L(0,1) = (b,d)$$

Note that $(c_1, c_2) * (1,0) = (c_1, c_2)$ hence $L((c_1 + ic_2)1) = (c_1 + ic_2)L(1)$ yields

$$(ac_1 + bc_2) + i(cc_1 + dc_2) = (c_1 + ic_2)(a + ic) = c_1a - c_2c + i(c_1c + c_2a)$$

We find two equations by equating the real and imaginary parts:

$$ac_1 + bc_2 = c_1a - c_2c \qquad cc_1 + dc_2 = c_1c + c_2a$$

Therefore, $bc_2 = -c_2c$ and $dc_2 = c_2a$ for all $(c_1, c_2) \in \mathbb{C}$. Suppose $c_1 = 0$ and $c_2 = 1$. We find $b = -c$ and $d = a$. We leave the converse proof to the reader. The proposition follows. □

In analogy with the real case we define $f'(z)$ as follows:

Definition 3.9.4.

Suppose $f : dom(f) \subseteq \mathbb{C} \to \mathbb{C}$ and $z \in dom(f)$ then we define $f'(z)$ by the limit below:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

The derivative function f' is defined pointwise for all such $z \in dom(f)$ that the limit above exists.

Note that $f'(z) = \lim_{h \to 0} \frac{f'(z)h}{h}$ hence

$$\lim_{h \to 0} \frac{f'(z)h}{h} = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \quad \Rightarrow \quad \lim_{h \to 0} \frac{f(z+h) - f(z) - f'(z)h}{h} = 0$$

Note that the limit above simply says that $L(v) = f'(z)v$ gives the is the best **complex-linear** approximation of $\Delta f = f(z+h) - f(z)$.

Proposition 3.9.5.

If f is a complex differentiable at z_o then linearization $L(h) = f'(z_o)h$ is a complex linear mapping.

Proof: let $c, h \in \mathbb{C}$ and note $L(ch) = f'(z_o)(ch) = cf'(z_o)h = cL(h)$. □

It turns out that complex differentiability automatically induces real differentiability:

Proposition 3.9.6.

If f is a complex differentiable at z_o then f is (real) differentiable at z_o with $L(h) = f'(z_o)h$.

Proof: note that $\lim_{h\to 0} \frac{f(z+h)-f(z)-f'(z)h}{h} = 0$ implies

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - f'(z)h}{|h|} = 0$$

but then $|h| = ||h||$ and we know $L(h) = f'(z_o)h$ is real-linear hence L is the best linear approximation to Δf at z_o and the proposition follows. □

Let's summarize what we've learned: if $f : dom(f) \to \mathbb{C}$ is complex differentiable at z_o and $f = u + iv$ then,

1. $L(h) = f'(z_o)h$ is complex linear.
2. $L(h) = f'(z_o)h$ is the best real linear approximation to f viewed as a mapping on $\mathbb{R}^2$.

The Jacobian matrix for $f = (u, v)$ has the form

$$J_f(p_o) = \begin{bmatrix} u_x(p_o) & u_y(p_o) \\ v_x(p_o) & v_y(p_o) \end{bmatrix}$$

Theorem 3.9.3 applies to $J_f(p_o)$ since L is a complex linear mapping. Therefore we find the Cauchy Riemann equations: $u_x = v_y$ and $u_y = -v_x$. We have proved the following theorem:

Theorem 3.9.7.

If $f = u + iv$ is a complex function which is **complex-differentiable** at z_o then the partial derivatives of u and v exist at z_o and satisfy the Cauchy-Riemann equations at z_o

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Example 3.9.8. *Let $f(z) = e^z$ where the definition of the complex exponential function is given by the following, for each $x, y \in \mathbb{R}$ and $z = x + iy$*

$$f(x + iy) = e^{x+iy} = e^x(\cos(y) + i\sin(y)) = e^x\cos(y) + ie^x\sin(y)$$

Identify for $f = u + iv$ we have $u(x,y) = e^x\cos(y)$ and $v(x,y) = e^x\sin(y)$. Calculate:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[e^x\cos(y)\right] = e^x\cos(y) \qquad \& \qquad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[e^x\cos(y)\right] = -e^x\sin(y),$$

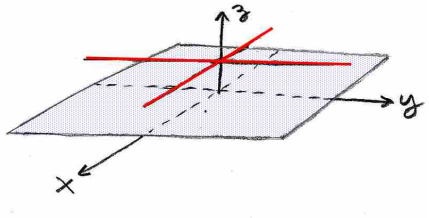
$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[e^x\sin(y)\right] = e^x\sin(y) \qquad \& \qquad \frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left[e^x\sin(y)\right] = e^x\cos(y).$$

Thus f satisfies the CR-equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. The complex exponential function is complex differentiable.

The converse of Theorem 3.9.7 is not true in general. It is possible to have functions $u, v : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ that satisfy the CR-equations at $z_o \in U$ and yet $f = u+iv$ fails to be complex differentiable at z_o.

Example 3.9.9. *Counter-example to converse of Theorem 3.9.7. Suppose* $f(x+iy) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$.

Clearly f has constant value 1 on the coordinate axes thus along the x-axes we can calculate the partial derivatives for u and v and they are both zero.



Likewise, along the y-axis we find u_y and v_y exist and are zero. At the origin we find u_x, u_y, v_x, v_y all exist and are zero. Therefore, the Cauchy-Riemann equations hold true at the origin. However, this function is not even continuous at the origin, thus it is not real differentiable!

The example above equally well serves as an example for a point where a function has partial derivatives which exist at all orders and yet the differential fails to exist. It's not a problem of complex variables in my opinion, it's a problem of advanced calculus. The key concept to reverse the theorem is continuous differentiability.

Theorem 3.9.10.

If u, v, u_x, u_y, v_x, v_y are continuous functions in some open disk of z_o and $u_x(z_o) = v_y(z_o)$ and $u_y(z_o) = -v_x(z_o)$ then $f = u + iv$ is complex differentiable at z_o.

Proof: we are given that a function $f : D_\epsilon(z_o) \subset \mathbb{R}^2 \to \mathbb{R}^2$ is continuous with continuous partial derivatives of its component functions u and v. Therefore, by Theorem 3.7.6 we know f is (real) differentiable at z_o. Therefore, we have a best linear approximation to the change in f near z_o which can be induced via multiplication of the Jacobian matrix:

$$L(v_1, v_2) = \begin{bmatrix} u_x(z_o) & u_y(z_o) \\ v_x(z_o) & v_y(z_o) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Note then that the given CR-equations show the matrix of L has the form

$$[L] = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $a = u_x(z_o)$ and $b = v_x(z_o)$. Consequently we find L is complex linear and it follows that f is complex differentiable at z_o since we have a complex linear map L such that

$$\lim_{h \to 0} \frac{f(z+h) - f(z) - L(h)}{||h||} = 0$$

note that the limit with h in the denominator is equivalent to the limit above which followed directly from the (real) differentiability at z_o. (the following is not needed for the proof of the theorem, but perhaps it is interesting anyway) Moreover, we can write

$$\begin{aligned} L(h_1, h_2) &= \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ &= \begin{bmatrix} u_x h_1 + u_y h_2 \\ -u_y h_1 + u_x h_2 \end{bmatrix} \\ &= u_x h_1 + u_y h_2 + i(-u_y h_1 + u_x h_2) \\ &= (u_x - i u_y)(h_1 + i h_2) \end{aligned}$$

Therefore we find $f'(z_o) = u_x - i u_y$ gives $L(h) = f'(z_o)z$. □

In the preceding section we found necessary and sufficient conditions for the component functions u, v to construct an complex differentiable function $f = u + iv$. The definition that follows is the next logical step: we say a function is analytic[14] at z_o if it is complex differentiable at each point in some open disk about z_o.

Definition 3.9.11.

Let $f = u + iv$ be a complex function. If there exists $\epsilon > 0$ such that f is complex differentiable for each $z \in D_\epsilon(z_o)$ then we say that f is **analytic** at z_o. If f is analytic for each $z_o \in U$ then we say f is analytic on U. If f is not analytic at z_o then we say that z_o is a **singular point**. Singular points may be outside the domain of the function. If f is analytic on the entire complex plane then we say f **is entire. Analytic functions are also called holomorphic functions**

If you look in my complex variables notes you can find proof of the following theorem (well, partial proof perhaps, but this result is shown in every good complex variables text)

[14] you may recall that a function on $\mathbb{R}$ was analyic at x_o if its Talyor series at x_o converged to the function in some neighborhood of x_o. This terminology is consistent but I leave the details for your complex analysis course

Theorem 3.9.12.

If $f:\mathbb{R}\to\mathbb{R}\subset\mathbb{C}$ is a function and $\tilde{f}:\mathbb{C}\to\mathbb{C}$ is an extension of f which is analytic then $\tilde{f}$ is unique. In particular, if there is an analytic extension of sine, cosine, hyperbolic sine or hyperbolic cosine then those extensions are unique.

This means if we demand analyticity then we actually had no freedom in our choice of the exponential. If we find a complex function which matches the exponential function on a line-segment (*in particular a closed interval in* $\mathbb{R}$ *viewed as a subset of* $\mathbb{C}$ *is a line-segment*) then there is just one complex function which agrees with the real exponential and is complex differentiable everywhere.

$$f(x)=e^x \qquad \text{extends uniquely to} \qquad \tilde{f}(z)=e^{Re(z)}(\cos(Im(z))+i\sin(Im(z))).$$

Note $\tilde{f}(x+0i)=e^x(\cos(0)+i\sin(0))=e^x$ thus $\tilde{f}|_{\mathbb{R}}=f$. Naturally, analyiticity is a desireable property for the complex-extension of known functions so this concept of analytic continuation is very nice. Existence aside, we should first construct sine, cosine etc... then we have to check they are both analytic and also that they actually agree with the real sine or cosine etc... If a function on $\mathbb{R}$ has vertical asymptotes, points of discontinuity or points where it is not smooth then the story is more complicated.

3.9.1 harmonic functions

We've discussed in some depth how to determine if a given function $f=u+iv$ is in fact analytic. In this section we study another angle on the story. We learn that the component functions u,v of an analytic function $f=u+iv$ are *harmonic conjugates* and they satisfy the phyically significant Laplace's equation $\nabla^2\phi=0$ where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. In addition we'll learn that if we have one solution of Laplace's equation then we can consider it to be the "u" of some yet undetermined analytic function $f=u+iv$. The remaining function v is then constructed through some integration guided by the CR-equations. The construction is similar to the problem of construction of a potential function for a given conservative force in calculus III.

Proposition 3.9.13.

If $f=u+iv$ is analytic on some domain $D\subseteq\mathbb{C}$ then u and v are solutions of Laplace's equation $\phi_{xx}+\phi_{yy}=0$ on D.

Proof: since $f=u+iv$ is analytic we know the CR-equations hold true; $u_x=v_y$ and $u_y=-v_x$. Moreover, f is continuously differentiable so we may commute partial derivatives by a theorem from multivariate calculus. Consider

$$u_{xx}+u_{yy}=(u_x)_x+(u_y)_y=(v_y)_x+(-v_x)_y=v_{yx}-v_{xy}=0$$

Likewise,

$$v_{xx}+v_{yy}=(v_x)_x+(v_y)_y=(-u_y)_x+(u_x)_y=-u_{yx}+u_{xy}=0$$

Of course these relations hold for all points inside D and the proposition follows. □

Example 3.9.14. *Note $f(z) = z^2$ is analytic with $u = x^2 - y^2$ and $v = 2xy$. We calculate,*

$$u_{xx} = 2, \quad u_{yy} = -2 \quad \Rightarrow \quad u_{xx} + u_{yy} = 0$$

Note $v_{xx} = v_{yy} = 0$ so v is also a solution to Laplace's equation.

Now let's see if we can reverse this idea.

Example 3.9.15. *Let $u(x, y) = x + c_1$ notice that u solves Laplace's equation. We seek to find a harmonic conjugate of u. We need to find v such that,*

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$$

Integrate these equations to deduce $v(x, y) = y + c_2$ for some constant $c_2 \in \mathbb{R}$. We thus construct an analytic function $f(x, y) = x + c_1 + i(y + c_2) = x + iy + c_1 + ic_2$. This is just $f(z) = z + c$ for $c = c_1 + ic_2$.

Example 3.9.16. *Suppose $u(x, y) = e^x \cos(y)$. Note that $u_{xx} = u$ whereas $u_{yy} = -u$ hence $u_{xx} + u_{yy} = 0$. We seek to find v such that*

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos(y) \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin(y)$$

Integrating $v_y = e^x \cos(y)$ with respect to y and $v_x = e^x \sin(y)$ with respect to x yields $v(x, y) = e^x \sin(y)$. We thus construct an analytic function $f(x, y) = e^x \cos(y) + ie^x \sin(y)$. Of course we should recognize the function we just constructed, it's just the complex exponential $f(z) = e^z$.

Notice we cannot just construct an analytic function from any given function of two variables. We have to start with a solution to Laplace's equation. This condition is rather restrictive. There is much more to say about harmonic functions, especially where applications are concerned. My goal here was just to give another perspective on analytic functions. Geometrically one thing we could see without further work at this point is that for an analytic function $f = u + iv$ the families of level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal. Note $grad(u) =< u_x, u_y >$ and $grad(v) =< v_x, v_y >$ have

$$grad(u) \cdot grad(v) = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0$$

This means the normal lines to the level curves for u and v are orthogonal. Hence the level curves of u and v are orthogonal. Another way to twist this, if you want to obtain orthogonal families of curves then analytic functions provide an easy way to create examples.

Remark 3.9.17.

This section covers a few lectures of the complex analysis course. I include it here in part to make connections. I always encourage students to understand math outside the comfort zone of isolated course components. Whenever we can understand material from several courses as part of a larger framework it is a step in the right direction.

Chapter 4

inverse and implicit function theorems

It is tempting to give a complete and rigourous proof of these theorems at the outset, but I will resist the temptation in lecture. I'm actually more interested that the student understand what the theorem claims before I show the real proof. I will sketch the proof and show many applications. A nearly complete proof is found in Edwards where he uses an iterative approximation technique founded on the contraction mapping principle, we will go through that a bit later in the course. I probably will not have typed notes on that material this semester, but Edward's is fairly readable and I think we'll profit from working through those sections. That said, we develop an intuition for just what these theorems are all about to start. That is the point of this chapter: to grasp what the linear algebra of the Jacobian suggests about the local behaviour of functions and equations.

4.1 inverse function theorem

Consider the problem of finding a **local** inverse for $f : dom(f) \subseteq \mathbb{R} \to \mathbb{R}$. If we are given a point $p \in dom(f)$ such that there exists an open interval I containing p with $f|_I$ a one-one function then we can reasonably construct an inverse function by the simple rule $f^{-1}(y) = x$ iff $f(x) = y$ for $x \in I$ and $y \in f(I)$. A sufficient condition to insure the existence of a local inverse is that the derivative function is either strictly positive or strictly negative on some neighborhood of p. If we are give a continuously differentiable function at p then it has a derivative which is continuous on some neighborhood of p. For such a function if $f'(p) \neq 0$ then there exists some interval centered at p for which the derivative is strictly positive or negative. It follows that such a function is strictly monotonic and is hence one-one thus there is a local inverse at p. We should all learn in calculus I that the derivative informs us about the local invertibility of a function. Natural question to ask for us here: does this extend to higher dimensions? If so, how?

The arguments I just made are supported by theorems that are developed in calculus I. Let me shift gears a bit and give a direct calculational explaination based on the linearization approximation.

If $x \approx p$ then $f(x) \approx f(p) + f'(p)(x-p)$. To find the formula for the inverse we solve $y = f(x)$ for x:

$$y \approx f(p) + f'(p)(x-p) \quad \Rightarrow \quad x \approx p + \frac{1}{f'(p)}\big[y - f(p)\big]$$

Therefore, $\boxed{f^{-1}(y) \approx p + \frac{1}{f'(p)}\big[y - f(p)\big]}$ for y near $f(p)$.

Example 4.1.1. *Just to help you believe me, consider $f(x) = 3x - 2$ then $f'(x) = 3$ for all x. Suppose we want to find the inverse function near $p = 2$ then the discussion preceding this example suggests,*

$$f^{-1}(y) = 2 + \frac{1}{3}(y - 4).$$

I invite the reader to check that $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$ for all $x, y \in \mathbb{R}$.

In the example above we found a global inverse exactly, but this is thanks to the linearity of the function in the example. Generally, inverting the linearization just gives the first approximation to the inverse.

Consider $F : dom(F) \subseteq \mathbb{R}^n \to \mathbb{R}^n$. If F is differentiable at $p \in \mathbb{R}^n$ then we can write $F(x) \approx F(p) + F'(p)(x-p)$ for $x \approx p$. Set $y = F(x)$ and solve for x via matrix algebra. This time we need to assume $F'(p)$ is an invertible matrix in order to isolate x,

$$y \approx F(p) + F'(p)(x-p) \quad \Rightarrow \quad x \approx p + (F'(p))^{-1}\big[y - f(p)\big]$$

Therefore, $\boxed{F^{-1}(y) \approx p + (F'(p))^{-1}\big[y - f(p)\big]}$ for y near $F(p)$. Apparently the condition to find a local inverse for a mapping on $\mathbb{R}^n$ is that the derivative matrix is nonsingular[1] in some neighborhood of the point. Experience has taught us from the one-dimensional case that we must insist the derivative is continuous near the point in order to maintain the validity of the approximation.

Recall from calculus II that as we attempt to approximate a function with a power series it takes an infinite series of power functions to recapture the formula exactly. Well, something similar is true here. However, the method of approximation is through an iterative approximation procedure which is built off the idea of Newton's method. The product of this iteration is a nested sequence of composite functions. To prove the theorem below one must actually provide proof the recursively generated sequence of functions converges. See pages 160-187 of Edwards for an in-depth exposition of the iterative approximation procedure. Then see pages 404-411 of Edwards for some material on uniform convergence[2] The main analytical tool which is used to prove the convergence is called the **contraction mapping principle**. The proof of the principle is relatively easy to follow and interestingly the main non-trivial step is an application of the geometric series. For

[1] nonsingular matrices are also called invertible matrices and a convenient test is that A is invertible iff $det(A) \neq 0$.

[2] actually that later chapter is part of why I chose Edwards' text, he makes a point of proving things in $\mathbb{R}^n$ in such a way that the proof naturally generalizes to function space. This is done by arguing with properties rather than formulas. The properties offen extend to infinite dimensions whereas the formulas usually do not.

the student of analysis this is an important topic which you should spend considerable time really trying to absorb as deeply as possible. The contraction mapping is at the base of a number of interesting and nontrivial theorems. Read Rosenlicht's *Introduction to Analysis* for a broader and better organized exposition of this analysis. In contrast, Edwards' uses analysis as a tool to obtain results for advanced calculus but his central goal is not a broad or well-framed treatment of analysis. Consequently, if analysis is your interest then you really need to read something else in parallel to get a better ideas about sequences of functions and uniform convergence. I have some notes from a series of conversations with a student about Rosenlicht, I'll post those for the interested student. These notes focus on the part of the material I require for this course. This is Theorem 3.3 on page 185 of Edwards' text:

Theorem 4.1.2. *(inverse function theorem)*

Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set W containing a and the derivative matrix $F'(a)$ is invertible. Then F is locally invertible at a. This means that there exists an open set $U \subseteq W$ containing a and V a open set containing $b = F(a)$ and a one-one, continuously differentiable mapping $G : V \to W$ such that $G(F(x)) = x$ for all $x \in U$ and $F(G(y)) = y$ for all $y \in V$. Moreover, the local inverse G can be obtained as the limit of the sequence of successive approximations defined by

$$G_o(y) = a \quad \text{and} \quad G_{n+1}(y) = G_n(y) - [F'(a)]^{-1}[F(G_n(y)) - y] \qquad \text{for all } y \in V.$$

The qualifier local is important to note. If we seek a global inverse then other ideas are needed. If the function is everywhere injective then logically $F(x) = y$ defines $F^{-1}(y) = x$ and F^{-1} so constructed in single-valued by virtue of the injectivity of F. However, for differentiable mappings, one might wonder how can the criteria of global injectivity be tested via the differential. Even in the one-dimensional case a vanishing derivative does not indicate a lack of injectivity; $f(x) = x^3$ has $f^{-1}(y) = \sqrt[3]{y}$ and yet $f'(0) = 0$ (therefore $f'(0)$ is not invertible). One the other hand, we'll see in the examples that follow that even if the derivative is invertible over a set it is possible for the values of the mapping to double-up and once that happens we cannot find a single-valued inverse function[3]

Remark 4.1.3. *James R. Munkres' Analysis on Manifolds good for a different proof.*

Another good place to read the inverse function theorem is in James R. Munkres *Analysis on Manifolds*. That text is careful and has rather complete arguments which are not entirely the same as the ones given in Edwards. Munkres' text does not use the contraction mapping principle, instead the arguments are more topological in nature.

[3] there are scientists and engineers who work with multiply-valued functions with great success, however, as a point of style if nothing else, we try to use functions in math.

To give some idea of what I mean by topological let be give an example of such an argument. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $F'(p)$ is invertible. Here's a sketch of the argument that $F'(x)$ is invertible for all x near p as follows:

1. the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(x) = det(F'(x))$ is formed by a multinomial in the component functions of $F'(x)$. This function is clearly continuous since we are given that the partial derivatives of the component functions of F are all continuous.

2. note we are given $F'(p)$ is invertible and hence $det(F'(p)) \neq 0$ thus the continuous function g is nonzero at p. It follows there is some open set U containing p for which $0 \notin g(U)$

3. we have $det(F'(x)) \neq 0$ for all $x \in U$ hence $F'(x)$ is invertible on U.

I would argue this is a topological argument because the key idea here is the continuity of g. Topology is the study of continuity in general.

Remark 4.1.4. *James J. Callahan's Advanced Calculus: a Geometric View, good reading.*

James J. Callahan's *Advanced Calculus: a Geometric View* has great merit in both visualization and well-thought use of linear algebraic techniques. In addition, many student will enjoy his staggered proofs where he first shows the proof for a simple low dimensional case and then proceeds to the general case.

Example 4.1.5. *Suppose* $F(x,y) = \big(\ \sin(y)+1,\ \sin(x)+2\ \big)$ *for* $(x,y) \in \mathbb{R}^2$. *Clearly* F *is continuously differentiable as all its component functions have continuous partial derivatives. Observe,*

$$F'(x,y) = [\ \partial_x F \mid \partial_y F\ ] = \begin{bmatrix} 0 & \cos(y) \\ \cos(x) & 0 \end{bmatrix}$$

Hence $F'(x,y)$ *is invertible at points* (x,y) *such that* $det(F'(x,y)) = -\cos(x)\cos(y) \neq 0$. *This means we* **may** *not be able to find local inverses at points* (x,y) *with* $x = \frac{1}{2}(2n+1)\pi$ *or* $y = \frac{1}{2}(2m+1)\pi$ *for some* $m,n \in \mathbb{Z}$. *Points where* $F'(x,y)$ *are singular are points where one or both of* $\sin(y)$ *and* $\sin(x)$ *reach extreme values thus the points where the Jacobian matrix are singular are in fact points where we cannot find a local inverse. Why? Because the function is clearly not 1-1 on any set which contains the points of singularity for* dF. *Continuing, recall from precalculus that sine has a standard inverse on* $[-\pi/2,\pi/2]$. *Suppose* $(x,y) \in [-\pi/2,\pi/2]^2$ *and seek to solve* $F(x,y) = (a,b)$ *for* (x,y)*:*

$$F(x,y) = \begin{bmatrix} \sin(y)+1 \\ \sin(x)+2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \ \Rightarrow\ \left\{ \begin{array}{l} \sin(y)+1 = a \\ \sin(x)+2 = b \end{array} \right\} \ \Rightarrow\ \left\{ \begin{array}{l} y = \sin^{-1}(a-1) \\ x = \sin^{-1}(b-2) \end{array} \right\}$$

It follows that $F^{-1}(a,b) = \big(\sin^{-1}(b-2),\ \sin^{-1}(a-1)\big)$ *for* $(a,b) \in [0,2]\times[1,3]$ *where you should note* $F([-\pi/2,\pi/2]^2) = [0,2]\times[1,3]$. *We've found a local inverse for* F *on the region* $[-\pi/2,\pi/2]^2$. *In other words, we just found a global inverse for the restriction of* F *to* $[-\pi/2,\pi/2]^2$. *Technically we ought not write* F^{-1}*, to be more precise we should write:*

$$(F|_{[-\pi/2,\pi/2]^2})^{-1}(a,b) = \big(\sin^{-1}(b-2),\ \sin^{-1}(a-1)\big).$$

It is customary to avoid such detail in many contexts. Inverse functions for sine, cosine, tangent etc... are good examples of this slight of langauge.

A **coordinate system** on $\mathbb{R}^n$ is an invertible mapping of $\mathbb{R}^n$ to $\mathbb{R}^n$. However, in practice the term coordinate system is used with less rigor. Often a coordinate system has various degeneracies. For example, in polar coordinates you could say $\theta = \pi/4$ or $\theta = 9\pi/4$ or generally $\theta = 2\pi k + \pi/4$ for any $k \in \mathbb{Z}$. Let's examine polar coordinates in view of the inverse function theorem.

Example 4.1.6. *Let $T(r,\theta) = \big(\ r\cos(\theta),\ r\sin(\theta)\ \big)$ for $(r,\theta) \in [0,\infty) \times (-\pi/2, \pi/2)$. Clearly T is continuously differentiable as all its component functions have continuous partial derivatives. To find the inverse we seek to solve $T(r,\theta) = (x,y)$ for (r,θ). Hence, consider $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Note that*

$$x^2 + y^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2(\cos^2(\theta) + \sin^2(\theta)) = r^2$$

and

$$\frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \tan(\theta).$$

It follows that $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$ for $(x,y) \in (0,\infty) \times \mathbb{R}$. We find

$$T^{-1}(x,y) = \left(\ \sqrt{x^2+y^2},\ \tan^{-1}(y/x)\ \right).$$

Let's see how the derivative fits with our results. Calcuate,

$$T'(r,\theta) = [\ \partial_r T \mid \partial_\theta T\ ] = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

note that $det(T'(r,\theta)) = r$ hence we the inverse function theorem provides the existence of a local inverse around any point except the origin. Notice the derivative does not detect the defect in the angular coordinate. Challenge, find the inverse function for $T(r,\theta) = \big(\ r\cos(\theta),\ r\sin(\theta)\ \big)$ with $dom(T) = [0,\infty) \times (\pi/2, 3\pi/2)$. Or, find the inverse for polar coordinates in a neighborhood of $(0,-1)$.

Example 4.1.7. *Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (ax, by, cz)$ for constants $a,b,c \in \mathbb{R}$ where $abc \neq 0$. Clearly T is continuously differentiable as all its component functions have continuous partial derivatives. We calculate $T'(x,y,z) = [\partial_x T|\partial_y T|\partial_z T] = [ae_1|be_2|ce_3]$. Thus $det(T'(x,y,z)) = abc \neq 0$ for all $(x,y,z) \in \mathbb{R}^3$ hence this function is locally invertible everywhere. Moreover, we calculate the inverse mapping by solving $T(x,y,z) = (u,v,w)$ for (x,y,z):*

$$(ax,by,cz) = (u,v,w) \quad \Rightarrow \quad (x,y,z) = (u/a, v/b, w/c) \quad \Rightarrow \quad \boxed{T^{-1}(u,v,w) = (u/a, v/b, w/c).}$$

Example 4.1.8. *Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $F(x) = Ax + b$ for some matrix $A \in \mathbb{R}^{n\times n}$ and vector $b \in \mathbb{R}^n$.* **Under what conditions is such a function invertible ?**. *Since the formula for this function gives each component function as a polynomial in the n-variables we can conclude the*

function is continuously differentiable. You can calculate that $F'(x) = A$. It follows that a sufficient condition for local inversion is $det(A) \neq 0$. It turns out that this is also a necessary condition as $det(A) = 0$ implies the matrix A has nontrivial solutions for $Av = 0$. We say $v \in Null(A)$ iff $Av = 0$. Note if $v \in Null(A)$ then $F(v) = Av + b = b$. This is not a problem when $det(A) \neq 0$ for in that case the null space is contains just zero; $Null(A) = \{0\}$. However, when $det(A) = 0$ we learn in linear algebra that $Null(A)$ contains infinitely many vectors so F is far from injective. For example, suppose $Null(A) = span\{e_1\}$ then you can show that $F(a_1, a_2, \ldots, a_n) = F(x, a_2, \ldots, a_n)$ for all $x \in \mathbb{R}$. Hence any point will have other points nearby which output the same value under F. Suppose $det(A) \neq 0$, to calculate the inverse mapping formula we should solve $F(x) = y$ for x,

$$y = Ax + b \quad \Rightarrow \quad x = A^{-1}(y - b) \quad \Rightarrow \quad \boxed{F^{-1}(y) = A^{-1}(y - b).}$$

Remark 4.1.9. *inverse function theorem holds for higher derivatives.*

In Munkres the inverse function theorem is given for r-times differentiable functions. In short, a C^r function with invertible differential at a point has a C^r inverse function local to the point. Edwards also has arguments for $r > 1$, see page 202 and arguments and surrounding arguments.

4.2 implicit function theorem

Consider the problem of solving $x^2 + y^2 = 1$ for y as a function of x.

$$x^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = 1 - x^2 \quad \Rightarrow \quad y = \pm\sqrt{1 - x^2}.$$

A function cannot have two outputs for a single input, when we write $\pm$ in the expression above it simply indicates our ignorance as to which is chosen. Once further information is given then we may be able to choose a $+$ or a $-$. For example:

1. if $x^2 + y^2 = 1$ and we want to solve for y near $(0, 1)$ then $y = \sqrt{1 - x^2}$ is the correct choice since $y > 0$ at the point of interest.

2. if $x^2 + y^2 = 1$ and we want to solve for y near $(0, -1)$ then $y = -\sqrt{1 - x^2}$ is the correct choice since $y < 0$ at the point of interest.

3. if $x^2 + y^2 = 1$ and we want to solve for y near $(1, 0)$ then it's impossible to find a single function which reproduces $x^2 + y^2 = 1$ on an **open disk** centered at $(1, 0)$.

What is the defect of case (3.) ? The trouble is that no matter how close we zoom in to the point there are always two y-values for each given x-value. Geometrically, this suggests either we have a discontinuity, a kink, or a vertical tangent in the graph. The given problem has a vertical tangent and hopefully you can picture this with ease since its just the unit-circle. In calculus I we studied

implicit differentiation, our starting point was to assume $y = y(x)$ and then we differentiated equations to work out implicit formulas for dy/dx. Take the unit-circle and differentiate both sides,

$$x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Note $\frac{dy}{dx}$ is not defined for $y = 0$. It's no accident that those two points $(-1, 0)$ and $(1, 0)$ are precisely the points at which we cannot solve for y as a function of x. Apparently, the singularity in the derivative indicates where we may have trouble solving an equation for one variable as a function of the remaining variable.

We wish to study this problem in general. Given n-equations in $(m+n)$-unknowns when can we solve for the last n-variables as functions of the first m-variables ? Given a continuously differentiable mapping $G = (G_1, G_2, \ldots, G_n) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ study the level set: (here $k_1, k_2, \ldots, k_n$ are constants)

$$\begin{aligned} G_1(x_1, \ldots, x_m, y_1, \ldots, y_n) &= k_1 \\ G_2(x_1, \ldots, x_m, y_1, \ldots, y_n) &= k_2 \\ &\vdots \\ G_n(x_1, \ldots, x_m, y_1, \ldots, y_n) &= k_n \end{aligned}$$

We wish to locally solve for $y_1, \ldots, y_n$ as functions of $x_1, \ldots x_m$. That is, find a mapping $h : \mathbb{R}^m \to \mathbb{R}^n$ such that $G(x, y) = k$ iff $y = h(x)$ near some point $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $G(a, b) = k$. In this section we use the notation $x = (x_1, x_2, \ldots x_m)$ and $y = (y_1, y_2, \ldots, y_n)$.

Before we turn to the general problem let's analyze the unit-circle problem in this notation. We are given $G(x, y) = x^2 + y^2$ and we wish to find $f(x)$ such that $y = f(x)$ solves $G(x, y) = 1$. Differentiate with respect to x and use the chain-rule:

$$\frac{\partial G}{\partial x}\frac{dx}{dx} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0$$

We find that $\boxed{dy/dx = -G_x/G_y} = -x/y$. Given this analysis we should suspect that if we are given some level curve $G(x, y) = k$ then we may be able to solve for y as a function of x near p if $G(p) = k$ and $G_y(p) \neq 0$. This suspicion is valid and it is one of the many consequences of the implicit function theorem.

We again turn to the linearization approximation. Suppose $G(x, y) = k$ where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ and suppose $G : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Suppose $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$ has $G(a, b) = k$. Replace G with its linearization based at (a, b):

$$G(x, y) \approx k + G'(a, b)(x - a, y - b)$$

here we have the matrix multiplication of the $n \times (m+n)$ matrix $G'(a,b)$ with the $(m+n) \times 1$ column vector $(x-a, y-b)$ to yield an n-component column vector. It is convenient to define partial derivatives with respect to a whole vector of variables,

$$\frac{\partial G}{\partial x} = \begin{bmatrix} \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial G_n}{\partial x_1} & \cdots & \frac{\partial G_n}{\partial x_m} \end{bmatrix} \qquad \frac{\partial G}{\partial y} = \begin{bmatrix} \frac{\partial G_1}{\partial y_1} & \cdots & \frac{\partial G_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial G_n}{\partial y_1} & \cdots & \frac{\partial G_n}{\partial y_n} \end{bmatrix}$$

In this notation we can write the $n \times (m+n)$ matrix $G'(a,b)$ as the concatenation of the $n \times m$ matrix $\frac{\partial G}{\partial x}(a,b)$ and the $n \times n$ matrix $\frac{\partial G}{\partial y}(a,b)$

$$G'(a,b) = \left[\frac{\partial G}{\partial x}(a,b) \middle| \frac{\partial G}{\partial y}(a,b) \right]$$

Therefore, for points close to (a,b) we have:

$$G(x,y) \approx k + \frac{\partial G}{\partial x}(a,b)(x-a) + \frac{\partial G}{\partial y}(a,b)(y-b)$$

The nonlinear problem $G(x,y) = k$ has been (locally) replaced by the linear problem of solving what follows for y:

$$k \approx k + \frac{\partial G}{\partial x}(a,b)(x-a) + \frac{\partial G}{\partial y}(a,b)(y-b) \tag{4.1}$$

Suppose the square matrix $\frac{\partial G}{\partial y}(a,b)$ is invertible at (a,b) then we find the following approximation for the implicit solution of $G(x,y) = k$ for y as a function of x:

$$y = b - \left[\frac{\partial G}{\partial y}(a,b) \right]^{-1} \left[\frac{\partial G}{\partial x}(a,b)(x-a) \right].$$

Of course this is not a formal proof, but it does suggest that $det\left[\frac{\partial G}{\partial y}(a,b)\right] \neq 0$ is a necessary condition for solving for the y variables.

As before suppose $G : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$. Suppose we have a continuously differentiable function $h : \mathbb{R}^m \to \mathbb{R}^n$ such that $h(a) = b$ and $G(x, h(x)) = k$. We seek to find the derivative of h in terms of the derivative of G. This is a generalization of the implicit differentiation calculation we perform in calculus I. I'm including this to help you understand the notation a bit more before I state the implicit function theorem. Differentiate with respect to x_l for $l \in \mathbb{N}_m$:

$$\frac{\partial}{\partial x_l}\Big[G(x, h(x)) \Big] = \sum_{i=1}^{m} \frac{\partial G}{\partial x_i}\frac{\partial x_i}{\partial x_l} + \sum_{j=1}^{n} \frac{\partial G}{\partial y_j}\frac{\partial h_j}{\partial x_l} = \frac{\partial G}{\partial x_l} + \sum_{j=1}^{n} \frac{\partial G}{\partial y_j}\frac{\partial h_j}{\partial x_l} = 0$$

we made use of the identity $\frac{\partial x_i}{\partial x_k} = \delta_{ik}$ to squash the sum of i to the single nontrivial term and the zero on the r.h.s follows from the fact that $\frac{\partial}{\partial x_l}(k) = 0$. Concatenate these derivatives from $k = 1$

up to $k = m$:

$$\left[\frac{\partial G}{\partial x_1} + \sum_{j=1}^{n} \frac{\partial G}{\partial y_j}\frac{\partial h_j}{\partial x_1}\middle|\frac{\partial G}{\partial x_2} + \sum_{j=1}^{n} \frac{\partial G}{\partial y_j}\frac{\partial h_j}{\partial x_2}\middle|\cdots\middle|\frac{\partial G}{\partial x_m} + \sum_{j=1}^{n} \frac{\partial G}{\partial y_j}\frac{\partial h_j}{\partial x_m}\right] = [0|0|\cdots|0]$$

Properties of matrix addition allow us to parse the expression above as follows:

$$\left[\frac{\partial G}{\partial x_1}\middle|\frac{\partial G}{\partial x_2}\middle|\cdots\middle|\frac{\partial G}{\partial x_m}\right] + \left[\sum_{j=1}^{n} \frac{\partial G}{\partial y_j}\frac{\partial h_j}{\partial x_1}\middle|\sum_{j=1}^{n} \frac{\partial G}{\partial y_j}\frac{\partial h_j}{\partial x_2}\middle|\cdots\middle|\sum_{j=1}^{n} \frac{\partial G}{\partial y_j}\frac{\partial h_j}{\partial x_m}\right] = [0|0|\cdots|0]$$

But, this reduces to

$$\frac{\partial G}{\partial x} + \left[\frac{\partial G}{\partial y}\frac{\partial h}{\partial x_1}\middle|\frac{\partial G}{\partial y}\frac{\partial h}{\partial x_2}\middle|\cdots\middle|\frac{\partial G}{\partial y}\frac{\partial h}{\partial x_m}\right] = 0 \in \mathbb{R}^{\,m\times n}$$

The concatenation property of matrix multiplication states $[Ab_1|Ab_2|\cdots|Ab_m] = A[b_1|b_2|\cdots|b_m]$ we use this to write the expression once more,

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\left[\frac{\partial h}{\partial x_1}\middle|\frac{\partial h}{\partial x_2}\middle|\cdots\middle|\frac{\partial h}{\partial x_m}\right] = 0 \quad \Rightarrow \quad \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{\partial h}{\partial x} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial h}{\partial x} = -\frac{\partial G}{\partial y}^{-1}\frac{\partial G}{\partial x}}$$

where in the last implication we made use of the assumption that $\frac{\partial G}{\partial y}$ is invertible.

Theorem 4.2.1. *(Theorem 3.4 in Edwards's Text see pg 190)*

Let $G : dom(G) \subseteq \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in a open ball about the point (a, b) where $G(a, b) = k$ (a constant vector in $\mathbb{R}^n$). If the matrix $\frac{\partial G}{\partial y}(a, b)$ is invertible then there exists an open ball U containing a in $\mathbb{R}^m$ and an open ball W containing (a, b) in $\mathbb{R}^m \times \mathbb{R}^n$ and a continuously differentiable mapping $h : U \to \mathbb{R}^n$ such that $G(x, y) = k$ iff $y = h(x)$ for all $(x, y) \in W$. Moreover, the mapping h is the limit of the sequence of successive approximations defined inductively below

$$h_o(x) = b, \quad h_{n+1} = h_n(x) - [\tfrac{\partial G}{\partial y}(a, b)]^{-1}G(x, h_n(x)) \qquad \text{for all } x \in U.$$

We will not attempt a proof of the last sentence for the same reasons we did not pursue the details in the inverse function theorem. However, we have already derived the first step in the iteration in our study of the linearization solution.

Proof: Let $G : dom(G) \subseteq \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in a open ball B about the point (a, b) where $G(a, b) = k$ ($k \in \mathbb{R}^n$ a constant). Furthermore, assume the matrix $\frac{\partial G}{\partial y}(a, b)$ is invertible. We seek to use the inverse function theorem to prove the implicit function theorem. Towards that end consider $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ defined by $F(x, y) = (x, G(x, y))$. To begin, observe that F is continuously differentiable in the open ball B which is centered at (a, b) since

G and x have continuous partials of their components in B. Next, calculate the derivative of $F = (x, G)$,

$$F'(x,y) = [\partial_x F | \partial_y F] = \left[\begin{array}{c|c} \partial_x x & \partial_y x \\ \hline \partial_x G & \partial_y G \end{array}\right] = \left[\begin{array}{c|c} I_m & 0_{m\times n} \\ \hline \partial_x G & \partial_y G \end{array}\right]$$

The determinant of the matrix above is the product of the deteminant of the blocks I_m and $\partial_y G$; $det(F'(x,y) = det(I_m)det(\partial_y G) = \partial_y G$. We are given that $\frac{\partial G}{\partial y}(a,b)$ is invertible and hence $det(\frac{\partial G}{\partial y}(a,b)) \neq 0$ thus $det(F'(x,y) \neq 0$ and we find $F'(a,b)$ is invertible. Consequently, the inverse function theorem applies to the function F at (a,b). Therefore, there exists $F^{-1} : V \subseteq \mathbb{R}^m \times \mathbb{R}^n \to U \subseteq \mathbb{R}^m \times \mathbb{R}^n$ such that F^{-1} is continuously differentiable. Note $(a,b) \in U$ and V contains the point $F(a,b) = (a, G(a,b)) = (a,k)$.

Our goal is to find the implicit solution of $G(x,y) = k$. We know that

$$F^{-1}(F(x,y)) = (x,y) \qquad \text{and} \qquad F(F^{-1}(u,v)) = (u,v)$$

for all $(x,y) \in U$ and $(u,v) \in V$. As usual to find the formula for the inverse we can solve $F(x,y) = (u,v)$ for (x,y) this means we wish to solve $(x, G(x,y)) = (u,v)$ hence $x = u$. The formula for v is more elusive, but we know it exists by the inverse function theorem. Let's say $y = H(u,v)$ where $H : V \to \mathbb{R}^n$ and thus $F^{-1}(u,v) = (u, H(u,v))$. Consider then,

$$(u,v) = F(F^{-1}(u,v) = F(u, H(u,v)) = (u, G(u, H(u,v))$$

Let $v = k$ thus $(u,k) = (u, G(u, H(u,k))$ for all $(u,v) \in V$. Finally, define $h(u) = H(u,k)$ for all $(u,k) \in V$ and note that $k = G(u, h(u))$. In particular, $(a,k) \in V$ and at that point we find $h(a) = H(a,k) = b$ by construction. It follows that $y = h(x)$ provides a continuously differentiable solution of $G(x,y) = k$ near (a,b).

Uniqueness of the solution follows from the uniqueness for the limit of the sequence of functions described in Edwards' text on page 192. However, other arguments for uniqueness can be offered, independent of the iterative method, for instance: see page 75 of Munkres *Analysis on Manifolds*. □

Remark 4.2.2. *notation and the implementation of the implicit function theorem.*

We assumed the variables y were to be written as functions of x variables to make explicit a local solution to the equation $G(x,y) = k$. This ordering of the variables is convenient to argue the proof, however the real theorem is far more general. We can select any subset of n input variables to make up the "y" so long as $\frac{\partial G}{\partial y}$ is invertible. I will use this generalization of the formal theorem in the applications that follow. Moreover, the notations x and y are unlikely to maintain the same interpretation as in the previous pages. Finally, we will for convenience make use of the notation $y = y(x)$ to express the existence of a function f such that $y = f(x)$ when appropriate. Also, $z = z(x,y)$ means there is some function h for which $z = h(x,y)$. If this notation confuses then invent names for the functions in your problem.

Example 4.2.3. *Suppose $G(x,y,z) = x^2+y^2+z^2$. Suppose we are given a point (a,b,c) such that $G(a,b,c) = R^2$ for a constant R.* **Problem: For which variable can we solve? What, if any, influence does the given point have on our answer?** *Solution: to begin, we have one equation and three unknowns so we should expect to find one of the variables as functions of the remaining two variables. The implicit function theorem applies as G is continuously differentiable.*

1. *if we wish to solve $z = z(x,y)$ then we need $G_z(a,b,c) = 2c \neq 0$.*
2. *if we wish to solve $y = y(x,z)$ then we need $G_y(a,b,c) = 2b \neq 0$.*
3. *if we wish to solve $x = x(y,z)$ then we need $G_x(a,b,c) = 2a \neq 0$.*

The point has no local solution for z if it is a point on the intersection of the xy-plane and the sphere $G(x,y,z) = R^2$. Likewise, we cannot solve for $y = y(x,z)$ on the $y = 0$ slice of the sphere and we cannot solve for $x = x(y,z)$ on the $x = 0$ slice of the sphere.

Notice, algebra verifies the conclusions we reached via the implicit function theorem:

$$z = \pm\sqrt{R^2-x^2-y^2} \qquad y = \pm\sqrt{R^2-x^2-z^2} \qquad x = \pm\sqrt{R^2-y^2-z^2}$$

When we are at zero for one of the coordinates then we cannot choose $+$ or $-$ since we need both on an open ball intersected with the sphere centered at such a point[4]. Remember, when I talk about local solutions I mean solutions which exist over the intersection of the solution set and an open ball in the ambient space ($\mathbb{R}^3$ in this context). The preceding example is the natural extension of the unit-circle example to $\mathbb{R}^3$. A similar result is available for the n-sphere in $\mathbb{R}^n$. I hope you get the point of the example, if we have one equation then if we wish to solve for a particular variable in terms of the remaining variables then all we need is continuous differentiability of the level function and a nonzero partial derivative at the point where we wish to find the solution. Now, the implicit function theorem doesn't find the solution for us, but it does provide the existence. In the section on implicit differentiation, existence is really all we need since focus our attention on rates of change rather than actually solutions to the level set equation.

Example 4.2.4. *Consider the equation $e^{xy} + z^3 - xyz = 2$.* **Can we solve this equation for $z = z(x,y)$ near $(0,0,1)$?** *Let $G(x,y,z) = e^{xy}+z^3-xyz$ and note $G(0,0,1) = e^0+1+0 = 2$ hence $(0,0,1)$ is a point on the solution set $G(x,y,z) = 2$. Note G is clearly continuously differentiable and*

$$G_z(x,y,z) = 3z^2 - xy \quad \Rightarrow \quad G_z(0,0,1) = 3 \neq 0$$

therefore, there exists a continuously differentiable function $h : dom(h) \subseteq \mathbb{R}^2 \to \mathbb{R}$ which solves $G(x,y,h(x,y)) = 2$ for (x,y) near $(0,0)$ and $h(0,0) = 1$.

I'll not attempt an explicit solution for the last example.

[4]if you consider $G(x,y,z) = R^2$ as a space then the open sets on the space are taken to be the intersection with the space and open balls in $\mathbb{R}^3$. This is called the subspace topology in topology courses.

Example 4.2.5. *Let $(x, y, z) \in S$ iff $x + y + z = 2$ and $y + z = 1$.* **Problem: For which variable(s) can we solve?** *Solution: define $G(x, y, z) = (x + y + z, y + z)$ we wish to study $G(x, y, z) = (2, 1)$. Notice the solution set is not empty since $G(1, 0, 1) = (1 + 0 + 1, 0 + 1) = (2, 1)$ Moreover, G is continuously differentiable. In this case we have two equations and three unknowns so we expect two variables can be written in terms of the remaining free variable. Let's examine the derivative of G:*

$$G'(x, y, z) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Suppose we wish to solve $x = x(z)$ and $y = y(z)$ then we should check invertiblility of[5]

$$\frac{\partial G}{\partial (x, y)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The matrix above is invertible hence the implicit function theorem applies and we can solve for x and y as functions of z. On the other hand, if we tried to solve for $y = y(x)$ and $z = z(x)$ then we'll get no help from the implicit function theorem as the matrix

$$\frac{\partial G}{\partial (y, z)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

is not invertible. Geometrically, we can understand these results from noting that $G(x, y, z) = (2, 1)$ is the intersection of the plane $x+y+z = 2$ and $y+z = 1$. Substituting $y+z = 1$ into $x+y+z = 2$ yields $x + 1 = 2$ hence $x = 1$ on the line of intersection. We can hardly use x as a free variable for the solution when the problem fixes x from the outset.

The method I just used to analyze the equations in the preceding example was a bit adhoc. In linear algebra we do much better for systems of linear equations. A procedure called Gaussian elimination naturally reduces a system of equations to a form in which it is manifestly obvious how to eliminate redundant variables in terms of a minimal set of basic free variables. The "y" of the implicit function proof discussions plays the role of the so-called **pivotal variables** whereas the "x" plays the role of the remaining **free variables**. These variables are generally intermingled in the list of total variables so to reproduce the pattern assumed for the implicit function theorem we would need to relabel variables from the outset of a calculation. In the following example, I show how reordering the variables allows us to solve for various pairs. In short, put the dependent variable first and the independent variables second so the Gaussian elimination shows the solution with minimal effort. Here's how:

[5] this notation should not be confused with $\frac{\partial(x,y)}{\partial(u,v)}$ which is used to denote a particular determinant associated with coordinate change of integrals, or pull-back of a differential form as explained on page 100 of H.M Edward's Advanced Calculus: A differential Forms Approach, we should discuss it in a later chapter.

Example 4.2.6. *Consider $G(x,y,u,v) = (3x+2y-u,\ 2x+y-v) = (-1,3)$. We have two equations with four variables. Let's investigate which pairs of variables can be taken as independent or dependent variables. The most efficient method to dispatch these questions is probably Gaussian elimination. I leave it to the reader to verify that:*

$$rref\left[\begin{array}{cccc|c} 3 & 2 & -1 & 0 & -1 \\ 2 & 1 & 0 & -1 & 3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 7 \\ 0 & 1 & -2 & 3 & -11 \end{array}\right]$$

We can immediately read from the result above that x, y can be taken to depend on u, v via the formulas:

$$x = -u + 2v + 7, \qquad y = 2u - 3v - 11$$

On the other hand, if we order the variables (u,v,x,y) then Gaussian elimination gives:

$$rref\left[\begin{array}{cccc|c} -1 & 0 & 3 & 2 & -1 \\ 0 & -1 & 2 & 1 & 3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 1 \\ 0 & 1 & -2 & -1 & -3 \end{array}\right]$$

Therefore, we find $u(x,y)$ and $v(x,y)$ as follows:

$$u = 3x + 2y + 1, \qquad v = 2x + y - 3.$$

To solve for x, u as functions of y, v consider:

$$rref\left[\begin{array}{cccc|c} 3 & -1 & 2 & 0 & -1 \\ 2 & 0 & 1 & -1 & 3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 1/2 & -1/2 & 3/2 \\ 0 & 1 & -1/2 & -3/2 & 11/2 \end{array}\right]$$

From which we can read,

$$x = -y/2 + v/2 + 3/2, \qquad u = y/2 + 3v/2 + 11/2.$$

I could solve the problem below in the efficient style above, but I will instead follow the method in which we discussed in the paragraphs surrounding Equation 4.1. In contrast to the general case, because the problem is linear the solution of Equation 4.1 is also a solution of the actual problem.

Example 4.2.7. *Solve the following system of equations near (1,2,3,4,5).*

$$G(x,y,z,a,b) = \begin{bmatrix} x+y+z+2a+2b \\ x+0+2z+2a+3b \\ 3x+2y+z+3a+4b \end{bmatrix} = \begin{bmatrix} 24 \\ 30 \\ 42 \end{bmatrix}$$

Differentiate to find the Jacobian:

$$G'(x,y,z,a,b) = \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 2 & 3 \\ 3 & 2 & 1 & 3 & 4 \end{bmatrix}$$

Let us solve $G(x,y,z,a,b) = (24,30,42)$ for $x(a,b), y(a,b), z(a,b)$ by the method of Equation 4.1. I'll omit the point-dependence of the Jacobian since it clearly has none.

$$G(x,y,z,a,b) = \begin{bmatrix} 24 \\ 30 \\ 42 \end{bmatrix} + \frac{\partial G}{\partial(x,y,z)} \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} + \frac{\partial G}{\partial(a,b)} \begin{bmatrix} a-4 \\ b-5 \end{bmatrix}$$

Let me make the notational chimera above explicit:

$$G(x,y,z,a,b) = \begin{bmatrix} 24 \\ 30 \\ 42 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a-4 \\ b-5 \end{bmatrix}$$

To solve $G(x,y,z,a,b) = (24,30,42)$ for (x,y,z) we may use the expression above. After a little calculation one finds:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} -4 & 1 & 2 \\ 5 & -2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

The constant term cancels and we find:

$$\begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -4 & 1 & 2 \\ 5 & -2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a-4 \\ b-5 \end{bmatrix}$$

Multiplying the matrices gives:

$$\begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 0 & 3 \\ 3 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a-4 \\ b-5 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a-4 \\ b-5 \end{bmatrix} = \begin{bmatrix} 5-b \\ 4-a \\ 9-a-b \end{bmatrix}$$

Therefore,

$$\boxed{x = 6-b, \quad y = 6-a, \quad z = 12-a-b.}$$

Is it possible to solve for any triple of the variables x,y,z,a,b for the given system? In fact, no. Let me explain by linear algebra. We can calculate: the augmented coefficient matrix for $G(x,y,z,a,b) = (24,30,42)$ Gaussian eliminates as follows:

$$rref \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 2 & 2 & 24 \\ 1 & 0 & 2 & 2 & 3 & 30 \\ 3 & 2 & 1 & 3 & 4 & 42 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 6 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 & 1 & 12 \end{array}\right].$$

First, note this is consistent with the answer we derived above. Second, examine the columns of $rref[G']$. You can ignore the 6-th column in the interest of this thought extending to nonlinear systems. The question of the suitability of a triple amounts to the invertibility of the submatrix of G' which corresponds to the triple. Examine:

$$\frac{\partial G}{\partial(y,z,a)} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}, \qquad \frac{\partial G}{\partial(x,z,b)} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix}$$

both of these are clearly singular since the third column is the sum of the first two columns. Alternatively, you can calculate the determinant of each of the matrices above is zero. In contrast,

$$\frac{\partial G}{\partial(z,a,b)} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

is non-singular. How to I know there is no linear dependence? Well, we could calculate the determinant is $1(8-6)-2(8-2)+2(6-2) = -2 \neq 0$. *Or, we could examine the row reduction above. The column correspondance property*[6] *states that linear dependences amongst columns of a matrix are preserved under row reduction. This means we can easily deduce dependence (if there is any) from the reduced matrix. Observe that column* 4 *is clearly the sum of columns* 2 *and* 3. *Likewise, column* 5 *is the sum of columns* 1 *and* 3. *On the other hand, columns* $3, 4, 5$ *admit no linear dependence. In general, more calculation would be required to "see" the independence of the far right columns. One reorders the columns and performs a new reduction to ascertain dependence. No such calculation is needed here since the problem is not that complicated.*

I find calculating the determinant of sub-Jacobian matrices is the simplest way for most students to quickly understand. I'll showcase this method in a series of examples attached to a later section. I have made use of some matrix theory in this section. If you didn't learn it in linear (or haven't taken linear yet) it's worth learning. These are nice tools to keep for later problems in life.

Remark 4.2.8. *independent constraints*

Gaussian elimination on a system of linear equations may produce a row of zeros. For example, $x + y = 0$ and $2x + 2y = 0$ gives $\text{rref}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The reason for this is quite obvious: the equations consider are not indpendent. In fact the second equation is a scalar multiple of the first. Generally, if there is some linear-dependence in a set of equations then we can expect this will happen. Although, if the equations are inhomogenous the last column might not be trivial because the system could be inconsistent (for example $x + y = 1$ and $2x + 2y = 5$).

Consider $G : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$. As we linearize $G = k$ we arrive at a homogeneous system which can be written briefly as $G'\vec{r} = 0$ (think about Equation 4.1 with the k's cancelled). We should **define** $G(\vec{r}) = k$ is a **system of** n **independent equations at** $\vec{r}_o$ iff $G(\vec{r}_o) = k$ and $\text{rref}[G'(\vec{r}_o)]$ has zero row. In other terminology, we could say the system of (possibly nonlinear) equations $G(\vec{r}) = k$ is built from n-independent equations near $\vec{r}_o$ iff the Jacobian matrix has **full-rank** at $\vec{r}_o$. If this full-rank condition is met then we can solve for n of the variables in terms of the remaining p variables. In general there will be many choices of how to do this, and some choices will be forbidden as we have seen in the examples already.

[6] I like to call it the CCP in my linear notes

4.3 implicit differentiation

Enough theory, let's calculate. In this section I apply previous theoretical constructions to specific problems. I also introduce standard notation for "constrained" partial differentiation which is also sometimes called "partial differentiation with a side condition". The typical problem is the following: given equations:

$$\begin{aligned} G_1(x_1,\dots,x_m,y_1,\dots,y_n) &= k_1 \\ G_2(x_1,\dots,x_m,y_1,\dots,y_n) &= k_2 \\ &\vdots \\ G_n(x_1,\dots,x_m,y_1,\dots,y_n) &= k_n \end{aligned}$$

calculate partial derivative of dependent variables with respect to independent variables. Continuing with the notation of the implicit function discussion we'll assume that y will be dependent on x. I want to recast some of our arguments via differentials[7]. Take the total differential of each equation above,

$$\begin{aligned} dG_1(x_1,\dots,x_m,y_1,\dots,y_n) &= 0 \\ dG_2(x_1,\dots,x_m,y_1,\dots,y_n) &= 0 \\ &\vdots \\ dG_n(x_1,\dots,x_m,y_1,\dots,y_n) &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} \partial_{x_1}G_1dx_1+\cdots+\partial_{x_m}G_1dx_m+\partial_{y_1}G_1dy_1+\cdots+\partial_{y_n}G_1dy_n &= 0 \\ \partial_{x_1}G_2dx_1+\cdots+\partial_{x_m}G_2dx_m+\partial_{y_1}G_2dy_1+\cdots+\partial_{y_n}G_2dy_n &= 0 \\ &\vdots \\ \partial_{x_1}G_ndx_1+\cdots+\partial_{x_m}G_ndx_m+\partial_{y_1}G_ndy_1+\cdots+\partial_{y_n}G_ndy_n &= 0 \end{aligned}$$

Notice, this can be nicely written in column vector notation as:

$$\partial_{x_1}Gdx_1+\cdots+\partial_{x_m}Gdx_m+\partial_{y_1}Gdy_1+\cdots+\partial_{y_n}Gdy_n=0$$

Or, in matrix notation:

$$[\partial_{x_1}G|\cdots|\partial_{x_m}G]\begin{bmatrix} dx_1 \\ \vdots \\ dx_m \end{bmatrix}+[\partial_{y_1}G|\cdots|\partial_{y_n}G]\begin{bmatrix} dy_1 \\ \vdots \\ dy_n \end{bmatrix}=0$$

[7] in contrast, In the previous section we mostly used derivative notation

Finally, solve for dy, we assume $[\partial_{y_1}G|\cdots|\partial_{y_n}G]^{-1}$ exists,

$$\begin{bmatrix} dy_1 \\ \vdots \\ dy_n \end{bmatrix} = -[\partial_{y_1}G|\cdots|\partial_{y_n}G]^{-1}[\partial_{x_1}G|\cdots|\partial_{x_m}G] \begin{bmatrix} dx_1 \\ \vdots \\ dx_m \end{bmatrix}$$

Given all of this we can calculate $\frac{\partial y_i}{\partial x_j}$ by simply reading the coeffient dx_j in the i-th row. I will make this idea quite explicit in the examples that follow.

Example 4.3.1. *Let's return to a common calculus III problem. Suppose $F(x,y,z) = k$ for some constant k.* **Find partial derivatives of x, y or z with repsect to the remaining variables.** *Solution: I'll use the method of differentials once more:*

$$dF = F_x dx + F_y dy + F_z dz = 0$$

We can solve for dx, dy or dz provided F_x, F_y or F_z is nonzero respective and these differential expressions reveal various partial derivatives of interest:

$$dx = -\frac{F_y}{F_x}dy - \frac{F_z}{F_x}dz \qquad \Rightarrow \qquad \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} \;\; \& \;\; \frac{\partial x}{\partial z} = -\frac{F_z}{F_x}$$

$$dy = -\frac{F_x}{F_y}dx - \frac{F_z}{F_y}dz \qquad \Rightarrow \qquad \frac{\partial y}{\partial x} = -\frac{F_x}{F_y} \;\; \& \;\; \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$

$$dz = -\frac{F_x}{F_z}dx - \frac{F_y}{F_z}dy \qquad \Rightarrow \qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \;\; \& \;\; \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

In each case above, the implicit function theorem allows us to solve for one variable in terms of the remaining two. If the partial derivative of F in the denominator are zero then the implicit function theorem does not apply and other thoughts are required. Often calculus text give the following as a homework problem:

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -\frac{F_y}{F_x}\frac{F_z}{F_y}\frac{F_x}{F_z} = -1.$$

In the equation above we have x appear as a dependent variable on y, z and also as an independent variable for the dependent variable z. These mixed expressions are actually of interest to engineering and physics. The less mbiguous notation below helps better handle such expressions:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

In each part of the expression we have clearly denoted which variables are taken to depend on the others and in turn what sort of partial derivative we mean to indicate. Partial derivatives are not taken alone, they must be done in concert with an understanding of the totality of the indpendent variables for the problem. We hold all the remaining indpendent variables fixed as we take a partial derivative.

The explicit independent variable notation is more important for problems where we can choose more than one set of indpendent variables for a given dependent variables. In the example that follows we study $w = w(x, y)$ but we could just as well consider $w = w(x, z)$. Generally it will not be the case that $\left(\frac{\partial w}{\partial x}\right)_y$ is the same as $\left(\frac{\partial w}{\partial x}\right)_z$. In calculation of $\left(\frac{\partial w}{\partial x}\right)_y$ we hold y constant as we vary x whereas in $\left(\frac{\partial w}{\partial x}\right)_z$ we hold z constant as we vary x. There is no reason these ought to be the same[8].

Example 4.3.2. *Suppose* $x+y+z+w = 3$ *and* $x^2-2xyz+w^3 = 5$. **Calculate partial derivatives of** z **and** w **with respect to the independent variables** x, y. *Solution: we begin by calculation of the differentials of both equations:*

$$\begin{aligned} dx + dy + dz + dw &= 0 \\ (2x - 2yz)dx - 2xzdy - 2xydz + 3w^2dw &= 0 \end{aligned}$$

We can solve for (dz, dw). *In this calculation we can treat the differentials as formal variables.*

$$\begin{aligned} dz + dw &= -dx - dy \\ -2xydz + 3w^2dw &= -(2x - 2yz)dx + 2xzdy \end{aligned}$$

I find matrix notation is often helpful,

$$\begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix} \begin{bmatrix} dz \\ dw \end{bmatrix} = \begin{bmatrix} -dx - dy \\ -(2x - 2yz)dx + 2xzdy \end{bmatrix}$$

Use Kramer's rule, multiplication by inverse, substitution, adding/subtracting equations etc... whatever technique of solving linear equations you prefer. Our goal is to solve for dz *and* dw *in terms of* dx *and* dy. *I'll use Kramer's rule this time:*

$$dz = \frac{det\left[\begin{array}{c|c} -dx - dy & 1 \\ -(2x - 2yz)dx + 2xzdy & 3w^2 \end{array}\right]}{det\begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix}} = \frac{3w^2(-dx - dy) + (2x - 2yz)dx - 2xzdy}{3w^2 + 2xy}$$

Collecting terms,

$$dz = \left(\frac{-3w^2 + 2x - 2yz}{3w^2 + 2xy}\right)dx + \left(\frac{-3w^2 - 2xz}{3w^2 + 2xy}\right)dy$$

From the expression above we can read various implicit derivatives,

$$\boxed{\left(\frac{\partial z}{\partial x}\right)_y = \frac{-3w^2 + 2x - 2yz}{3w^2 + 2xy} \qquad \& \qquad \left(\frac{\partial z}{\partial y}\right)_x = \frac{-3w^2 - 2xz}{3w^2 + 2xy}}$$

The notation above indicates that z *is understood to be a function of independent variables* x, y. $\left(\frac{\partial z}{\partial x}\right)_y$ *means we take the derivative of* z *with respect to* x *while holding* y *fixed. The appearance*

[8] a good exercise would be to do the example over but instead aim to calculate partial derivatives for y, w with respect to independent variables x, z

of the dependent variable w can be removed by using the equations $G(x,y,z,w) = (3,5)$. Similar ambiguities exist for implicit differentiation in calculus I. Apply Kramer's rule once more to solve for dw:

$$dw = \frac{det\left[\begin{array}{c|c} 1 & -dx-dy \\ -2xy & -(2x-2yz)dx+2xzdy \end{array}\right]}{det\begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix}} = \frac{-(2x-2yz)dx+2xzdy-2xy(dx+dy)}{3w^2+2xy}$$

Collecting terms,

$$dw = \left(\frac{-2x+2yz-2xy}{3w^2+2xy}\right)dx + \left(\frac{2xzdy-2xydy}{3w^2+2xy}\right)dy$$

We can read the following from the differential above:

$$\boxed{\left(\frac{\partial w}{\partial x}\right)_y = \frac{-2x+2yz-2xy}{3w^2+2xy} \qquad \& \qquad \left(\frac{\partial w}{\partial y}\right)_x = \frac{2xzdy-2xydy}{3w^2+2xy}}$$

You should ask: where did we use the implicit function theorem in the preceding example? Notice our underlying hope is that we can solve for $z = z(x,y)$ and $w = w(x,y)$. The implicit function theorem states this is possible precisely when $\frac{\partial G}{\partial(z,w)} = \begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix}$ is non singular. Interestingly this is the same matrix we must consider to isolate dz and dw. The calculations of the example are only meaningful if the $det\begin{bmatrix} 1 & 1 \\ -2xy & 3w^2 \end{bmatrix} \neq 0$. In such a case the implicit function theorem applies and it is reasonable to suppose z, w can be written as functions of x, y.

4.3.1 computational techniques for partial differentiation with side conditions

In this section I show you how I teach this to calculus III. In other words, we set-aside the explicit mention of the implicit function theorem and work out some typical calculations. If one desires rigor then the answer is found from the implicit function theorems careful application, that is how to justify what follows. These notes are taken from my calculus III notes, but I thought it wise to include them here since most calculus texts do not bother to show these calculations (which is sad since they actually matter to the application of multivariate analysis to many real world applications) To begin, we define[9] the total differential.

Definition 4.3.3.

If $f = f(x_1, x_2, \ldots, x_n)$ then $df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n$.

[9] I invite the reader to verify the notation "defined" in this section is in fact totally sympatico with our previous definitions

Example 4.3.4. *Suppose $E = pv + t^2$ then $dE = vdp + pdv + 2tdt$. In this example the dependent variable is E whereas the independent variables are p, v and t.*

Example 4.3.5. Problem: *what are $\partial F/\partial x$ and $\partial F/\partial y$ if we know that $F = F(x, y)$ and $dF = (x^2 + y)dx - \cos(xy)dy$.*
Solution: *if $F = F(x, y)$ then the total differential has the form $dF = F_x dx + F_y dy$. We simply compare the general form to the given $dF = (x^2 + y)dx - \cos(xy)dy$ to obtain:*

$$\frac{\partial F}{\partial x} = x^2 + y, \qquad \frac{\partial F}{\partial y} = -\cos(xy).$$

Example 4.3.6. *Suppose $w = xyz$ then $dw = yzdx + xzdy + xydz$. On the other hand, we can solve for $z = z(x, y, w)$*

$$z = \frac{w}{xy} \qquad \Rightarrow \qquad dz = -\frac{w}{x^2 y}dx - \frac{w}{xy^2}dy + \frac{1}{xy}dw. \ \star$$

If we solve $dw = yzdx + xzdy + xydz$ directly for dz we obtain:

$$dz = -\frac{z}{x}dx - \frac{z}{y}dy + \frac{1}{xy}dw \quad \star\star.$$

Are $\star$ and $\star\star$ consistent? Well, yes. Note $\frac{w}{x^2y} = \frac{xyz}{x^2y} = \frac{z}{x}$ and $\frac{w}{xy^2} = \frac{xyz}{xy^2} = \frac{z}{y}$.

Which variables are independent/dependent in the example above? It depends. In this initial portion of the example we treated x, y, z as independent whereas w was dependent. But, in the last half we treated x, y, w as independent and z was the dependent variable. Consider this, if I ask you what the value of $\frac{\partial z}{\partial x}$ is in the example above then this question is ambiguous!

$$\underbrace{\frac{\partial z}{\partial x} = 0}_{z \text{ indpendent of } x} \qquad \text{verses} \qquad \underbrace{\frac{\partial z}{\partial x} = \frac{-z}{x}}_{z \text{ depends on } x}$$

Obviously this sort of ambiguity is rather unpleasant. A natural solution to this trouble is simply to write a bit more when variables are used in multiple contexts. In particular,

$$\underbrace{\left.\frac{\partial z}{\partial x}\right|_{y,z} = 0}_{\text{means } x,y,z \text{ independent}} \qquad \text{is different than} \qquad \underbrace{\left.\frac{\partial z}{\partial x}\right|_{y,w} = \frac{-z}{x}}_{\text{means } x,y,w \text{ independent}} .$$

The key concept is that all the other independent variables are held fixed as an indpendent variable is partial differentiated. Holding y, z fixed as x varies means z does not change hence $\left.\frac{\partial z}{\partial x}\right|_{y,z} = 0$. On the other hand, if we hold y, w fixed as x varies then the change in z need not be trivial; $\left.\frac{\partial z}{\partial x}\right|_{y,w} = \frac{-z}{x}$. Let me expand on how this notation interfaces with the total differential.

Definition 4.3.7.

If w, x, y, z are variables then

$$dw = \left.\frac{\partial w}{\partial x}\right|_{y,z} dx + \left.\frac{\partial w}{\partial y}\right|_{x,z} dy + \left.\frac{\partial w}{\partial z}\right|_{x,y} dz.$$

Alternatively,

$$dx = \left.\frac{\partial x}{\partial w}\right|_{y,z} dw + \left.\frac{\partial x}{\partial y}\right|_{w,z} dy + \left.\frac{\partial x}{\partial z}\right|_{w,y} dz.$$

The larger idea here is that we can identify partial derivatives from the coefficients in equations of differentials. I'd say a differential equation but you might get the wrong idea... Incidentally, there is a whole theory of solving differential equations by clever use of differentials, I have books if you are interested.

Example 4.3.8. *Suppose $w = x + y + z$ and $x + y = wz$ then calculate $\left.\frac{\partial w}{\partial x}\right|_y$ and $\left.\frac{\partial w}{\partial x}\right|_z$. Notice we must choose dependent and independent variables to make sense of partial derivatives in question.*

1. *suppose w, z both depend on x, y. Calculate,*

$$\left.\frac{\partial w}{\partial x}\right|_y = \left.\frac{\partial}{\partial x}\right|_y (x + y + z) = \left.\frac{\partial x}{\partial x}\right|_y + \left.\frac{\partial y}{\partial x}\right|_y + \left.\frac{\partial z}{\partial x}\right|_y = 1 + 0 + \left.\frac{\partial z}{\partial x}\right|_y \quad \star$$

To calculate further we need to eliminate w by substituting $w = x + y + z$ into $x + y = wz$; thus $x + y = (x + y + z)z$ hence $dx + dy = (dx + dy + dz)z + (x + y + z)dz$

$$(2z + x + y)dz = (1 - z)dx + (1 - z)dy \quad \star\star$$

Therefore,

$$dz = \frac{1 - z}{2z + x + y}dx + \frac{1 - z}{2z + x + y}dy = \left.\frac{\partial z}{\partial x}\right|_y dx + \left.\frac{\partial z}{\partial y}\right|_x dy \quad \Rightarrow \quad \left.\frac{\partial z}{\partial x}\right|_y = \frac{1 - z}{2z + x + y}.$$

Returning to $\star$ we derive

$$\boxed{\left.\frac{\partial w}{\partial x}\right|_y = 1 + \frac{1 - z}{2z + x + y}.}$$

2. *suppose w, y both depend on x, z. Calculate,*

$$\left.\frac{\partial w}{\partial x}\right|_z = \left.\frac{\partial}{\partial x}\right|_z (x + y + z) = \left.\frac{\partial x}{\partial x}\right|_z + \left.\frac{\partial y}{\partial x}\right|_z + \left.\frac{\partial z}{\partial x}\right|_z = 1 + \left.\frac{\partial y}{\partial x}\right|_z + 0$$

To complete this calculation we need to eliminate w as before, using $\star\star$,

$$(1 - z)dy = (1 - z)dx - (2z + x + y)dz \quad \Rightarrow \quad \left.\frac{\partial y}{\partial x}\right|_z = 1.$$

Therefore,

$$\boxed{\left.\frac{\partial w}{\partial x}\right|_z = 2.}$$

I hope you can begin to see how the game is played. Basically the example above generalizes the idea of implicit differentiation to several equations of many variables. This is actually a pretty important type of calculation for engineering. The study of thermodynamics is full of variables which are intermittently used as either dependent or independent variables. The so-called equation of state can be given in terms of about a dozen distinct sets of state variables.

Example 4.3.9. *The ideal gas law states that for a fixed number of particles n the pressure P, volume V and temperature T are related by $PV = nRT$ where R is a constant. Calculate,*

$$\left.\frac{\partial P}{\partial V}\right|_T = \frac{\partial}{\partial V}\left[\frac{nRT}{V}\right]\Big|_T = -\frac{nRT}{V^2},$$

$$\left.\frac{\partial V}{\partial T}\right|_P = \frac{\partial}{\partial T}\left[\frac{nRT}{P}\right]\Big|_T = \frac{nR}{P},$$

$$\left.\frac{\partial T}{\partial P}\right|_V = \frac{\partial}{\partial P}\left[\frac{PV}{nR}\right]\Big|_T = \frac{V}{nR}.$$

You might expect that $\left.\frac{\partial P}{\partial V}\right|_T \left.\frac{\partial V}{\partial T}\right|_P \left.\frac{\partial T}{\partial P}\right|_V = 1$. Is it true?

$$\left.\frac{\partial P}{\partial V}\right|_T \left.\frac{\partial V}{\partial T}\right|_P \left.\frac{\partial T}{\partial P}\right|_V = -\frac{nRT}{V^2}\cdot\frac{nR}{P}\cdot\frac{V}{nR} = \frac{-nRT}{PV} = -1.$$

This is an example where naive cancellation of partials fails.

Example 4.3.10. *Suppose $F(x,y) = 0$ then $dF = F_x dx + F_y dy = 0$ and it follows that $dx = -\frac{F_y}{F_x}dy$ or $dy = -\frac{F_x}{F_y}dx$. Hence, $\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$ and $\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}$. Therefore,*

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = \frac{F_y}{F_x}\cdot\frac{F_x}{F_y} = 1$$

for (x,y) such that $F_x \neq 0$ and $F_y \neq 0$. The condition $F_x \neq 0$ suggests we can solve for $y = y(x)$ whereas the condition $F_y \neq 0$ suggests we can solve for $x = x(y)$.

4.4 the constant rank theorem

The implicit function theorem required we work with independent constraints. However, one does not always have that luxury. There is a theorem which deals with the slightly more general case. The base idea is that if the Jacobian has rank k then it locally injects a k-dimensional image into the codomain. If we are using a map as a parametrization then the rank k condition suggests the mapping does parametrize a k-fold, at least locally. On the other hand, if we are using the map to

define a space as a level set then $F : \mathbb{R}^n \to \mathbb{R}^p$ has $F^{-1}(C)$ as a $(n-k)$-fold. Previously, we would have insisted $k = p$. I've run out of time for 2013 notes so sadly I have no reference for this claim. However, the troubing Section 10.7.2 quotes the Theorem we desire in a somewhat unfortunate language for our current purposes. In any event, theorems aside, I think the red comments are worth some discussion.

Remark 4.4.1.

I have put remarks about the rank of the derivative in red for the examples below.

Example 4.4.2. *Let $f(t) = (t, t^2, t^3)$ then $f'(t) = (1, 2t, 3t^2)$. In this case we have*

$$f'(t) = [df_t] = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}$$

The Jacobian here is a single column vector. It has rank 1 provided the vector is nonzero. We see that $f'(t) \neq (0,0,0)$ for all $t \in \mathbb{R}$. This corresponds to the fact that this space curve has a well-defined tangent line for each point on the path.

Example 4.4.3. *Let $f(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$ be a mapping from $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. I'll denote the coordinates in the domain by $(x_1, x_2, x_3, y_1, y_2, y_3)$ thus $f(\vec{x}, \vec{y}) = x_1y_1 + x_2y_2 + x_3y_3$. Calculate,*

$$[df_{(\vec{x},\vec{y})}] = \nabla f(\vec{x}, \vec{y})^T = [y_1, y_2, y_3, x_1, x_2, x_3]$$

The Jacobian here is a single row vector. It has rank 6 provided all entries of the input vectors are nonzero.

Example 4.4.4. *Let $f(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y}$ be a mapping from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. I'll denote the coordinates in the domain by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ thus $f(\vec{x}, \vec{y}) = \sum_{i=1}^n x_iy_i$. Calculate,*

$$\frac{\partial}{x_j}\left[\sum_{i=1}^n x_iy_i\right] = \sum_{i=1}^n \frac{\partial x_i}{x_j} y_i = \sum_{i=1}^n \delta_{ij}y_i = y_j$$

Likewise,

$$\frac{\partial}{y_j}\left[\sum_{i=1}^n x_iy_i\right] = \sum_{i=1}^n x_i \frac{\partial y_i}{y_j} = \sum_{i=1}^n x_i\delta_{ij} = x_j$$

Therefore, noting that $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_n} f, \partial_{y_1} f, \ldots, \partial_{y_n} f)$,

$$[df_{(\vec{x},\vec{y})}]^T = (\nabla f)(\vec{x}, \vec{y}) = \vec{y} \times \vec{x} = (y_1, \ldots, y_n, x_1, \ldots, x_n)$$

The Jacobian here is a single row vector. It has rank 2n provided all entries of the input vectors are nonzero.

Example 4.4.5. *Suppose* $F(x,y,z) = (xyz, y, z)$ *we calculate,*

$$\frac{\partial F}{\partial x} = (yz, 0, 0) \qquad \frac{\partial F}{\partial y} = (xz, 1, 0) \qquad \frac{\partial F}{\partial z} = (xy, 0, 1)$$

Remember these are actually column vectors in my sneaky notation; $(v_1, \ldots, v_n) = [v_1, \ldots, v_n]^T$. *This means the* **derivative** *or* **Jacobian matrix** *of* F *at* (x,y,z) *is*

$$F'(x,y,z) = [dF_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note, $rank(F'(x,y,z)) = 3$ *for all* $(x,y,z) \in \mathbb{R}^3$ *such that* $y, z \neq 0$. *There are a variety of ways to see that claim, one way is to observe* $det[F'(x,y,z)] = yz$ *and this determinant is nonzero so long as neither* y *nor* z *is zero. In linear algebra we learn that a square matrix is invertible iff it has nonzero determinant iff it has linearly indpendent column vectors.*

Example 4.4.6. *Suppose* $F(x,y,z) = (x^2 + z^2, yz)$ *we calculate,*

$$\frac{\partial F}{\partial x} = (2x, 0) \qquad \frac{\partial F}{\partial y} = (0, z) \qquad \frac{\partial F}{\partial z} = (2z, y)$$

The derivative is a 2×3 *matrix in this example,*

$$F'(x,y,z) = [dF_{(x,y,z)}] = \begin{bmatrix} 2x & 0 & 2z \\ 0 & z & y \end{bmatrix}$$

The maximum rank for F' *is* 2 *at a particular point* (x,y,z) *because there are at most two linearly independent vectors in* $\mathbb{R}^2$. *You can consider the three square submatrices to analyze the rank for a given point. If any one of these is nonzero then the rank (dimension of the column space) is two.*

$$M_1 = \begin{bmatrix} 2x & 0 \\ 0 & z \end{bmatrix} \qquad M_2 = \begin{bmatrix} 2x & 2z \\ 0 & y \end{bmatrix} \qquad M_3 = \begin{bmatrix} 0 & 2z \\ z & y \end{bmatrix}$$

We'll need either $det(M_1) = 2xz \neq 0$ *or* $det(M_2) = 2xy \neq 0$ *or* $det(M_3) = -2z^2 \neq 0$. *I believe the only point where all three of these fail to be true simulataneously is when* $x = y = z = 0$. *This mapping has maximal rank at all points except the origin.*

Example 4.4.7. *Suppose* $F(x,y) = (x^2 + y^2, xy, x + y)$ *we calculate,*

$$\frac{\partial F}{\partial x} = (2x, y, 1) \qquad \frac{\partial F}{\partial y} = (2y, x, 1)$$

The derivative is a 3×2 *matrix in this example,*

$$F'(x,y) = [dF_{(x,y)}] = \begin{bmatrix} 2x & 2y \\ y & x \\ 1 & 1 \end{bmatrix}$$

The maximum rank is again 2, this time because we only have two columns. The rank will be two if the columns are not linearly dependent. We can analyze the question of rank a number of ways but I find determinants of submatrices a comforting tool in these sort of questions. If the columns are linearly dependent then all three sub-square-matrices of F' will be zero. Conversely, if even one of them is nonvanishing then it follows the columns must be linearly independent. The submatrices for this problem are:

$$M_1 = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix} \qquad M_2 = \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix} \qquad M_3 = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$$

You can see $det(M_1) = 2(x^2 - y^2)$, $det(M_2) = 2(x - y)$ and $det(M_3) = y - x$. Apparently we have $rank(F'(x, y, z)) = 2$ for all $(x, y) \in \mathbb{R}^2$ with $y \neq x$. In retrospect this is not surprising.

Example 4.4.8. *Let $F(x, y) = (x, y, \sqrt{R^2 - x^2 - y^2})$ for a constant R. We calculate,*

$$\nabla\sqrt{R^2 - x^2 - y^2} = \left(\frac{-x}{\sqrt{R^2-x^2-y^2}}, \frac{-y}{\sqrt{R^2-x^2-y^2}} \right)$$

Also, $\nabla x = (1, 0)$ and $\nabla y = (0, 1)$ thus

$$F'(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{R^2-x^2-y^2}} & \frac{-y}{\sqrt{R^2-x^2-y^2}} \end{bmatrix}$$

This matrix clearly has rank 2 where is is well-defined. Note that we need $R^2 - x^2 - y^2 > 0$ for the derivative to exist. Moreover, we could define $G(y, z) = (\sqrt{R^2 - y^2 - z^2}, y, z)$ and calculate,

$$G'(y, z) = \begin{bmatrix} 1 & 0 \\ \frac{-y}{\sqrt{R^2-y^2-z^2}} & \frac{-z}{\sqrt{R^2-y^2-z^2}} \\ 0 & 1 \end{bmatrix}.$$

Observe that $G'(y, z)$ exists when $R^2 - y^2 - z^2 > 0$. Geometrically, F parametrizes the sphere above the equator at $z = 0$ whereas G parametrizes the right-half of the sphere with $x > 0$. These parametrizations overlap in the first octant where both x and z are positive. In particular, $dom(F') \cap dom(G') = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0 \text{ and } x^2 + y^2 < R^2\}$

Example 4.4.9. *Let $F(x, y, z) = (x, y, z, \sqrt{R^2 - x^2 - y^2 - z^2})$ for a constant R. We calculate,*

$$\nabla\sqrt{R^2 - x^2 - y^2 - z^2} = \left(\frac{-x}{\sqrt{R^2-x^2-y^2-z^2}}, \frac{-y}{\sqrt{R^2-x^2-y^2-z^2}}, \frac{-z}{\sqrt{R^2-x^2-y^2-z^2}} \right)$$

Also, $\nabla x = (1, 0, 0)$, $\nabla y = (0, 1, 0)$ and $\nabla z = (0, 0, 1)$ thus

$$F'(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-x}{\sqrt{R^2-x^2-y^2-z^2}} & \frac{-y}{\sqrt{R^2-x^2-y^2-z^2}} & \frac{-z}{\sqrt{R^2-x^2-y^2-z^2}} \end{bmatrix}$$

This matrix clearly has rank 3 where is is well-defined. Note that we need $R^2-x^2-y^2-z^2 > 0$ for the derivative to exist. This mapping gives us a parametrization of the 3-sphere $x^2+y^2+z^2+w^2 = R^2$ for $w > 0$. (drawing this is a little trickier)

Example 4.4.10. *Let $f(x,y,z) = (x+y, y+z, x+z, xyz)$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ yz & xz & xy \end{bmatrix}$$

This matrix clearly has rank 3 and is well-defined for all of $\mathbb{R}^3$.

Example 4.4.11. *Let $f(x,y,z) = xyz$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \end{bmatrix}$$

This matrix fails to have rank 3 if x, y or z are zero. In other words, $f'(x,y,z)$ has rank 3 in $\mathbb{R}^3$ provided we are at a point which is not on some coordinate plane. (the coordinate planes are $x = 0, y = 0$ and $z = 0$ for the yz, zx and xy coordinate planes respective)

Example 4.4.12. *Let $f(x,y,z) = (xyz, 1-x-y)$. You can calculate,*

$$[df_{(x,y,z)}] = \begin{bmatrix} yz & xz & xy \\ -1 & -1 & 0 \end{bmatrix}$$

This matrix has rank 3 if either $xy \neq 0$ or $(x-y)z \neq 0$. In contrast to the preceding example, the derivative does have rank 3 on certain points of the coordinate planes. For example, $f'(1,1,0)$ and $f'(0,1,1)$ both give $rank(f') = 3$.

Example 4.4.13. *Let $X(u,v) = (x,y,z)$ where x, y, z denote functions of u, v and I prefer to omit the explicit depedendence to reduce clutter in the equations to follow.*

$$\frac{\partial X}{\partial u} = X_u = (x_u, y_u, z_u) \quad and \quad \frac{\partial X}{\partial v} = X_v = (x_v, y_v, z_v)$$

Then the Jacobian is the 3×2 matrix

$$\left[dX_{(u,v)}\right] = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$$

The matrix $\left[dX_{(u,v)}\right]$ has rank 2 if at least one of the determinants below is nonzero,

$$det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \quad det \begin{bmatrix} x_u & x_v \\ z_u & z_v \end{bmatrix} \quad det \begin{bmatrix} y_u & y_v \\ z_u & z_v \end{bmatrix}$$

Chapter 5

two views of manifolds in $\mathbb{R}^n$

In this chapter we describe spaces inside $\mathbb{R}^n$ which are k-dimensional [1]. Technically, to make this precise we would need to study manifolds with boundary. Careful discussion of manifolds with boundary in euclidean space can be found in Munkres *Analysis on Manifolds*. In the interest of focusing on examples, I'll be a bit fuzzy about the defintion of a k-dimensional subspace S of euclidean space. This much we can say: there are two ways to envision the geometry of S:

(1.) **Parametrically**: provide a **patch** R such that $R : U \subseteq \mathbb{R}^k \to S \subseteq \mathbb{R}^n$. Here U is called the **parameter space** and R^{-1} is called a **coordinate chart**. The cannonical example:

$$R(x_1, \dots x_k) = (x_1, \dots x_k, 0, \dots, 0).$$

(2.) **Implicitly**: provide a **level function** $G : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$ such that $S = G^{-1}\{c\} = S$. This viewpoint casts S as points in $x \in \mathbb{R}^k \times \mathbb{R}^p$ for which $G(x) = k$. The cannonical example:

$$G(x_1, \dots, x_{k+p}) = (x_{k+1}, \dots, x_{k+p}) = (0, \dots, 0).$$

The cannonical examples of (1.) and (2.) are both the $x_1 \dots x_k$-coordinate plane embedded in $\mathbb{R}^n$. Just to take it down a notch. If $n = 3$ then we could look at the xy-plane in either view as follows:

$$(1.)\ R(x, y) = (x, y, 0) \qquad (2.)\ G(x, y, z) = z = 0.$$

Which viewpoint should we adopt? What is the dimension of a given space S? How should we find tangent space to S? How should we find the normal space to S? These are the questions we set-out to answer in this chapter.

Orthogonal complements help us to understand how all of this fits together. This is possible since we deal with embedded manifolds for which the euclidean dot-product of $\mathbb{R}^n$ is available to sort out the geometry. Finally, we use this geometry and a few simple lemmas to justify the method of Lagrange multipliers. Lagrange's technique paired with the theory of multivariate Taylor polynomials form

[1] I'll try to stick with this notation for this chapter, $n \geq k$ and $n = p + k$

the basis for analyzing extrema for multivariate functions. In this chapter we deal with the question of extrema on the edges of a set. The second half of the story is found in the next chapter where we deal with the interior points via the theory of quadratic forms applied to the second-order approximation to a function of several variables.

5.1 definition of level set

A level set is the solution set of some equation or system of equations. We confine our interest to level sets of $\mathbb{R}^n$. For example, the set of all (x, y) that satisfy

$$G(x, y) = c$$

is called a **level curve** in $\mathbb{R}^2$. Often we can use k to label the curve. You should also recall **level surfaces** in $\mathbb{R}^3$ are defined by an equation of the form

$$G(x, y, z) = c.$$

The set of all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ which solve $G(x_1, x_2, x_3, x_4) = c$ is a **level volume** in $\mathbb{R}^4$. We can obtain lower dimensional objects by simultaneously imposing several equations at once. For example, suppose $G_1(x, y, z) = z = 1$ and $G_2(x, y, z) = x^2 + y^2 + z^2 = 5$, points (x, y, z) which solve both of these equations are on the intersection of the plane $z = 1$ and the sphere $x^2 + y^2 + z^2 = 5$. Let $G = (G_1, G_2)$, note that $G(x, y, z) = (1, 5)$ describes a circle in $\mathbb{R}^3$. More generally:

Definition 5.1.1.

Suppose $G : dom(G) \subseteq \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$. Let c be a vector of constants in $\mathbb{R}^p$ and suppose $S = \{x \in \mathbb{R}^k \times \mathbb{R}^p \mid G(x) = c\}$ is non-empty and G is continuously differentiable on an open set containing S. We say S is an **k-dimensional level** set iff $G'(x)$ has p linearly independent rows at each $x \in S$.

The condition of linear independence of the rows is give to eliminate possible redundancy in the system of equations. In the case that $p = 1$ the criteria reduces to $G'(x) \neq 0$ over the level set of dimension $n - 1$. Intuitively we think of each equation in $G(x) = c$ as removing one of the dimensions of the ambient space $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^p$. It is worthwhile to cite a useful result from linear algebra at this point:

Proposition 5.1.2.

Let $A \in \mathbb{R}^{m \times n}$. The number of linearly independent columns in A is the same as the number of linearly independent rows in A. This invariant of A is called the **rank** of A.

Given the wisdom of linear algebra we see that we should require a k-dimensional level set $S = G^{-1}(c)$ to have a level function $G : \mathbb{R}^n \to \mathbb{R}^p$ whose derivative is of rank $n - k = p$ over all of S. We can either analyze linear independence of columns or rows.

Example 5.1.3. *Consider $G(x,y,z) = x^2 + y^2 - z^2$ and suppose $S = G^{-1}\{0\}$. Calculate,*

$$G'(x,y,z) = [2x, 2y, -2z]$$

Notice that $(0,0,0) \in S$ and $G'(0,0,0) = [0,0,0]$ hence G' is not rank one at the origin. At all other points in S we have $G'(x,y,z) \neq 0$ which means this is almost a $3-1 = 2$-dimensional level set. However, almost is not good enough in math. Under our definition the cone S is not a 2-dimensional level set since it fails to meet the full-rank criteria at the point of the cone.

A p-dimensional level set is an example of a p-dimensional manifold. The example above with the origin included is a manifold paired with a singular point, such spaces are known as **orbifolds**. The study of orbifolds has attracted considerable effort in recent years as the singularities of such orbifolds can be used to do physics in string theory. I digress. Let us examine another level set:

Example 5.1.4. *Let $G(x,y,z) = (x,y)$ and define $S = G^{-1}(a,b)$ for some fixed pair of constants $a, b \in \mathbb{R}$. We calculate that $G'(x,y,z) = I_2 \in \mathbb{R}^{2\times 2}$. We clearly have rank two at all points in S hence S is a $3-2 = 1$-dimensional level set. Perhaps you realize S is the vertical line which passes through $(a,b,0)$ in the xy-plane.*

5.2 tangents and normals to a level set

There are many ways to define a tangent space for some subset of $\mathbb{R}^n$. One natural definition is that the tangent space to $p \in S$ is simply the set of all tangent vectors to curves on S which pass through the point p. In this section we study the geometry of curves on a level-set. We'll see how the tangent space is naturally a vector space in the particular context of level-sets in $\mathbb{R}^n$.

Throughout this section we assume that S is a k-dimensional level set defined by $G : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$ where $G^{-1}(c) = S$. This means that we can apply the implicit function theorem to S and for any given point $p = (p_x, p_y) \in S$ where $p_x \in \mathbb{R}^k$ and $p_y \in \mathbb{R}^p$. There exists a local continuously differentiable solution $h : U \subseteq \mathbb{R}^k \to \mathbb{R}^p$ such that $h(p_x) = p_y$ and for all $x \in U$ we have $G(x, h(x)) = c$. We can view $G(x,y) = c$ for x near p as the graph of $y = h(x)$ for $x \in U$. With the set-up above in mind, suppose that $\gamma : \mathbb{R} \to U \subseteq S$. If we write $\gamma = (\gamma_x, \gamma_y)$ then it follows $\gamma = (\gamma_x, h \circ \gamma_x)$ over the subset $U \times h(U)$ of S. More explicitly, for all $t \in \mathbb{R}$ such that $\gamma(t) \in U \times h(U)$ we have

$$\gamma(t) = (\gamma_x(t), h(\gamma_x(t))).$$

Therefore, if $\gamma(0) = p$ then $\gamma(0) = (p_x, h(p_x))$. Differentiate, use the chain-rule in the second factor to obtain:

$$\gamma'(t) = (\gamma_x'(t), h'(\gamma_x(t))\gamma_x'(t)).$$

We find that the tangent vector to $p \in S$ of γ has a rather special form which was forced on us by the implicit function theorem:

$$\gamma'(0) = (\gamma_x'(0), h'(p_x)\gamma_x'(0)).$$

Or to cut through the notation a bit, if $\gamma'(0) = v = (v_x, v_y)$ then $v = (v_x, h'(p_x)v_x)$. The second component of the vector is not free of the first, it essentially redundant. This makes us suspect that the tangent space to S at p is k-dimensional.

Theorem 5.2.1.

Let $G : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$ be a level-mappping which defines a k-dimensional level set S by $G^{-1}(c) = S$. Suppose $\gamma_1, \gamma_2 : \mathbb{R} \to S$ are differentiable curves with $\gamma_1'(0) = v_1$ and $\gamma_2'(0) = v_2$ then there exists a differentiable curve $\gamma : \mathbb{R} \to S$ such that $\gamma'(0) = v_1 + v_2$ and $\gamma(0) = p$. Moreover, there exists a differentiable curve $\beta : \mathbb{R} \to S$ such that $\beta'(0) = cv_1$ and $\beta(0) = p$.

Proof: It is convenient to define a map which gives a **local parametrization** of S at p. Since we have a description of S locally as a graph $y = h(x)$ (near p) it is simple to construct the parameterization. Define $\Phi : U \subseteq \mathbb{R}^k \to S$ by $\Phi(x) = (x, h(x))$. Clearly $\Phi(U) = U \times h(U)$ and there is an inverse mapping $\Phi^{-1}(x, y) = x$ is well-defined since $y = h(x)$ for each $(x, y) \in U \times h(U)$. Let $w \in \mathbb{R}^k$ and observe that

$$\psi(t) = \Phi(\Phi^{-1}(p) + tw) = \Phi(p_x + tw) = (p_x + tw, h(p_x + tw))$$

is a curve from $\mathbb{R}$ to $U \subseteq S$ such that $\psi(0) = (p_x, h(p_x)) = (p_x, p_y) = p$ and using the chain rule on the final form of $\psi(t)$:

$$\psi'(0) = (w, h'(p_x)w).$$

The construction above shows that any vector of the form $(v_x, h'(p_x)v_x)$ is the tangent vector of a particular differentiable curve in the level set (differentiability of ψ follows from the differentiability of h and the other maps which we used to construct ψ). In particular we can apply this to the case $w = v_{1x} + v_{2x}$ and we find $\gamma(t) = \Phi(\Phi^{-1}(p) + t(v_{1x} + v_{2x}))$ has $\gamma'(0) = v_1 + v_2$ and $\gamma(0) = p$. Likewise, apply the construction to the case $w = cv_{1x}$ to write $\beta(t) = \Phi(\Phi^{-1}(p) + t(cv_{1x}))$ with $\beta'(0) = cv_1$ and $\beta(0) = p$. □

The idea of the proof is encapsulated in the picture below. This idea of mapping lines in a flat domain to obtain standard curves in a curved domain is an idea which plays over and over as you study manifold theory. The particular redundancy of the x and y sub-vectors is special to the discussion level-sets, however anytime we have a local parametrization we'll be able to construct curves with tangents of our choosing by essentially the same construction. In fact, there are infinitely many curves which produce a particular tangent vector in the tangent space of a manifold.

picture.

Theorem 5.2.1 shows that the definition given below is logical. In particular, it is not at all obvious that the sum of two tangent vectors ought to again be a tangent vector. However, that is just what the Theorem 5.2.1 told us for level-sets[2].

[2]technically, there is another logical gap which I currently ignore. I wonder if you can find it.

Definition 5.2.2.

Suppose S is a k-dimensional level-set defined by $S = G^{-1}\{c\}$ for $G : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$. We define the **tangent space at** $p \in S$ to be the set of pairs:

$$T_pS = \{(p, v) \mid \text{there exists differentiable } \gamma : \mathbb{R} \to S \text{ and } \gamma(0) = p \text{ where } v = \gamma'(0)\}$$

Moreover, we define (i.) **addition** and (ii.) **scalar multiplication** of vectors by the rules

$$(i.)\ \ (p, v_1) + (p, v_2) = (p, v_1 + v_2) \qquad (ii.)\ \ c(p, v_1) = (p, cv_1)$$

for all $(p, v_1), (p, v_2) \in T_pS$ and $c \in \mathbb{R}$.

When I picture T_pS in my mind I think of vectors pointing out from the base-point p. To make an explicit connection between the pairs of the above definition and the classical geometric form of the tangent space we simply take the image of T_pS under the mapping $\Psi(x, y) = x + y$ thus $\Psi(T_pS) = \{p + v \mid (p, v) \in T_pS\}$. I often picture T_pS as $\psi(T_pS)$[3]

We could set out to calculate tangent spaces in view of the definition above, but we are actually interested in more than just the tangent space for a level-set. In particular. we want a concrete description of all the vectors which are not in the tangent space.

Definition 5.2.3.

Suppose S is a k-dimensional level-set defined by $S = G^{-1}\{c\}$ for $G : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$ and T_pS is the tangent space at p. Note that $T_pS \leq V_p$ where $V_p = \{p\} \times \mathbb{R}^k \times \mathbb{R}^p$ is given the natural vector space structure which we already exhibited on the subspace T_pS. We define the **inner product** on V_p as follows: for all $(p, v), (p, w) \in V_p$,

$$(p, v) \cdot (p, w) = v \cdot w.$$

The length of a vector (p, v) is naturally defined by $||(p, v)|| = ||v||$. Moreover, we say two vectors $(p, v), (p, w) \in V_p$ are **orthogonal** iff $v \cdot w = 0$. Given a set of vectors $R \subseteq V_p$ we define the **orthogonal complement** by

$$R^{\perp} = \{(p, v) \in V_p \mid (p, v) \cdot (p, r) \ \text{ for all } (p, r) \in R\}.$$

Suppose $W_1, W_2 \subseteq V_p$ then we say W_1 is **orthogonal** to W_2 iff $w_1 \cdot w_2 = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. We denote orthogonality by writing $W_1 \perp W_2$. If every $v \in V_p$ can be written as $v = w_1 + w_2$ for a pair of $w_1 \in W_1$ and $w_2 \in W_2$ where $W_1 \perp W_2$ then we say that V_p is the **direct sum** of W_1 and W_2 which is denoted by $V_p = W_1 \oplus W_2$.

There is much more to say about orthogonality, however, our focus is not in that vein. We just need the langauge to properly define the normal space. The calculation below is probably the most

[3]In truth, as you continue to study manifold theory you'll find at least three seemingly distinct objects which are all called "tangent vectors"; equivalence classes of curves, derivations, contravariant tensors.

important calculation to understand for a level-set. Suppose we have a curve $\gamma : \mathbb{R} \to S$ where $S = G^{-1}(c)$ is a k-dimensional level-set in $\mathbb{R}^k \times \mathbb{R}^p$. Observe that for all $t \in \mathbb{R}$,

$$G(\gamma(t)) = c \quad \Rightarrow \quad G'(\gamma(t))\gamma'(t) = 0.$$

In particular, suppose for $t = 0$ we have $\gamma(0) = p$ and $v = \gamma'(0)$ which makes $(p, v) \in T_pS$ with

$$G'(p)v = 0.$$

Recall $G : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$ has an $p \times n$ derivative matrix where the j-th row is the gradient vector of the j-th component function. The equation $G'(p)v = 0$ gives us p-independent equations as we examine it componentwise. In particular, it reveals that (p, v) is orthogonal to $\nabla G_j(p)$ for $j = 1, 2, \ldots, p$. We have derived the following theorem:

Theorem 5.2.4.

Let $G : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$ be a level-mappping which defines a k-dimensional level set S by $G^{-1}(c) = S$. The gradient vectors $\nabla G_j(p)$ are perpendicular to the tangent space at p; for each $j \in \mathbb{N}_p$

$$(p, \nabla(G_j(p))^T) \in (T_pS)^\perp.$$

It's time to do some counting. Observe that the mapping $\phi : \mathbb{R}^k \to T_pS$ defined by $\phi(v) = (p, v)$ is an isomorphism of vector spaces hence $dim(T_pS) = k$. But, by the same isomorphism we can see that $V_p = \phi(\mathbb{R}^k \times \mathbb{R}^p)$ hence $dim(V_p) = p + k$. In linear algebra we learn that if we have a k-dimensional subspace W of an n-dimensional vector space V then the orthogonal complement $W^\perp$ is a subspace of V with **codimension** k. The term **codimension** is used to indicate a loss of dimension from the ambient space, in particular $dim(W^\perp) = n - k$. We should note that the direct sum of W and $W^\perp$ covers the whole space; $W \oplus W^\perp = V$. In the case of the tangent space, the codimension of $T_pS \leq V_p$ is found to be $p + k - k = p$. Thus $dim(T_pS)^\perp = p$. Any basis for this space must consist of p linearly independent vectors which are all orthogonal to the tangent space. Naturally, the subset of vectors $\{(p, (\nabla G_j(p))^T)_{j=1}^p$ forms just such a basis since it is given to be linearly independent by the $rank(G'(p)) = p$ condition. It follows that:

$$\boxed{(T_pS)^\perp \approx Row(G'(p))}$$

where equality can be obtained by the slightly tedious equation $\boxed{(T_pS)^\perp = \phi(Col(G'(p)^T))}$. That equation simply does the following:

1. transpose $G'(p)$ to swap rows to columns
2. construct column space by taking span of columns in $G'(p)^T$
3. adjoin p to make pairs of vectors which live in V_p

many wiser authors wouldn't bother. The comments above are primarily about notation. Certainly hiding these details would make this section prettier, however, would it make it better? Finally, I once more refer the reader to linear algebra where we learn that $(Row(A))^\perp = Null(A^T)$. Let me walk you through the proof: let $A \in \mathbb{R}^{m\times n}$. Observe $v \in Null(A^T)$ iff $A^T v = 0$ for $v \in \mathbb{R}^m$ iff $v^T A = 0$ iff $v^T col_j(A) = 0$ for $j = 1, 2, \ldots, n$ iff $v \cdot col_j(A) = 0$ for $j = 1, 2, \ldots, n$ iff $v \in Col(A)^\perp$. Another useful identity for the "perp" is that $(A^\perp)^\perp = A$. With those two gems in mind consider that:

$$(T_pS)^\perp \approx Row(G'(p)) \quad \Rightarrow \quad T_pS \approx Row(G'(p))^\perp = Null(G'(p)^T)$$

Let me once more replace $\approx$ by a more tedious, but explicit, procedure:

$$\boxed{T_pS = \phi(Null(G'(p)^T))}$$

Theorem 5.2.5.

Let $G : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$ be a level-mappping which defines a k-dimensional level set S by $G^{-1}(c) = S$. The **tangent space** T_pS and the **normal space** N_pS at $p \in S$ are given by

$$T_pS = \{p\} \times Null(G'(p)^T) \qquad \& \qquad N_pS = \{p\} \times Col(G'(p)^T).$$

Moreover, $V_p = T_pS \oplus N_pS$. Every vector can be uniquely written as the sum of a tangent vector and a normal vector.

The fact that there are only tangents and normals is the key to the method of Lagrange multipliers. It forces two seemingly distinct objects to be in the same direction as one another.

Example 5.2.6. *Let $g : \mathbb{R}^4 \to \mathbb{R}$ be defined by $g(x, y, z, t) = t+x^2+y^2-2z^2$ note that $g(x, y, z, t) = 0$ gives a three dimensional subset of $\mathbb{R}^4$, let's call it M. Notice $\nabla g =< 2x, 2y, -4z, 1 >$ is nonzero everywhere. Let's focus on the point $(2, 2, 1, 0)$ note that $g(2, 2, 1, 0) = 0$ thus the point is on M. The tangent plane at $(2, 2, 1, 0)$ is formed from the union of all tangent vectors to $g = 0$ at the point $(2, 2, 1, 0)$. To find the equation of the tangent plane we suppose $\gamma : \mathbb{R} \to M$ is a curve with $\gamma' \neq 0$ and $\gamma(0) = (2, 2, 1, 0)$. By assumption $g(\gamma(s)) = 0$ since $\gamma(s) \in M$ for all $s \in \mathbb{R}$. Define $\gamma'(0) =< a, b, c, d >$, we find a condition from the chain-rule applied to $g \circ \gamma = 0$ at $s = 0$,*

$$\begin{aligned} \frac{d}{ds}\big(\, g\circ\gamma(s)\,\big) = \big(\nabla g\big)(\gamma(s))\cdot\gamma'(s) = 0 \qquad &\Rightarrow \qquad \nabla g(2,2,1,0)\cdot < a,b,c,d >= 0 \\ &\Rightarrow \qquad < 4,4,-4,1 > \cdot < a,b,c,d >= 0 \\ &\Rightarrow \qquad 4a + 4b - 4c + d = 0 \end{aligned}$$

Thus the equation of the tangent plane is $4(x - 2) + 4(y - 2) - 4(z - 1) + t = 0$. In invite the reader to find a vector in the tangent plane and check it is orthogonal to $\nabla g(2, 2, 1, 0)$. However, this should not be surprising, the condition the chain rule just gave us is just the statement that $< a, b, c, d >\in Null(\nabla g(2, 2, 1, 0)^T)$ and that is precisely the set of vector orthogonal to $\nabla g(2, 2, 1, 0)$.

Example 5.2.7. *Let $G:\mathbb{R}^4\to\mathbb{R}^2$ be defined by $G(x,y,z,t)=(z+x^2+y^2-2, z+y^2+t^2-2)$. In this case $G(x,y,z,t)=(0,0)$ gives a two-dimensional manifold in $\mathbb{R}^4$ let's call it M. Notice that $G_1=0$ gives $z+x^2+y^2=2$ and $G_2=0$ gives $z+y^2+t^2=2$ thus $G=0$ gives the intersection of both of these three dimensional manifolds in $\mathbb{R}^4$ (no I can't "see" it either). Note,*

$$\nabla G_1 =< 2x, 2y, 1, 0 > \qquad \nabla G_2 =< 0, 2y, 1, 2t >$$

It turns out that the inverse mapping theorem says $G=0$ describes a manifold of dimension 2 if the gradient vectors above form a linearly independent set of vectors. For the example considered here the gradient vectors are linearly dependent at the origin since $\nabla G_1(0)=\nabla G_2(0)=(0,0,1,0)$. In fact, these gradient vectors are colinear along along the plane $x=t=0$ since $\nabla G_1(0,y,z,0)=\nabla G_2(0,y,z,0) =< 0,2y,1,0 >$. We again seek to contrast the tangent plane and its normal at some particular point. Choose $(1,1,0,1)$ which is in M since $G(1,1,0,1)=(0+1+1-2, 0+1+1-2)=(0,0)$. Suppose that $\gamma:\mathbb{R}\to M$ is a path in M which has $\gamma(0)=(1,1,0,1)$ whereas $\gamma'(0) =< a,b,c,d >$. Note that $\nabla G_1(1,1,0,1) =< 2,2,1,0 >$ and $\nabla G_2(1,1,0,1) =< 0,2,1,1 >$. Applying the chain rule to both G_1 and G_2 yields:

$$(G_1\circ\gamma)'(0)=\nabla G_1(\gamma(0))\cdot < a,b,c,d >= 0 \qquad \Rightarrow \qquad < 2,2,1,0 > \cdot < a,b,c,d >= 0$$
$$(G_2\circ\gamma)'(0)=\nabla G_2(\gamma(0))\cdot < a,b,c,d >= 0 \qquad \Rightarrow \qquad < 0,2,1,1 > \cdot < a,b,c,d >= 0$$

This is two equations and four unknowns, we can solve it and write the vector in terms of two free variables correspondant to the fact the tangent space is two-dimensional. Perhaps it's easier to use matrix techiques to organize the calculation:

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We calculate, $rref\begin{bmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 1/2 & 1/2 \end{bmatrix}$. It's natural to chose c,d as free variables then we can read that $a=d/2$ and $b=-c/2-d/2$ hence

$$< a,b,c,d >=< d/2, -c/2-d/2, c, d >= \tfrac{c}{2} < 0,-1,2,0 > +\tfrac{d}{2} < 1,-1,0,2 >$$

We can see a basis for the tangent space. In fact, I can give parametric equations for the tangent space as follows:

$$X(u,v)=(1,1,0,1)+u < 0,-1,2,0 > +v < 1,-1,0,2 >$$

Not surprisingly the basis vectors of the tangent space are perpendicular to the gradient vectors $\nabla G_1(1,1,0,1) =< 2,2,1,0 >$ and $\nabla G_2(1,1,0,1) =< 0,2,1,1 >$ which span the **normal plane** *N_p to the tangent plane T_p at $p=(1,1,0,1)$. We find that T_p is orthogonal to N_p. In summary $T_p^{\perp}=N_p$ and $T_p\oplus N_p=\mathbb{R}^4$. This is just a fancy way of saying that the normal and the tangent plane only intersect at zero and they together span the entire ambient space.*

5.3 tangent and normal space from patches

I use the term **parametrization** in courses more basic than this, however, perhaps the term **patch** would be better. It's certainly easier to say and in our current context has the same meaning. I suppose the term **parametrization** is used in a bit less technical sense, so it fits calculus III better. In any event, we should make a definition of patched k-dimensional surface for the sake of concrete discussion in this section.

Definition 5.3.1.

Suppose $R : dom(R) \subseteq \mathbb{R}^k \to S \subseteq \mathbb{R}^n$. We say S is an **k-dimensional patch** iff $R'(t)$ has rank k for each $t \in dom(R)$. We also call S a k-dimensional parametrized subspace of $\mathbb{R}^n$.

The condition $R'(t)$ is just a slick way to say that the k-tangent vectors to S obtained by partial differentiation with respect to $t_1, \ldots, t_k$ are linearly independent at $t = (t_1, \ldots, t_k)$. I spent considerable effort justifying the formulae for the level-set case. I believe what follows should be intuitively clear given our previous efforts. Or, if that leaves you unsatisfied then read on to the examples. It's really not that complicated. This theorem is dual to Theorem 5.2.5.

Theorem 5.3.2.

Suppose $R : dom(R) \subseteq \mathbb{R}^k \to S \subseteq \mathbb{R}^n$ defines a **k-dimensional patch** of S. The **tangent space** T_pS and the **normal space** at $p = R(t) \in S$ are given by

$$T_pS = \{p\} \times Col(R'(t)) \qquad \& \qquad N_pS = \{p\} \times Null(R'(t)^T).$$

Moreover, $V_p = T_pS \oplus N_pS$. Every vector can be uniquely written as the sum of a tangent vector and a normal vector.

Once again, the vector space structure of T_pS and N_pS is given by the addition of vectors based at p. Let us begin with a reasonably simple example.

Example 5.3.3. *Let $R : \mathbb{R}^2 \to \mathbb{R}^3$ with $R(x, y) = (x, y, xy)$ define $S \subset \mathbb{R}^3$. We calculate,*

$$R'(x, y) = [\partial_x R | \partial_y R] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{bmatrix}$$

If $p = (a, b, ab) \in S$ then $T_pS = \{(a, b, ab)\} \times span\{(1, 0, b), (0, 1, a)\}$. The normal space is found from $Null(R'(a, b)^T)$. A short calculation shows that

$$Null \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & a \end{bmatrix} = span\{(-b, -a, 1)\}$$

As a quick check, note $(1, 0, b) \bullet (-b, -a, 1) = 0$ and $(0, 1, a) \bullet (-b, -a, 1) = 0$. We conclude, for $p = (a, b, ab)$ the normal space is simply:

$$N_pS = \{(a, b, ab)\} \times span\{(-b, -a, 1)\}.$$

In the previous example, we could rightly call T_pS the tangent plane at p and N_pS the normal line through p. Moreover, we could have used three-dimensional vector analysis to find the normal line from the cross-product. However, that will not be possible in what follows:

Example 5.3.4. *Let $R : \mathbb{R}^2 \to \mathbb{R}^4$ with $R(s,t) = (s^2, t^2, t, s)$ define $S \subset \mathbb{R}^4$. We calculate,*

$$R'(s,t) = [\partial_s R | \partial_t R] = \begin{bmatrix} 2s & 0 \\ 0 & 2t \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

If $p = (1,9,3,1) \in S$ then $T_pS = \{(1,9,3,1)\} \times span\{(2,0,0,1),(0,6,3,0)\}$. The normal space is found from $Null(R'(1,3)^T)$. A short calculation shows that

$$Null \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 6 & 3 & 0 \end{bmatrix} = span\{(-1,0,0,2),(0,-3,6,0)\}$$

We conclude, for $p = (1,9,3,1)$ the normal space is simply:

$$N_pS = \{(1,9,3,1)\} \times span\{(-1,0,0,2),(0,-3,6,0)\}.$$

5.4 summary of tangent and normal spaces

Let me briefly draw together what we did thus far in this chapter: the notation below given in I is also used in $II.$ and $III.$

(I.) a set S has dimension k if

(a) $\{\partial_1 R(t), \ldots, \partial_k R(t)\}$ is pointwise linearly independent at each $t \in U$ where $R : U \to S$ is a patch.

(b) $\text{rank}(F'(x)) = p$ for all $x \in \tilde{S}$ where $\tilde{S}$ is open and contains $S = F^{-1}\{c\}$ for continuously differentiable $F : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^p$

(II.) the tangent space at x_o for the k-dimensional set S is found from:

(a) attaching the span of the vectors $\{\partial_1 R(t_o), \ldots, \partial_k R(t_o)\}$ to $x_o = R(t_o) \in S$.

(b) attaching the $Row(F'(x_o))^\perp$ to $x_o \in S$.

(III.) the normal space to a k-dimensional set S (embedded in $\mathbb{R}^n$) is found from:

(a) attaching $\{\partial_1 R(t_o), \ldots, \partial_k R(t_o)\}^\perp$ to $x_o = R(t_o)$.

(b) attaching $Row(F'(x_o))$ to $x_o \in S$.

5.5 method of Lagrange mulitpliers

Let us begin with a statement of the problem we wish to solve.

Problem: given an objective function $f : \mathbb{R}^n \to \mathbb{R}$ and continuously differentiable constraint function $G : \mathbb{R}^n \to \mathbb{R}^p$, find extreme values for the objective function f relative to the constraint $G(x) = c$.

Note that $G(x) = c$ is a vector notation for p-scalar equations. If we suppose $rank(G'(x)) = p$ then the constraint surface $G(x) = c$ will form an $(n-p)$-dimensional level set. Let us make that supposition throughout the remainder of this section.

In order to solve a problem it is sometimes helpful to find necessary conditions by assuming an answer exists. Let us do that here. Suppose x_o maps to the local extrema of $f(x_o)$ on $S = G^{-1}\{c\}$. This means there exists an open ball around x_o for which $f(x_o)$ is either an upper or lower bound of all the values of f over the ball intersected with S. One clear implication of this data is that if we take any continuously differentiable curve on S which passes through x_o, say $\gamma : \mathbb{R} \to \mathbb{R}^n$ with $\gamma(0) = x_o$ and $G(\gamma(t)) = c$ for all t, then the composite $f \circ \gamma$ is a function on $\mathbb{R}$ which takes an extreme value at $t = 0$. Fermat's theorem from calculus I applies and as $f \circ \gamma$ is differentiable near $t = 0$ we find $(f \circ \gamma)'(0) = 0$ is a necessary condition. But, this means we have two necessary conditions on γ:

1. $G(\gamma(t)) = c$

2. $(f \circ \gamma)'(0) = 0$

Let us expand a bit on both of these conditions:

1. $G'(x_o)\gamma'(0) = 0$

2. $f'(x_o)\gamma'(0) = 0$

The first of these conditions places $\gamma'(0) \in T_{x_o}S$ but then the second condition says that $f'(x_o) = (\nabla f)(x_o)^T$ is orthogonal to $\gamma'(0)$ hence $(\nabla f)(x_o)^T \in N_{x_o}$. Now, recall from the last section that the gradient vectors of the component functions to G span the normal space, this means any vector in N_{x_o} can be written as a linear combination of the gradient vectors. In particular, this means there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that

$$(\nabla f)(x_o)^T = \lambda_1(\nabla G_1)(x_o)^T + \lambda_2(\nabla G_2)(x_o)^T + \cdots + \lambda_p(\nabla G_p)(x_o)^T$$

We may summarize the method of Lagrange multipliers as follows:

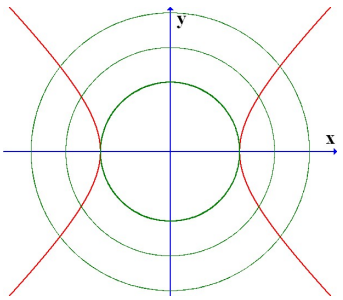
1. **choose n-variables which aptly describe your problem.**
2. **identify your objective function and write all constraints as level surfaces.**
3. **solve $\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2 + \cdots + \lambda_p \nabla G_p$ subject to the constraint $G(x) = c$.**
4. **test the validity of your proposed extremal points.**

The obvious gap in the method is the supposition that an extrema exists for the restriction $f|_S$. Well examine a few examples before I reveal a sufficient condition. We'll also see how absence of that sufficient condition does allow the method to fail.

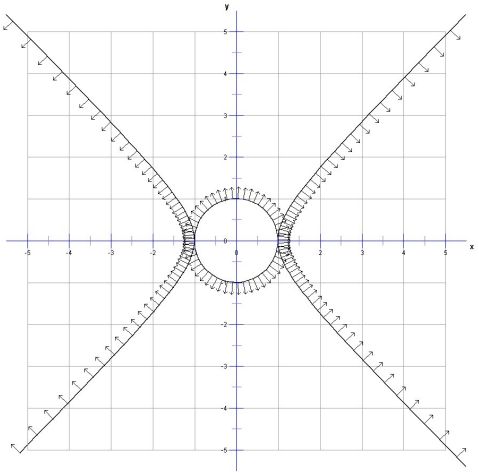
Example 5.5.1. *Suppose we wish to find maximum and minimum distance to the origin for points on the curve $x^2 - y^2 = 1$. In this case we can use the distance-squared function as our objective $f(x, y) = x^2 + y^2$ and the single constraint function is $g(x, y) = x^2 - y^2$. Observe that $\nabla f =< 2x, 2y >$ whereas $\nabla g =< 2x, -2y >$. We seek solutions of $\nabla f = \lambda \nabla g$ which gives us $< 2x, 2y >= \lambda < 2x, -2y >$. Hence $2x = 2\lambda x$ and $2y = -2\lambda y$. We must solve these equations subject to the condition $x^2 - y^2 = 1$. Observe that $x = 0$ is not a solution since $0 - y^2 = 1$ has no real solution. On the other hand, $y = 0$ does fit the constraint and $x^2 - 0 = 1$ has solutions $x = \pm 1$. Consider then*

$$2x = 2\lambda x \quad \text{and} \quad 2y = -2\lambda y \qquad \Rightarrow \qquad x(1 - \lambda) = 0 \quad \text{and} \quad y(1 + \lambda) = 0$$

Since $x \neq 0$ on the constraint curve it follows that $1 - \lambda = 0$ hence $\lambda = 1$ and we learn that $y(1 + 1) = 0$ hence $y = 0$. Consequently, $(1, 0$ and $(-1, 0)$ are the two point where we expect to find extreme-values of f. In this case, the method of Lagrange multipliers served it's purpose, as you can see in the graph. Below the green curves are level curves of the objective function whereas the particular red curve is the given constraint curve.



The picture below is a screen-shot of the Java applet created by David Lippman and Konrad Polthier to explore 2D and 3D graphs. Especially nice is the feature of adding vector fields to given objects, many other plotters require much more effort for similar visualization. See more at the website: http://dlippman.imathas.com/g1/GrapherLaunch.html.



Note how the gradient vectors to the objective function and constraint function line-up nicely at those points.

In the previous example, we actually got lucky. There are examples of this sort where we could get false maxima due to the nature of the constraint function.

Example 5.5.2. *Suppose we wish to find the points on the unit circle $g(x,y) = x^2 + y^2 = 1$ which give extreme values for the objective function $f(x,y) = x^2 - y^2$. Apply the method of Lagrange multipliers and seek solutions to $\nabla f = \lambda \nabla g$:*

$$< 2x, -2y >= \lambda < 2x, 2y >$$

We must solve $2x = 2x\lambda$ which is better cast as $(1-\lambda)x = 0$ and $-2y = 2\lambda y$ which is nicely written as $(1+\lambda)y = 0$. On the basis of these equations alone we have several options:

1. *if $\lambda = 1$ then $(1+1)y = 0$ hence $y = 0$*
2. *if $\lambda = -1$ then $(1-(1))x = 0$ hence $x = 0$*

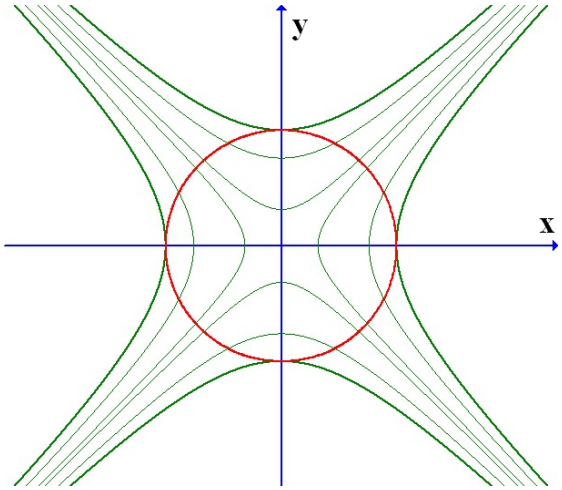
But, we also must fit the constraint $x^2 + y^2 = 1$ hence we find four solutions:

1. *if $\lambda = 1$ then $y = 0$ thus $x^2 = 1 \quad \Rightarrow \quad x = \pm 1 \quad \Rightarrow \quad (\pm 1, 0)$*
2. *if $\lambda = -1$ then $x = 0$ thus $y^2 = 1 \quad \Rightarrow \quad y = \pm 1 \quad \Rightarrow \quad (0, \pm 1)$*

We test the objective function at these points to ascertain which type of extrema we've located:

$$f(0, \pm 1) = 0^2 - (\pm 1)^2 = -1 \qquad \& \qquad f(\pm 1, 0) = (\pm 1)^2 - 0^2 = 1$$

When constrained to the unit circle we find the objective function attains a maximum value of 1 at the points $(1,0)$ and $(-1,0)$ and a minimum value of -1 at $(0,1)$ and $(0,-1)$. Let's illustrate the answers as well as a few non-answers to get perspective. Below the green curves are level curves of the objective function whereas the particular red curve is the given constraint curve.



The success of the last example was no accident. The fact that the constraint curve was a circle which is a closed and bounded subset of $\mathbb{R}^2$ means that is is a **compact** subset of $\mathbb{R}^2$. A well-known theorem of analysis states that any real-valued continuous function on a compact domain attains both maximum and minimum values. The objective function is continuous and the domain is compact hence the theorem applies and the method of Lagrange multipliers succeeds. In contrast, the constraint curve of the preceding example was a hyperbola which is not compact. We have no assurance of the existence of any extrema. Indeed, we only found minima but no maxima in Example 5.5.1.

The generality of the method of Lagrange multipliers is naturally limited to smooth constraint curves and smooth objective functions. We must insist the gradient vectors exist at all points of inquiry. Otherwise, the method breaks down. If we had a constraint curve which has sharp corners then the method of Lagrange breaks down at those corners. In addition, if there are points of discontinuity in the constraint then the method need not apply. This is not terribly surprising, even in calculus I the main attack to analyze extrema of function on $\mathbb{R}$ assumed continuity, differentiability and sometimes twice differentiability. Points of discontinuity require special attention in whatever context you meet them.

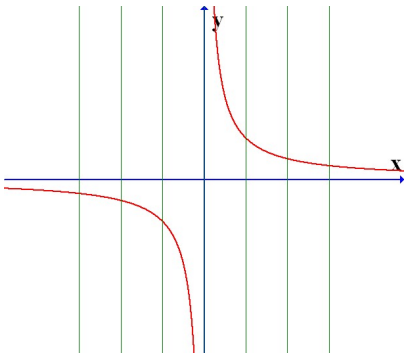
At this point it is doubtless the case that some of you are, to misquote an ex-student of mine, "not-impressed". Perhaps the following examples better illustrate the dangers of non-compact constraint curves.

Example 5.5.3. *Suppose we wish to find extrema of $f(x,y) = x$ when constrained to $xy = 1$. Identify $g(x,y) = xy = 1$ and apply the method of Lagrange multipliers and seek solutions to $\nabla f = \lambda \nabla g$:*

$$< 1, 0 >= \lambda < y, x > \;\Rightarrow\; 1 = \lambda y \;\; \text{and} \;\; 0 = \lambda x$$

If $\lambda = 0$ then $1 = \lambda y$ is impossible to solve hence $\lambda \neq 0$ and we find $x = 0$. But, if $x = 0$ then $xy = 1$ is not solvable. Therefore, we find no solutions. Well, I suppose we have succeeded here

in a way. We just learned there is no extreme value of x on the hyperbola $xy = 1$. Below the green curves are level curves of the objective function whereas the particular red curve is the given constraint curve.



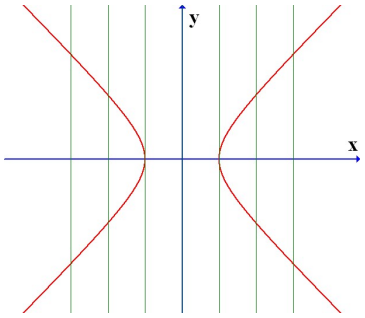
Example 5.5.4. *Suppose we wish to find extrema of $f(x, y) = x$ when constrained to $x^2 - y^2 = 1$. Identify $g(x, y) = x^2 - y^2 = 1$ and apply the method of Lagrange multipliers and seek solutions to $\nabla f = \lambda \nabla g$:*

$$< 1, 0 >= \lambda < 2x, -2y > \quad \Rightarrow \quad 1 = 2\lambda x \quad \text{and} \quad 0 = -2\lambda y$$

If $\lambda = 0$ then $1 = 2\lambda x$ is impossible to solve hence $\lambda \neq 0$ and we find $y = 0$. If $y = 0$ and $x^2 - y^2 = 1$ then we must solve $x^2 = 1$ whence $x = \pm 1$. We are tempted to conclude that:

1. *the objective function $f(x, y) = x$ attains a maximum on $x^2 - y^2 = 1$ at $(1, 0)$ since $f(1, 0) = 1$*
2. *the objective function $f(x, y) = x$ attains a minimum on $x^2 - y^2 = 1$ at $(-1, 0)$ since $f(1, 0) = -1$*

But, both conclusions are false. Note $\sqrt{2}^2 - 1^2 = 1$ hence $(\pm\sqrt{2}, 1)$ are points on the constraint curve and $f(\sqrt{2}, 1) = \sqrt{2}$ and $f(-\sqrt{2}, 1) = -\sqrt{2}$. The error of the method of Lagrange multipliers in this context is the supposition that there exists extrema to find, in this case there are no such points. It is possible for the gradient vectors to line-up at points where there are no extrema. Below the green curves are level curves of the objective function whereas the particular red curve is the given constraint curve.



Incidentally, if you want additional discussion of Lagrange multipliers for two-dimensional problems one very nice source I certainly profitted from was the YouTube video by Edward Frenkel of Berkley. See his website http://math.berkeley.edu/ frenkel/ for links.

Example 5.5.5. *Find points on the circle $x^2+y^2=1$ which are closest to the parabola $y^2=2(4-x)$.*

Let $f(x,y,u,v) = (x-u)^2 + (y-v)^2$ and construct $G(x,y,u,v) = (x^2+y^2-1,\ v^2+2u-8)$. Max/Min f subject to constraints $G=0$.

$$\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2$$

$$\langle 2(x-u), 2(y-v), -2(x-u), -2(x-v)\rangle = \lambda_1\langle 2x, 2y, 0, 0\rangle + \lambda_2\langle 0, 0, 2, 2v\rangle$$

yields,

$$\left.\begin{aligned} 2(x-u) &= 2\lambda_1 x \\ 2(y-v) &= 2\lambda_1 y \end{aligned}\right\} \longrightarrow \frac{y}{x} = \frac{y-v}{x-u} \quad (I)$$

$$-2(x-u) = 2\lambda_2 \longrightarrow -\lambda_2 = x-u$$

$$-2(y-v) = 2\lambda_2 v \longrightarrow y-v = (x-u)v \quad (II)$$

combining (I) & (II)

$$\frac{y}{x} = \frac{y-v}{x-u} = \frac{(x-u)v}{(x-u)} = v \ \therefore\ y = xv \quad (III)$$

Given together with (II) $\longrightarrow$

$$xv - v = (x-u)v$$
$$\Rightarrow xv - v = xv - uv$$
$$\Rightarrow v = uv$$
$$\Rightarrow u = 1.$$

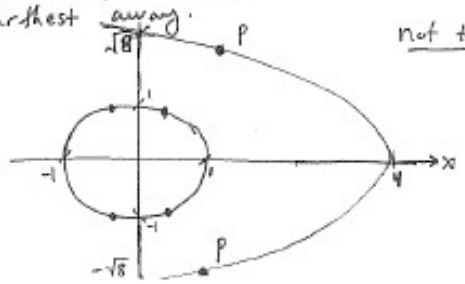
But, $v^2 = 2(4-u) = 2(3) = 6 \ \therefore\ v = \pm\sqrt{6}$.

Hence, $y = (\pm\sqrt{6})x \Rightarrow x^2+y^2 = x^2+6x^2 = 1$

$7x^2 = 1 \ \therefore\ x = \pm 1/\sqrt{7}$

We find points $(1, \pm\sqrt{6})$ on the parabola and $(\pm 1/\sqrt{7}, \pm\sqrt{6}/\sqrt{7})$ on the circle are closest or furthest away.

not to scale!



Example 5.5.6. *Find points on $x^2 + y^2 + z^2 = 1$ and plane $x + y + z = 3$ which are closest.*

$g_1(x,y,z) = (1 - x^2 - y^2 - z^2) = 0$ gives sphere. as $g_1^{-1}\{0\}$.

$g_2(u,v,w) = 3 - u - v - w = 0$ gives plane as $g_2^{-1}\{0\}$.

Consider $f(\vec{x},\vec{u}) = \|\vec{x} - \vec{u}\|^2$. We'd like to find min/max for f subject the constraints

$$G(\vec{x},\vec{u}) = \langle g_1(\vec{x}), g_2(\vec{u}) \rangle = \langle 0, 0 \rangle.$$

To say $G = 0$ is to place $\vec{x}$ on the sphere and $\vec{u}$ on the plane. Let $G_1(\vec{x},\vec{u}) = g_1(\vec{x})$ and $G_2(\vec{x},\vec{u}) = g_2(\vec{u})$ then

$$\nabla G_1 = \langle \nabla g_1, 0 \rangle = \langle g_{1x}, g_{1y}, g_{1z}, 0, 0, 0 \rangle$$
$$\nabla G_2 = \langle 0, \nabla g_2 \rangle = \langle 0, 0, 0, g_{2u}, g_{2v}, g_{2w} \rangle$$

Likewise, $f(\vec{x},\vec{u}) = \sum_{j=1}^{n} (x_j - u_j)^2 = (x-u)^2 + (y-v)^2 + (z-w)^2$

$$\nabla f(x,y,z,u,v,w) = \langle 2(x-u), 2(y-v), 2(z-w), -2(x-u), -2(y-v), -2(z-w) \rangle$$

Then $\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2$ yields,

$$\left.\begin{aligned} 2(x-u) &= \lambda_1 g_{1x} = -2\lambda_1 x \\ 2(y-v) &= \lambda_1 g_{1y} = -2\lambda_1 y \\ 2(z-w) &= \lambda_1 g_{1z} = -2\lambda_1 z \end{aligned}\right\} \quad \frac{x-u}{x} = \frac{y-v}{y} = \frac{z-w}{z}$$

$$\left.\begin{aligned} -2(x-u) &= \lambda_2 g_{2u} = -\lambda_2 \\ -2(y-v) &= \lambda_2 g_{2v} = -\lambda_2 \\ -2(z-w) &= \lambda_2 g_{2w} = -\lambda_2 \end{aligned}\right\} \quad x-u = y-v = z-w$$

Thus, $\frac{x-u}{x} = \frac{x-u}{y} = \frac{x-u}{z} \Rightarrow x = y = z \Rightarrow u = v = w$

$3x^2 = 1$, $x = \pm 1/\sqrt{3}$; $3u = 3$, $u = 1$

We obtain the point $(1,1,1)$ on the plane is closest to $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and furthest from $(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$ on the sphere.

Notice that on page 116 of Edwards he derives this as a mere special case of the fascinatingly general Example 10 of that section.

Chapter 6

critical point analysis for several variables

In the typical calculus sequence you learn the first and second derivative tests in calculus I. Then in calculus II you learn about power series and Taylor's Theorem. Finally, in calculus III, in many popular texts, you learn an essentially ad-hoc procedure for judging the nature of critical points as minimum, maximum or saddle. These topics are easily seen as disconnected events. In this chapter, we connect them. We learn that the geometry of quadratic forms is ellegantly revealed by eigenvectors and more than that this geometry is precisely what elucidates the proper classifications of critical points of multivariate functions with real values.

6.1 multivariate power series

We set aside the issue of convergence for now. We will suppose the series discussed in this section exist on and converge on some domain, but we do not seek to treat that topic here. Our focus is computational. How should we calculate the Taylor series for $f(x,y)$ at (a,b)? Or, what about $f(x)$ at $x_o \in \mathbb{R}^n$?.

6.1.1 taylor's polynomial for one-variable

If $f : U \subseteq \mathbb{R} \to \mathbb{R}$ is analytic at $x_o \in U$ then we can write

$$f(x) = f(x_o) + f'(x_o)(x-x_o) + \frac{1}{2}f''(x_o)(x-x_o)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_o)}{n!}(x-x_o)^n$$

We could write this in terms of the operator $D = \frac{d}{dt}$ and the evaluation of $t = x_o$

$$f(x) = \left[\sum_{n=0}^{\infty} \frac{1}{n!}(x-t)^n D^n f(t)\right]_{t=x_o} =$$

I remind the reader that a function is called **entire** if it is analytic on all of $\mathbb{R}$, for example $e^x, \cos(x)$ and $\sin(x)$ are all entire. In particular, you should know that:

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

Since $e^x = \cosh(x) + \sinh(x)$ it also follows that

$$\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$$

$$\sinh(x) = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$$

The geometric series is often useful, for $a, r \in \mathbb{R}$ with $|r| < 1$ it is known

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

This generates a whole host of examples, for instance:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots$$

$$\frac{x^3}{1-2x} = x^3(1 + 2x + (2x)^2 + \cdots) = x^3 + 2x^4 + 4x^5 + \cdots$$

Moreover, the term-by-term integration and differentiation theorems yield additional results in conjuction with the geometric series:

$$\tan^{-1}(x) = \int \frac{dx}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$

$$\ln(1-x) = \int \frac{d}{dx}\ln(1-x)dx = \int \frac{-1}{1-x}dx = -\int \sum_{n=0}^{\infty} x^n dx = \sum_{n=0}^{\infty} \frac{-1}{n+1}x^{n+1}$$

Of course, these are just the basic building blocks. We also can twist things and make the student use algebra,

$$e^{x+2} = e^x e^2 = e^2(1 + x + \frac{1}{2}x^2 + \cdots)$$

or trigonmetric identities,

$$\sin(x) = \sin(x - 2 + 2) = \sin(x - 2)\cos(2) + \cos(x - 2)\sin(2)$$

$$\Rightarrow \quad \sin(x) = \cos(2)\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x-2)^{2n+1} + \sin(2)\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-2)^{2n}.$$

Feel free to peruse my most recent calculus II materials to see a host of similarly sneaky calculations.

6.1.2 taylor's multinomial for two-variables

Suppose we wish to find the taylor polynomial centered at $(0, 0)$ for $f(x, y) = e^x \sin(y)$. It is a simple as this:

$$f(x,y) = \left(1 + x + \frac{1}{2}x^2 + \cdots\right)\left(y - \frac{1}{6}y^3 + \cdots\right) = y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \cdots$$

the resulting expression is called a multinomial since it is a polynomial in multiple variables. If all functions $f(x, y)$ could be written as $f(x, y) = F(x)G(y)$ then multiplication of series known from calculus II would often suffice. However, many functions do not possess this very special form. For example, how should we expand $f(x, y) = \cos(xy)$ about $(0, 0)$?. We need to derive the two-dimensional Taylor's theorem.

We already know Taylor's theorem for functions on $\mathbb{R}$,

$$g(x) = g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \cdots + \frac{1}{k!}g^{(k)}(a)(x-a)^k + R_k$$

and... If the remainder term vanishes as $k \to \infty$ then the function g is represented by the Taylor series given above and we write:

$$g(x) = \sum_{k=0}^{\infty} \frac{1}{k!}g^{(k)}(a)(x-a)^k.$$

Consider the function of two variables $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ which is smooth with smooth partial derivatives of all orders. Furthermore, let $(a, b) \in U$ and construct a line through (a, b) with direction vector (h_1, h_2) as usual:

$$\phi(t) = (a, b) + t(h_1, h_2) = (a + th_1, b + th_2)$$

for $t \in \mathbb{R}$. Note $\phi(0) = (a, b)$ and $\phi'(t) = (h_1, h_2) = \phi'(0)$. Construct $g = f \circ \phi : \mathbb{R} \to \mathbb{R}$ and choose $dom(g)$ such that $\phi(t) \in U$ for $t \in dom(g)$. This function g is a real-valued function of a

real variable and we will be able to apply Taylor's theorem from calculus II on g. However, to differentiate g we'll need tools from calculus III to sort out the derivatives. In particular, as we differentiate g, note we use the chain rule for functions of several variables:

$$\begin{aligned} g'(t) = (f \circ \phi)'(t) &= f'(\phi(t))\phi'(t) \\ &= \nabla f(\phi(t)) \cdot (h_1, h_2) \\ &= h_1 f_x(a + th_1, b + th_2) + h_2 f_y(a + th_1, b + th_2) \end{aligned}$$

Note $g'(0) = h_1 f_x(a, b) + h_2 f_y(a, b)$. Differentiate again (I omit $(\phi(t))$ dependence in the last steps),

$$\begin{aligned} g''(t) &= h_1 f'_x(a + th_1, b + th_2) + h_2 f'_y(a + th_1, b + th_2) \\ &= h_1 \nabla f_x(\phi(t)) \cdot (h_1, h_2) + h_2 \nabla f_y(\phi(t)) \cdot (h_1, h_2) \\ &= h_1^2 f_{xx} + h_1 h_2 f_{yx} + h_2 h_1 f_{xy} + h_2^2 f_{yy} \\ &= h_1^2 f_{xx} + 2h_1 h_2 f_{xy} + h_2^2 f_{yy} \end{aligned}$$

Thus, making explicit the point dependence, $g''(0) = h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)$. We may construct the Taylor series for g up to quadratic terms:

$$\begin{aligned} g(0 + t) &= g(0) + tg'(0) + \frac{1}{2} g''(0) + \cdots \\ &= f(a, b) + t[h_1 f_x(a, b) + h_2 f_y(a, b)] + \frac{t^2}{2} \big[h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)\big] + \cdots \end{aligned}$$

Note that $g(t) = f(a + th_1, b + th_2)$ hence $g(1) = f(a + h_1, b + h_2)$ and consequently,

$$\begin{aligned} f(a + h_1, b + h_2) = f(a, b) &+ h_1 f_x(a, b) + h_2 f_y(a, b) + \\ &+ \frac{1}{2} \left[h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)\right] + \cdots \end{aligned}$$

Omitting point dependence on the 2^{nd} derivatives,

$$\boxed{f(a + h_1, b + h_2) = f(a, b) + h_1 f_x(a, b) + h_2 f_y(a, b) + \tfrac{1}{2} \big[h_1^2 f_{xx} + 2h_1 h_2 f_{xy} + h_2^2 f_{yy}\big] + \cdots}$$

Sometimes we'd rather have an expansion about (x, y). To obtain that formula simply substitute $x - a = h_1$ and $y - b = h_2$. Note that the point (a, b) is fixed in this discussion so the derivatives are not modified in this substitution,

$$\begin{aligned} f(x, y) = f(a, b) &+ (x - a) f_x(a, b) + (y - b) f_y(a, b) + \\ &+ \frac{1}{2} \left[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)\right] + \cdots \end{aligned}$$

At this point we ought to recognize the first three terms give the tangent plane to $z = f(z, y)$ at $(a, b, f(a, b))$. The higher order terms are nonlinear corrections to the linearization, these quadratic

terms form a *quadratic form*. If we computed third, fourth or higher order terms we will find that, using $a = a_1$ and $b = a_2$ as well as $x = x_1$ and $y = x_2$,

$$\boxed{f(x,y) = \sum_{n=0}^{\infty} \sum_{i_1=0}^{2} \sum_{i_2=0}^{2} \cdots \sum_{i_n=0}^{2} \frac{1}{n!} \frac{\partial^{(n)} f(a_1,a_2)}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}} (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2}) \cdots (x_{i_n} - a_{i_n})}$$

Example 6.1.1. *Expand $f(x,y) = \cos(xy)$ about $(0,0)$. We calculate derivatives,*

$$f_x = -y\sin(xy) \qquad f_y = -x\sin(xy)$$

$$f_{xx} = -y^2\cos(xy) \qquad f_{xy} = -\sin(xy) - xy\cos(xy) \qquad f_{yy} = -x^2\cos(xy)$$

$$f_{xxx} = y^3\sin(xy) \qquad f_{xxy} = -y\cos(xy) - y\cos(xy) + xy^2\sin(xy)$$

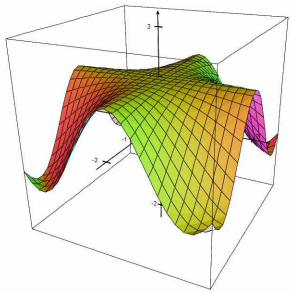
$$f_{xyy} = -x\cos(xy) - x\cos(xy) + x^2y\sin(xy) \qquad f_{yyy} = x^3\sin(xy)$$

Next, evaluate at $x = 0$ and $y = 0$ to find $f(x,y) = 1 + \cdots$ to third order in x, y about $(0,0)$. We can understand why these derivatives are all zero by approaching the expansion a different route: simply expand cosine directly in the variable (xy),

$$f(x,y) = 1 - \frac{1}{2}(xy)^2 + \frac{1}{4!}(xy)^4 + \cdots = 1 - \frac{1}{2}x^2y^2 + \frac{1}{4!}x^4y^4 + \cdots .$$

Apparently the given function only has nontrivial derivatives at $(0,0)$ at orders $0, 4, 8, \ldots$. We can deduce that $f_{xxxxy}(0,0) = 0$ without furthter calculation.

This is actually a very interesting function, I think it defies our analysis in the later portion of this chapter. The second order part of the expansion reveals nothing about the nature of the critical point $(0,0)$. Of course, any student of trigonometry should recognize that $f(0,0) = 1$ is likely a local maximum, it's certainly not a local minimum. The graph reveals that $f(0,0)$ is a local maxium for f restricted to certain rays from the origin whereas it is constant on several special directions (the coordinate axes).



And, if you were wondering, yes, we could also derive this from subsitution of $u = xy$ into the standard expansion for $\cos(u) = 1 - \frac{1}{2}u^2 + \frac{1}{4!}u^4 + \cdots$. Often such subsitutions are the quickest way to generate interesting examples.

6.1.3 taylor's multinomial for many-variables

Suppose $f : dom(f) \subseteq \mathbb{R}^n \to \mathbb{R}$ is a function of n-variables and we seek to derive the Taylor series centered at $a = (a_1, a_2, \ldots, a_n)$. Once more consider the composition of f with a line in $dom(f)$. In particular, let $\phi : \mathbb{R} \to \mathbb{R}^n$ be defined by $\phi(t) = a + th$ where $h = (h_1, h_2, \ldots, h_n)$ gives the direction of the line and clearly $\phi'(t) = h$. Let $g : dom(g) \subseteq \mathbb{R} \to \mathbb{R}$ be defined by $g(t) = f(\phi(t))$ for all $t \in \mathbb{R}$ such that $\phi(t) \in dom(f)$. Differentiate, use the multivariate chain rule, recall here that $\nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + \cdots + e_n \frac{\partial}{\partial x_n} = \sum_{i=1}^n e_i \partial_i$,

$$g'(t) = \nabla f(\phi(t)) \cdot \phi'(t) = \nabla f(\phi(t)) \cdot h = \sum_{i=1}^n h_i (\partial_i f)(\phi(t))$$

If we omit the explicit dependence on $\phi(t)$ then we find the simple formula $g'(t) = \sum_{i=1}^n h_i \partial_i f$. Differentiate a second time,

$$g''(t) = \frac{d}{dt}\left[\sum_{i=1}^n h_i \partial_i f(\phi(t))\right] = \sum_{i=1}^n h_i \frac{d}{dt}\left[\big(\partial_i f\big)(\phi(t))\right] = \sum_{i=1}^n h_i \big(\nabla \partial_i f\big)(\phi(t)) \cdot \phi'(t)$$

Omitting the $\phi(t)$ dependence and once more using $\phi'(t) = h$ we find

$$g''(t) = \sum_{i=1}^n h_i \nabla \partial_i f \cdot h$$

Recall that $\nabla = \sum_{j=1}^n e_j \partial_j$ and expand the expression above,

$$g''(t) = \sum_{i=1}^n h_i \left(\sum_{j=1}^n e_j \partial_j \partial_i f \right) \cdot h = \sum_{i=1}^n \sum_{j=1}^n h_i h_j \partial_j \partial_i f$$

where we should remember $\partial_j \partial_i f$ depends on $\phi(t)$. It should be clear that if we continue and take k-derivatives then we will obtain:

$$g^{(k)}(t) = \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_k=1}^n h_{i_1} h_{i_2} \cdots h_{i_k} \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f$$

More explicitly,

$$g^{(k)}(t) = \sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_k=1}^n h_{i_1} h_{i_2} \cdots h_{i_k} (\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f)(\phi(t))$$

Hence, by Taylor's theorem, provided we are sufficiently close to $t = 0$ as to bound the remainder[1]

$$g(t) = \sum_{k=0}^\infty \frac{1}{k!} \left(\sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_k=1}^n h_{i_1} h_{i_2} \cdots h_{i_k} (\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f)(\phi(t)) \right) t^k$$

[1] there exist smooth examples for which no neighborhood is small enough, the bump function in one-variable has higher-dimensional analogues, we focus our attention to functions for which it is possible for the series below to converge

Recall that $g(t) = f(\phi(t)) = f(a+th)$. Put[2] $t = 1$ and bring in the $\frac{1}{k!}$ to derive

$$\boxed{f(a+h) = \sum_{k=0}^{\infty} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} \frac{1}{k!} \big(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k} f\big)(a)\ h_{i_1}h_{i_2}\cdots h_{i_k}.}$$

Naturally, we sometimes prefer to write the series expansion about a as an expresssion in $x = a+h$. With this substitution we have $h = x-a$ and $h_{i_j} = (x-a)_{i_j} = x_{i_j} - a_{i_j}$ thus

$$\boxed{f(x) = \sum_{k=0}^{\infty} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} \frac{1}{k!} \big(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k} f\big)(a)\ (x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2})\cdots(x_{i_k} - a_{i_k}).}$$

Example 6.1.2. *Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ let's unravel the Taylor series centered at $(0,0,0)$ from the general formula boxed above. Utilize the notation $x = x_1, y = x_2$ and $z = x_3$ in this example.*

$$f(x) = \sum_{k=0}^{\infty} \sum_{i_1=1}^{3} \sum_{i_2=1}^{3} \cdots \sum_{i_k=1}^{3} \frac{1}{k!} \big(\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k} f\big)(0)\ x_{i_1}x_{i_2}\cdots x_{i_k}.$$

The terms to order 2 are as follows:

$$\begin{aligned} f(x) &= f(0) + f_x(0)x + f_y(0)y + f_z(0)z \\ &\quad +\tfrac{1}{2}\Big(f_{xx}(0)x^2 + f_{yy}(0)y^2 + f_{zz}(0)z^2 + \\ &\qquad +f_{xy}(0)xy + f_{xz}(0)xz + f_{yz}(0)yz + f_{yx}(0)yx + f_{zx}(0)zx + f_{zy}(0)zy \Big) + \cdots \end{aligned}$$

Partial derivatives commute for smooth functions hence,

$$\begin{aligned} f(x) &= f(0) + f_x(0)x + f_y(0)y + f_z(0)z \\ &\quad +\tfrac{1}{2}\Big(f_{xx}(0)x^2 + f_{yy}(0)y^2 + f_{zz}(0)z^2 + 2f_{xy}(0)xy + 2f_{xz}(0)xz + 2f_{yz}(0)yz \Big) \\ &\quad +\tfrac{1}{3!}\Big(f_{xxx}(0)x^3 + f_{yyy}(0)y^3 + f_{zzz}(0)z^3 + 3f_{xxy}(0)x^2y + 3f_{xxz}(0)x^2z \\ &\qquad +3f_{yyz}(0)y^2z + 3f_{xyy}(0)xy^2 + 3f_{xzz}(0)xz^2 + 3f_{yzz}(0)yz^2 + 6f_{xyz}(0)xyz \Big) + \cdots \end{aligned}$$

Example 6.1.3. *Suppose $f(x,y,z) = e^{xyz}$. Find a quadratic approximation to f near $(0,1,2)$. Observe:*

$$f_x = yze^{xyz} \qquad f_y = xze^{xyz} \qquad f_z = xye^{xyz}$$

$$f_{xx} = (yz)^2e^{xyz} \qquad f_{yy} = (xz)^2e^{xyz} \qquad f_{zz} = (xy)^2e^{xyz}$$

$$f_{xy} = ze^{xyz} + xyz^2e^{xyz} \qquad f_{yz} = xe^{xyz} + x^2yze^{xyz} \qquad f_{xz} = ye^{xyz} + xy^2ze^{xyz}$$

[2] if $t = 1$ is not in the domain of g then we should rescale the vector h so that $t = 1$ places $\phi(1)$ in $dom(f)$, if f is smooth on some neighborhood of a then this is possible

Evaluating at $x = 0, y = 1$ *and* $z = 2$*,*

$$f_x(0,1,2) = 2 \qquad f_y(0,1,2) = 0 \qquad f_z(0,1,2) = 0$$

$$f_{xx}(0,1,2) = 4 \qquad f_{yy}(0,1,2) = 0 \qquad f_{zz}(0,1,2) = 0$$

$$f_{xy}(0,1,2) = 2 \qquad f_{yz}(0,1,2) = 0 \qquad f_{xz}(0,1,2) = 1$$

Hence, as $f(0,1,2) = e^0 = 1$ *we find*

$$f(x,y,z) = 1 + 2x + 2x^2 + 2x(y-1) + 2x(z-2) + \cdots$$

Another way to calculate this expansion is to make use of the adding zero trick,

$$f(x,y,z) = e^{x(y-1+1)(z-2+2)} = 1 + x(y-1+1)(z-2+2) + \frac{1}{2}\big[x(y-1+1)(z-2+2)\big]^2 + \cdots$$

Keeping only terms with two or less of x*,* $(y-1)$ *and* $(z-2)$ *variables,*

$$f(x,y,z) = 1 + 2x + x(y-1)(2) + x(1)(z-2) + \frac{1}{2}x^2(1)^2(2)^2 + \cdots$$

Which simplifies once more to $f(x,y,z) = 1 + 2x + 2x(y-1) + x(z-2) + 2x^2 + \cdots$*.*

6.2 a brief introduction to the theory of quadratic forms

Definition 6.2.1.

Generally, a **quadratic form** Q is a function $Q : \mathbb{R}^n \to \mathbb{R}$ whose formula can be written $Q(\vec{x}) = \vec{x}^T A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ where $A \in \mathbb{R}^{n\times n}$ such that $A^T = A$. In particular, if $\vec{x} = (x, y)$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = ax^2 + bxy + byx + cy^2 = ax^2 + 2bxy + y^2.$$

The $n = 3$ case is similar,denote $A = [A_{ij}]$ and $\vec{x} = (x, y, z)$ so that

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = A_{11}x^2 + 2A_{12}xy + 2A_{13}xz + A_{22}y^2 + 2A_{23}yz + A_{33}z^2.$$

Generally, if $[A_{ij}] \in \mathbb{R}^{n\times n}$ and $\vec{x} = [x_i]^T$ then the associated quadratic form is

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \sum_{i,j} A_{ij}x_i x_j = \sum_{i=1}^{n} A_{ii}x_i^2 + \sum_{i<j} 2A_{ij}x_i x_j.$$

In case you wondering, yes you could write a given quadratic form with a different matrix which is not symmetric, but we will find it convenient to insist that our matrix is symmetric since that choice is always possible for a given quadratic form.

It is at times useful to use the dot-product to express a given quadratic form:

$$\vec{x}^T A\vec{x} = \vec{x} \cdot (A\vec{x}) = (A\vec{x}) \cdot \vec{x} = \vec{x}^T A^T \vec{x}$$

Some texts actually use the middle equality above to define a symmetric matrix.

Example 6.2.2.

$$2x^2 + 2xy + 2y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 6.2.3.

$$2x^2 + 2xy + 3xz - 2y^2 - z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 1 & 3/2 \\ 1 & -2 & 0 \\ 3/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proposition 6.2.4.

The values of a quadratic form on $\mathbb{R}^n - \{0\}$ is completely determined by it's values on the $(n-1)$-sphere $S_{n-1} = \{\vec{x} \in \mathbb{R}^n \mid ||\vec{x}|| = 1\}$. In particular, $Q(\vec{x}) = ||\vec{x}||^2 Q(\hat{x})$ where $\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$.

Proof: Let $Q(\vec{x}) = \vec{x}^T A\vec{x}$. Notice that we can write any nonzero vector as the product of its magnitude $||x||$ and its direction $\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$,

$$Q(\vec{x}) = Q(||\vec{x}||\hat{x}) = (||\vec{x}||\hat{x})^T A||\vec{x}||\hat{x} = ||\vec{x}||^2\hat{x}^T A\hat{x} = ||x||^2 Q(\hat{x}).$$

Therefore $Q(\vec{x})$ is simply proportional to $Q(\hat{x})$ with proportionality constant $||\vec{x}||^2$. □

The proposition above is very interesting. It says that if we know how Q works on unit-vectors then we can extrapolate its action on the remainder of $\mathbb{R}^n$. If $f : S \to \mathbb{R}$ then we could say $f(S) > 0$ iff $f(s) > 0$ for all $s \in S$. Likewise, $f(S) < 0$ iff $f(s) < 0$ for all $s \in S$. The proposition below follows from the proposition above since $||\vec{x}||^2$ ranges over all nonzero positive real numbers in the equations above.

Proposition 6.2.5.

If Q is a quadratic form on $\mathbb{R}^n$ and we denote $\mathbb{R}^n_* = \mathbb{R}^n - \{0\}$

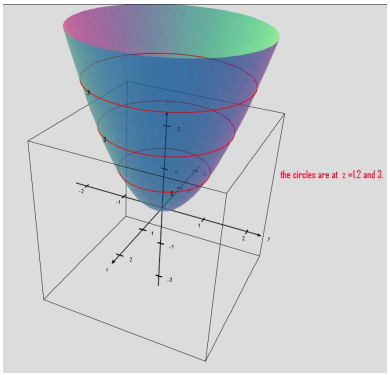
1.(negative definite) $Q(\mathbb{R}^n_*) < 0$ iff $Q(S_{n-1}) < 0$

2.(positive definite) $Q(\mathbb{R}^n_*) > 0$ iff $Q(S_{n-1}) > 0$

3.(non-definite) $Q(\mathbb{R}^n_*) = \mathbb{R} - \{0\}$ iff $Q(S_{n-1})$ has both positive and negative values.

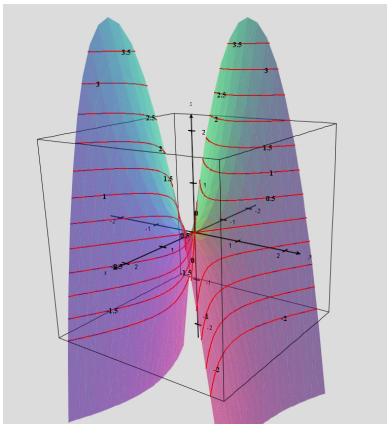
Before I get too carried away with the theory let's look at a couple examples.

Example 6.2.6. *Consider the quadric form $Q(x, y) = x^2 + y^2$. You can check for yourself that $z = Q(x, y)$ is a cone and Q has positive outputs for all inputs except $(0, 0)$. Notice that $Q(v) = ||v||^2$ so it is clear that $Q(S_1) = 1$. We find agreement with the preceding proposition. Next, think about the application of $Q(x, y)$ to level curves; $x^2 + y^2 = k$ is simply a circle of radius $\sqrt{k}$ or just the origin. Here's a graph of $z = Q(x, y)$:*



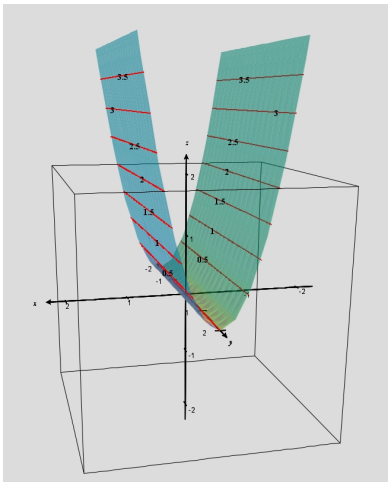
Notice that $Q(0, 0) = 0$ is the absolute minimum for Q. Finally, let's take a moment to write $Q(x, y) = [x, y]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = \lambda_2 = 1$.

Example 6.2.7. *Consider the quadric form $Q(x,y) = x^2 - 2y^2$. You can check for yourself that $z = Q(x,y)$ is a hyperboloid and Q has non-definite outputs since sometimes the x^2 term dominates whereas other points have $-2y^2$ as the dominent term. Notice that $Q(1,0) = 1$ whereas $Q(0,1) = -2$ hence we find $Q(S_1)$ contains both positive and negative values and consequently we find agreement with the preceding proposition. Next, think about the application of $Q(x,y)$ to level curves; $x^2 - 2y^2 = k$ yields either hyperbolas which open vertically ($k > 0$) or horizontally ($k < 0$) or a pair of lines $y = \pm\frac{x}{2}$ in the $k = 0$ case. Here's a graph of $z = Q(x,y)$:*



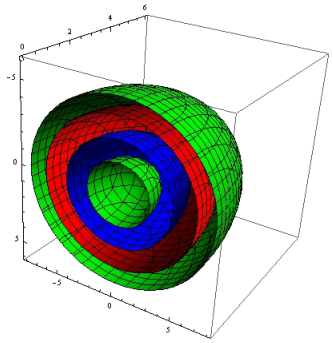
The origin is a **saddle point**. *Finally, let's take a moment to write* $Q(x,y) = [x,y]\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ *in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = -2$.*

Example 6.2.8. *Consider the quadric form $Q(x,y) = 3x^2$. You can check for yourself that $z = Q(x,y)$ is parabola-shaped trough along the y-axis. In this case Q has positive outputs for all inputs except $(0,y)$, we would call this form* **positive semi-definite**. *A short calculation reveals that $Q(S_1) = [0,3]$ thus we again find agreement with the preceding proposition (case 3). Next, think about the application of $Q(x,y)$ to level curves; $3x^2 = k$ is a pair of vertical lines: $x = \pm\sqrt{k/3}$ or just the y-axis. Here's a graph of $z = Q(x,y)$:*



Finally, let's take a moment to write $Q(x,y) = [x,y]\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ *in this case the matrix is diagonal and we note that the e-values are* $\lambda_1 = 3$ *and* $\lambda_2 = 0$.

Example 6.2.9. *Consider the quadric form* $Q(x,y,z) = x^2+2y^2+3z^2$. *Think about the application of* $Q(x,y,z)$ *to level surfaces;* $x^2+2y^2+3z^2 = k$ *is an ellipsoid. I can't graph a function of three variables, however, we can look at level surfaces of the function. I use Mathematica to plot several below:*



Finally, let's take a moment to write $Q(x,y,z) = [x,y,z]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *in this case the matrix is diagonal and we note that the e-values are* $\lambda_1 = 1$ *and* $\lambda_2 = 2$ *and* $\lambda_3 = 3$.

6.2.1 diagonalizing forms via eigenvectors

The examples given thus far are the simplest cases. We don't really need linear algebra to understand them. In contrast, e-vectors and e-values will prove a useful tool to unravel the later examples[3]

Definition 6.2.10.

Let $A \in \mathbb{R}^{n\times n}$. If $v \in \mathbb{R}^{n\times 1}$ is **nonzero** and $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ then we say v is an **eigenvector** with **eigenvalue** λ of the matrix A.

Proposition 6.2.11.

Let $A \in \mathbb{R}^{n\times n}$ then λ is an eigenvalue of A iff $det(A-\lambda I) = 0$. We say $P(\lambda) = det(A-\lambda I)$ the **characteristic polynomial** and $det(A-\lambda I) = 0$ is the **characteristic equation**.

Proof: Suppose λ is an eigenvalue of A then there exists a nonzero vector v such that $Av = \lambda v$ which is equivalent to $Av - \lambda v = 0$ which is precisely $(A-\lambda I)v = 0$. Notice that $(A-\lambda I)0 = 0$

[3] this is the one place in this course where we need eigenvalues and eigenvector calculations, I include these to illustrate the structure of quadratic forms in general, however, as linear algebra is not a prerequisite you may find some things in this section mysterious. The homework and study guide will elaborate on what is required this semester

thus the matrix $(A-\lambda I)$ is singular as the equation $(A-\lambda I)x=0$ has more than one solution. Consequently $det(A-\lambda I)=0$.

Conversely, suppose $det(A-\lambda I)=0$. It follows that $(A-\lambda I)$ is singular. Clearly the system $(A-\lambda I)x=0$ is consistent as $x=0$ is a solution hence we know there are infinitely many solutions. In particular there exists at least one vector $v\neq 0$ such that $(A-\lambda I)v=0$ which means the vector v satisfies $Av=\lambda v$. Thus v is an eigenvector with eigenvalue λ for A. □

Example 6.2.12. *Let* $A=\begin{bmatrix}3&1\\3&1\end{bmatrix}$ *find the e-values and e-vectors of A.*

$$det(A-\lambda I)=det\begin{bmatrix}3-\lambda&1\\3&1-\lambda\end{bmatrix}=(3-\lambda)(1-\lambda)-3=\lambda^2-4\lambda=\lambda(\lambda-4)=0$$

We find $\lambda_1=0$ *and* $\lambda_2=4$. *Now find the e-vector with e-value* $\lambda_1=0$, *let* $u_1=[u,v]^T$ *denote the e-vector we wish to find. Calculate,*

$$(A-0I)u_1=\begin{bmatrix}3&1\\3&1\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}3u+v\\3u+v\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Obviously the equations above are redundant and we have infinitely many solutions of the form $3u+v=0$ *which means* $v=-3u$ *so we can write,* $u_1=\begin{bmatrix}u\\-3u\end{bmatrix}=u\begin{bmatrix}1\\-3\end{bmatrix}$. *In applications we often make a choice to select a particular e-vector. Most modern graphing calculators can calculate e-vectors. It is customary for the e-vectors to be chosen to have length one. That is a useful choice for certain applications as we will later discuss. If you use a calculator it would likely give* $u_1=\frac{1}{\sqrt{10}}\begin{bmatrix}1\\-3\end{bmatrix}$ *although the* $\sqrt{10}$ *would likely be approximated unless your calculator is smart.*

Continuing we wish to find eigenvectors $u_2=[u,v]^T$ *such that* $(A-4I)u_2=0$. *Notice that* u,v *are disposable variables in this context, I do not mean to connect the formulas from the* $\lambda=0$ *case with the case considered now.*

$$(A-4I)u_1=\begin{bmatrix}-1&1\\3&-3\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}-u+v\\3u-3v\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Again the equations are redundant and we have infinitely many solutions of the form $v=u$. *Hence,* $u_2=\begin{bmatrix}u\\u\end{bmatrix}=u\begin{bmatrix}1\\1\end{bmatrix}$ *is an eigenvector for any* $u\in\mathbb{R}$ *such that* $u\neq 0$.

Theorem 6.2.13.

A matrix $A\in\mathbb{R}^{n\times n}$ is symmetric iff there exists an orthonormal eigenbasis for A.

There is a geometric proof of this theorem in Edwards[4] (see Theorem 8.6 pgs 146-147) . I prove half of this theorem in my linear algebra notes by a non-geometric argument (full proof is in Appendix C of Insel,Spence and Friedberg). It might be very interesting to understand the connection between the geometric verse algebraic arguments. We'll content ourselves with an example here:

Example 6.2.14. *Let* $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. *Observe that* $det(A - \lambda I) = -\lambda(\lambda+1)(\lambda-3)$ *thus* $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 3$. *We can calculate orthonormal e-vectors of* $v_1 = [1,0,0]^T$, $v_2 = \frac{1}{\sqrt{2}}[0,1,-1]^T$ *and* $v_3 = \frac{1}{\sqrt{2}}[0,1,1]^T$. *I invite the reader to check the validity of the following equation:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Its really neat that to find the inverse of a matrix of orthonormal e-vectors we need only take the transpose; note $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Proposition 6.2.15.

If Q is a quadratic form on $\mathbb{R}^n$ with matrix A and e-values $\lambda_1, \lambda_2, \ldots, \lambda_n$ with orthonormal e-vectors $v_1, v_2, \ldots, v_n$ then

$$Q(v_i) = {\lambda_i}^2$$

for $i = 1, 2, \ldots, n$. Moreover, if $P = [v_1|v_2|\cdots|v_n]$ then

$$Q(\vec{x}) = (P^T\vec{x})^T P^T A P P^T \vec{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where we defined $\vec{y} = P^T\vec{x}$.

Let me restate the proposition above in simple terms: we can transform a given quadratic form to a diagonal form by finding orthonormalized e-vectors and performing the appropriate coordinate transformation. Since P is formed from orthonormal e-vectors we know that P will be either a rotation or reflection. This proposition says we can remove "cross-terms" by transforming the quadratic forms with an appropriate rotation.

Example 6.2.16. *Consider the quadric form* $Q(x,y) = 2x^2 + 2xy + 2y^2$. *It's not immediately obvious (to me) what the level curves* $Q(x,y) = k$ *look like. We'll make use of the preceding proposition to understand those graphs. Notice* $Q(x,y) = [x,y]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. *Denote the matrix*

[4] think about it, there is a 1-1 correspondance between symmetric matrices and quadratic forms

of the form by A and calculate the e-values/vectors:

$$det(A-\lambda I) = det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) = 0$$

Therefore, the e-values are $\lambda_1 = 1$ and $\lambda_2 = 3$.

$$(A-I)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A-3I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

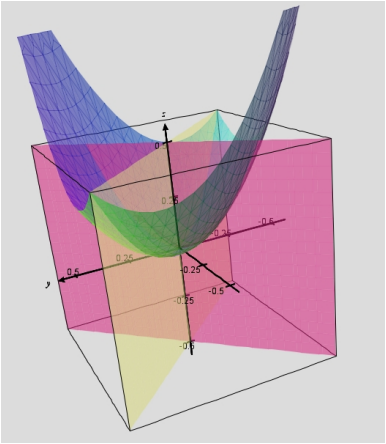
I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x &= \frac{1}{2}(\bar{x}+\bar{y}) \\ y &= \frac{1}{2}(-\bar{x}+\bar{y}) \end{array} \quad or \quad \begin{array}{ll} \bar{x} &= \frac{1}{2}(x-y) \\ \bar{y} &= \frac{1}{2}(x+y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield[5]*:*

$$\tilde{Q}(\bar{x}, \bar{y}) = \bar{x}^2 + 3\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x, y) = k$ is an ellipse. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x, y) = 2x^2 + 2xy + 2y^2 = k$ is an ellipse rotated by 45 degrees. Or, if you like, we can plot $z = Q(x, y)$:



[5] technically $\tilde{Q}(\bar{x}, \bar{y})$ is $Q(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))$

Example 6.2.17. *Consider the quadric form $Q(x,y) = x^2+2xy+y^2$. It's not immediately obvious (to me) what the level curves $Q(x,y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x,y) = [x,y]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A-\lambda I) = det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (\lambda-1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda-2) = 0$$

Therefore, the e-values are $\lambda_1 = 0$ and $\lambda_2 = 2$.

$$(A-0)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u+v=0$ to give $v=-u$ choose $u=1$ then normalize to get the vector above. Next,

$$(A-2I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

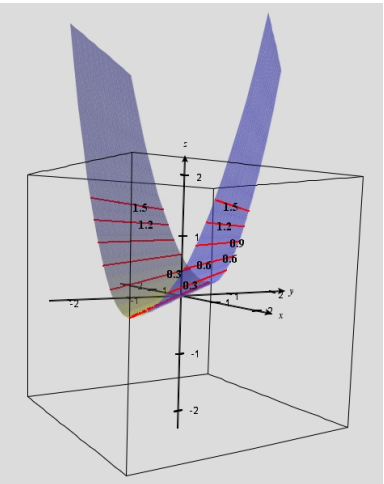
I just solved $u-v=0$ to give $v=u$ choose $u=1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x},\bar{y}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x &= \tfrac{1}{2}(\bar{x}+\bar{y}) \\ y &= \tfrac{1}{2}(-\bar{x}+\bar{y}) \end{aligned} \quad or \quad \begin{aligned} \bar{x} &= \tfrac{1}{2}(x-y) \\ \bar{y} &= \tfrac{1}{2}(x+y) \end{aligned}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x},\bar{y}) = 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x,y) = k$ is a pair of paralell lines. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = x^2+2xy+y^2 = k$ is a line with slope -1. Indeed, with a little algebraic insight we could have anticipated this result since $Q(x,y) = (x+y)^2$ so $Q(x,y) = k$ implies $x+y = \sqrt{k}$ thus $y = \sqrt{k}-x$. Here's a plot which again verifies what we've already found:



Example 6.2.18. *Consider the quadric form $Q(x,y) = 4xy$. It's not immediately obvious (to me) what the level curves $Q(x,y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x,y) = [x,y]\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A - \lambda I) = det\begin{bmatrix} -\lambda & 2 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 - 4 = (\lambda + 2)(\lambda - 2) = 0$$

Therefore, the e-values are $\lambda_1 = -2$ and $\lambda_2 = 2$.

$$(A + 2I)\vec{u}_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

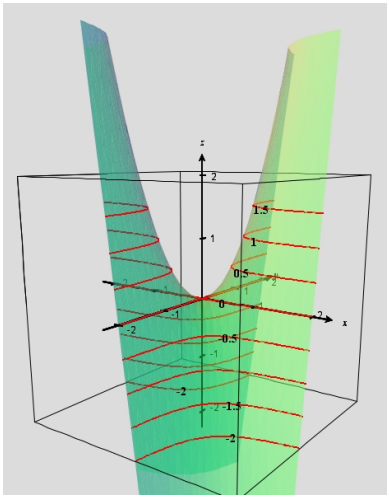
I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x &= \frac{1}{2}(\bar{x} + \bar{y}) \\ y &= \frac{1}{2}(-\bar{x} + \bar{y}) \end{array} \quad or \quad \begin{array}{ll} \bar{x} &= \frac{1}{2}(x - y) \\ \bar{y} &= \frac{1}{2}(x + y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x}, \bar{y}) = -2\bar{x}^2 + 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x,y) = k$ is a hyperbola. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = 4xy = k$ is a hyperbola rotated by 45 degrees. The graph $z = 4xy$ is thus a hyperbolic paraboloid:



The fascinating thing about the mathematics here is that if you don't want to graph $z = Q(x, y)$, but you do want to know the general shape then you can determine which type of quadraic surface you're dealing with by simply calculating the eigenvalues of the form.

Remark 6.2.19.

I made the preceding triple of examples all involved the same rotation. This is purely for my lecturing convenience. In practice the rotation could be by all sorts of angles. In addition, you might notice that a different ordering of the e-values would result in a redefinition of the barred coordinates. [6]

We ought to do at least one 3-dimensional example.

Example 6.2.20. *Consider the quadric form Q defined below:*

$$Q(x, y, z) = [x, y, z] \begin{bmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Denote the matrix of the form by A and calculate the e-values/vectors:

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} 6-\lambda & -2 & 0 \\ -2 & 6-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} \\ &= [(\lambda - 6)^2 - 4](5 - \lambda) \\ &= (5 - \lambda)[\lambda^2 - 12\lambda + 32](5 - \lambda) \\ &= (\lambda - 4)(\lambda - 8)(5 - \lambda) \end{aligned}$$

Therefore, the e-values are $\lambda_1 = 4$, $\lambda_2 = 8$ and $\lambda_3 = 5$. After some calculation we find the following orthonormal e-vectors for A:

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

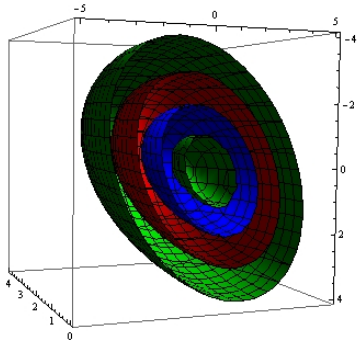
Let $P = [\vec{u}_1|\vec{u}_2|\vec{u}_3]$ and introduce new coordinates $\vec{y} = [\bar{x}, \bar{y}, \bar{z}]^T$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x &= \tfrac{1}{2}(\bar{x} + \bar{y}) \\ y &= \tfrac{1}{2}(-\bar{x} + \bar{y}) \\ z &= \bar{z} \end{aligned} \quad or \quad \begin{aligned} \bar{x} &= \tfrac{1}{2}(x - y) \\ \bar{y} &= \tfrac{1}{2}(x + y) \\ \bar{z} &= z \end{aligned}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x}, \bar{y}, \bar{z}) = 4\bar{x}^2 + 8\bar{y}^2 + 5\bar{z}^2$$

It is clear that in the barred coordinate system the level surface $Q(x, y, z) = k$ is an ellipsoid. If we draw the barred coordinate system superposed over the xyz-coordinate system then you'll see that the graph of $Q(x, y, z) = k$ is an ellipsoid rotated by 45 degrees around the $z-axis$. Plotted below are a few representative ellipsoids:



In summary, the behaviour of a quadratic form $Q(x) = x^T Ax$ is governed by it's set of eigenvalues[7] $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. Moreover, the form can be written as $Q(y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_k y_k^2$ by choosing the coordinate system which is built from the orthonormal eigenbasis of $col(A)$. In this coordinate system the shape of the level-sets of Q becomes manifest from the signs of the e-values.)

Remark 6.2.21.

If you would like to read more about conic sections or quadric surfaces and their connection to e-values/vectors I reccommend sections 9.6 and 9.7 of Anton's linear algebra text. I have yet to add examples on how to include translations in the analysis. It's not much more trouble but I decided it would just be an unecessary complication this semester. Also, section 7.1,7.2 and 7.3 in Lay's linear algebra text show a bit more about how to use this math to solve concrete applied problems. You might also take a look in Gilbert Strang's linear algebra text, his discussion of tests for positive-definite matrices is much more complete than I will give here.

6.3 second derivative test in many-variables

There is a connection between the shape of level curves $Q(x_1, x_2, \ldots, x_n) = k$ and the graph $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ of f. I'll discuss $n = 2$ but these comments equally well apply to $w = f(x, y, z)$ or higher dimensional examples. Consider a critical point (a, b) for $f(x, y)$ then the Taylor expansion about (a, b) has the form

$$f(a + h, b + k) = f(a, b) + Q(h, k)$$

where $Q(h, k) = \frac{1}{2}h^2 f_{xx}(a, b) + hk f_{xy}(a, b) + \frac{1}{2}h^2 f_{yy}(a, b) = [h, k][Q](h, k)$. Since $[Q]^T = [Q]$ we can find orthonormal e-vectors $\vec{u}_1, \vec{u}_2$ for $[Q]$ with e-values λ_1 and λ_2 respective. Using $U = [\vec{u}_1|\vec{u}_2]$ we

[7] this set is called the spectrum of the matrix

can introduce rotated coordinates $(\bar{h}, \bar{k}) = U(h, k)$. These will give

$$Q(\bar{h}, \bar{k}) = \lambda_1 \bar{h}^2 + \lambda_2 \bar{k}^2$$

Clearly if $\lambda_1 > 0$ and $\lambda_2 > 0$ then $f(a, b)$ yields the local minimum whereas if $\lambda_1 < 0$ and $\lambda_2 < 0$ then $f(a, b)$ yields the local maximum. Edwards discusses these matters on pgs. 148-153. In short, supposing $f \approx f(p) + Q$, if all the e-values of Q are positive then f has a local minimum of $f(p)$ at p whereas if all the e-values of Q are negative then f reaches a local maximum of $f(p)$ at p. Otherwise Q has both positive and negative e-values and we say Q is non-definite and the function has a saddle point. If all the e-values of Q are positive then Q is said to be **positive-definite** whereas if all the e-values of Q are negative then Q is said to be **negative-definite**. Edwards gives a few nice tests for ascertaining if a matrix is positive definite without explicit computation of e-values. Finally, if one of the e-values is zero then the graph will be like a trough.

Example 6.3.1. *Suppose* $f(x, y) = exp(-x^2 - y^2 + 2y - 1)$ *expand* f *about the point* $(0, 1)$*:*

$$f(x, y) = exp(-x^2)exp(-y^2 + 2y - 1) = exp(-x^2)exp(-(y-1)^2)$$

expanding,

$$f(x, y) = (1 - x^2 + \cdots)(1 - (y-1)^2 + \cdots) = 1 - x^2 - (y-1)^2 + \cdots$$

Recenter about the point $(0, 1)$ *by setting* $x = h$ *and* $y = 1 + k$ *so*

$$f(h, 1 + k) = 1 - h^2 - k^2 + \cdots$$

If (h, k) *is near* $(0, 0)$ *then the dominant terms are simply those we've written above hence the graph is like that of a quadraic surface with a pair of negative e-values. It follows that* $f(0, 1)$ *is a local maximum. In fact, it happens to be a global maximum for this function.*

Example 6.3.2. *Suppose* $f(x, y) = 4-(x-1)^2+(y-2)^2+Aexp(-(x-1)^2-(y-2)^2)+2B(x-1)(y-2)$ *for some constants* A, B*. Analyze what values for* A, B *will make* $(1, 2)$ *a local maximum, minimum or neither. Expanding about* $(1, 2)$ *we set* $x = 1 + h$ *and* $y = 2 + k$ *in order to see clearly the local behaviour of* f *at* $(1, 2)$*,*

$$\begin{aligned} f(1 + h, 2 + k) &= 4 - h^2 - k^2 + Aexp(-h^2 - k^2) + 2Bhk \\ &= 4 - h^2 - k^2 + A(1 - h^2 - k^2) + 2Bhk \cdots \\ &= 4 + A - (A + 1)h^2 + 2Bhk - (A + 1)k^2 + \cdots \end{aligned}$$

There is no nonzero linear term in the expansion at $(1, 2)$ *which indicates that* $f(1, 2) = 4 + A$ *may be a local extremum. In this case the quadratic terms are nontrivial which means the graph of this function is well-approximated by a quadraic surface near* $(1, 2)$*. The quadratic form* $Q(h, k) = -(A + 1)h^2 + 2Bhk - (A + 1)k^2$ *has matrix*

$$[Q] = \begin{bmatrix} -(A+1) & B \\ B & -(A+1)^2 \end{bmatrix}.$$

The characteristic equation for Q is

$$det([Q] - \lambda I) = det \begin{bmatrix} -(A+1) - \lambda & B \\ B & -(A+1)^2 - \lambda \end{bmatrix} = (\lambda + A + 1)^2 - B^2 = 0$$

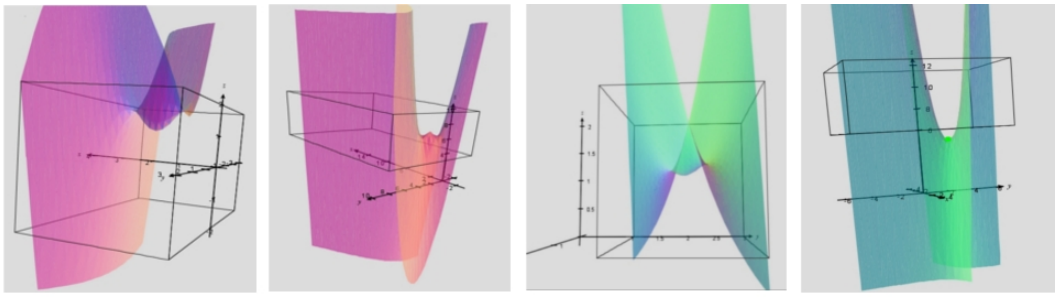
We find solutions $\lambda_1 = -A - 1 + B$ and $\lambda_2 = -A - 1 - B$. The possibilities break down as follows:

1. *if $\lambda_1, \lambda_2 > 0$ then $f(1,2)$ is local minimum.*
2. *if $\lambda_1, \lambda_2 < 0$ then $f(1,2)$ is local maximum.*
3. *if just one of λ_1, λ_2 is zero then f is constant along one direction and min/max along another so technically it is a local extremum.*
4. *if $\lambda_1 \lambda_2 < 0$ then $f(1,2)$ is not a local etremum, however it is a saddle point.*

In particular, the following choices for A, B will match the choices above

1. *Let $A = -3$ and $B = 1$ so $\lambda_1 = 3$ and $\lambda_2 = 1$;*
2. *Let $A = 3$ and $B = 1$ so $\lambda_1 = -3$ and $\lambda_2 = -5$*
3. *Let $A = -3$ and $B = -2$ so $\lambda_1 = 0$ and $\lambda_2 = 4$*
4. *Let $A = 1$ and $B = 3$ so $\lambda_1 = 1$ and $\lambda_2 = -5$*

Here are the graphs of the cases above, note the analysis for case 3 is more subtle for Taylor approximations as opposed to simple quadraic surfaces. In this example, case 3 was also a local minimum. In contrast, in Example 6.2.17 the graph was like a trough. The behaviour of f away from the critical point includes higher order terms whose influence turns the trough into a local minimum.



Example 6.3.3. *Suppose $f(x,y) = \sin(x)\cos(y)$ to find the Taylor series centered at $(0,0)$ we can simply multiply the one-dimensional result $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$ and $\cos(y) = 1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \cdots$ as follows:*

$$\begin{aligned} f(x,y) &= (x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots)(1 - \tfrac{1}{2!}y^2 + \tfrac{1}{4!}y^4 + \cdots) \\ &= x - \tfrac{1}{2}xy^2 + \tfrac{1}{24}xy^4 - \tfrac{1}{6}x^3 - \tfrac{1}{12}x^3y^2 + \cdots \\ &= x + \cdots \end{aligned}$$

The origin $(0,0)$ is a critical point since $f_x(0,0) = 0$ and $f_y(0,0) = 0$, however, this particular critical point escapes the analysis via the quadratic form term since $Q = 0$ in the Taylor series for this function at $(0,0)$. This is analogous to the inconclusive case of the 2nd derivative test in calculus III.

Example 6.3.4. *Suppose $f(x,y,z) = xyz$. Calculate the multivariate Taylor expansion about the point $(1,2,3)$. I'll actually calculate this one via differentiation, I have used tricks and/or calculus II results to shortcut any differentiation in the previous examples. Calculate first derivatives*

$$f_x = yz \qquad f_y = xz \qquad f_z = xy,$$

and second derivatives,

$$\begin{aligned} f_{xx} &= 0 & f_{xy} &= z & f_{xz} &= y \\ f_{yx} &= z & f_{yy} &= 0 & f_{yz} &= x \\ f_{zx} &= y & f_{zy} &= x & f_{zz} &= 0, \end{aligned}$$

and the nonzero third derivatives,

$$f_{xyz} = f_{yzx} = f_{zxy} = f_{zyx} = f_{yxz} = f_{xzy} = 1.$$

It follows,

$$\begin{aligned} &f(a+h,b+k,c+l) = \\ &\quad = f(a,b,c) \ + \ f_x(a,b,c)h \ + \ f_y(a,b,c)k \ + \ f_z(a,b,c)l \ + \\ &\qquad \tfrac{1}{2}(\ f_{xx}hh + f_{xy}hk + f_{xz}hl + f_{yx}kh + f_{yy}kk + f_{yz}kl + f_{zx}lh + f_{zy}lk + f_{zz}ll\ ) + \cdots \end{aligned}$$

Of course certain terms can be combined since $f_{xy} = f_{yx}$ etc... for smooth functions (we assume smooth in this section, moreover the given function here is clearly smooth). In total,

$$f(1+h,2+k,3+l) = 6 + 6h + 3k + 2l + \frac{1}{2}\big(3hk + 2hl + 3kh + kl + 2lh + lk\big) + \frac{1}{3!}(6)hkl$$

Of course, we could also obtain this from simple algebra:

$$f(1+h,2+k,3+l) = (1+h)(2+k)(3+l) = 6 + 6h + 3k + l + 3hk + 2hl + kl + hkl.$$

Example 6.3.5. *Find the extreme values of $w = x + y$ subject to the condition $x^2 + y^2 + z^2 \leq 1$.*

Notice $f(x,y,z) = x + z$ has $\nabla f = \langle 1, 0, 1 \rangle$ thus f has no critical pts. If follows f must attain max/min on boundary $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$. Use method of Lagrange,

$$\nabla f = \lambda \nabla g \quad \text{where } g = 0$$

$$\langle 1, 0, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$\left.\begin{aligned} 1 &= 2\lambda x \\ 0 &= 2\lambda y \\ 1 &= 2\lambda z \end{aligned}\right\} \longrightarrow y = 0 \quad \& \quad 1 = \frac{x}{z} \Rightarrow z = x$$

$$\Rightarrow x^2 + 0^2 + x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{2}$$

$$\Rightarrow x = \pm 1/\sqrt{2}.$$

Thus $(1/\sqrt{2}, 0, 1/\sqrt{2})$ or $(-1/\sqrt{2}, 0, -1/\sqrt{2})$ yield extremal values of f on $g = 0$.

$$f(1/\sqrt{2}, 0, 1/\sqrt{2}) = 1/\sqrt{2} + 1/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}.$$

$$f(-1/\sqrt{2}, 0, -1/\sqrt{2}) = -1/\sqrt{2} - 1/\sqrt{2} = -2/\sqrt{2} = -\sqrt{2}.$$

The max. value is $\sqrt{2}$ reached at $(1/\sqrt{2}, 0, 1/\sqrt{2})$ and the min. value is $-\sqrt{2}$ reached at $(-1/\sqrt{2}, 0, -1/\sqrt{2})$.

Chapter 7

convergence and estimation

If you look at the 2011 notes you'll find at least some of this, however, I leave this here as a reminder that this chapter is currently missing from my exposition. That said, I will give a lecture which shows how differentiating under the integral is an excellent technique. The analysis which justifies such a move will also be sketched. That lecture is based on Keith Conrad's blurb: DIFFERENTIATING UNDER THE INTEGRAL SIGN. Incidentally, as a student of pure or applied math the value of Conrad's blurbs ought not be ignored. See this blurb overview page Professor Conrad's exposition is often better than what you'll find in your textbook.

7.1 sequences of functions

7.2 power series

7.3 matrix exponential

7.4 uniform convergence

7.5 differentiating under the integral

7.6 contraction mappings and Newton's method

7.7 multivariate mean value theorem

7.8 proof of the inverse function theorem

Chapter 8

introduction to variational calculus

8.1 history

The problem of variational calculus is almost as old as modern calculus. Variational calculus seeks to answer questions such as:

Remark 8.1.1.

1. what is the shortest path between two points on a surface ?
2. what is the path of least time for a mass sliding without friction down some path between two given points ?
3. what is the path which minimizes the energy for some physical system ?
4. given two points on the x-axis and a particular area what curve has the longest perimeter and bounds that area between those points and the x-axis?

You'll notice these all involve a variable which is not a real variable or even a vector-valued-variable. Instead, the answers to the questions posed above will be **paths** or **curves** depending on how you wish to frame the problem. In variational calculus the variable is a function and we wish to find extreme values for a **functional**. In short, a functional is an abstract function of functions. A functional takes as an input a function and gives as an output a number. The space from which these functions are taken varies from problem to problem. Often we put additional **contraints** or **conditions** on the **space of admissable solutions**. To read about the full generality of the problem you should look in a text such as Hans Sagan's. Our treatment is introductory in this chapter, my aim is to show you why it is plausible and then to show you how we use variational calculus.

We will see that the problem of finding an extreme value for a functional is equivalent to solving the Euler-Lagrange equations or Euler equations for the functional. Euler predates Lagrange in his

discovery of the equations bearing their names. Eulers's initial attack of the problem was to chop the hypothetical solution curve up into a polygonal path. The unknowns in that approach were the coordinates of the vertices in the polygonal path. Then through some ingenious calculations he arrived at the Euler-Lagrange equations. Apparently there were logical flaws in Euler's original treatment. Lagrange later derived the same equations using the viewpoint that the variable was a function and the **variation** was one of shifting by an arbitrary function. The treatment of variational calculus in Edwards is neither Euler nor Lagrange's approach, it is a refined version which takes in the contributions of generations of mathematicians working on the subject and then merges it with careful functional analysis. I'm no expert of the full history, I just give you a rough sketch of what I've gathered from reading a few variational calculus texts.

Physics played a large role in the development of variational calculus. Lagrange was a physicist as well as a mathematician. At the present time, every physicist takes course(s) in *Lagrangian Mechanics*. Moreover, the use of variational calculus is fundamental since Hamilton's principle says that all physics can be derived from the principle of least action. In short this means that nature is lazy. The solutions realized in the physical world are those which minimize the action. The action

$$S[y] = \int L(y, y', t)\, dt$$

is constructed from the Lagrangian $L = T - U$ where T is the kinetic energy and U is the potential energy. In the case of classical mechanics the Euler Lagrange equations are precisely Newton's equations. The Hamiltonian $H = T + U$ is similar to the Lagrangian except that the fundamental variables are taken to be momentum and position in contrast to velocity and position in Lagrangian mechanics. Hamiltonians and Lagrangians are used to set-up new physical theories. Euler-Lagrange equations are said to give the so-called *classical limit* of modern field theories. The concept of a force is not so useful to quantum theories, instead the concept of energy plays the central role. Moreover, the problem of quantizing and then renormalizing field theory brings in very sophisiticated mathematics. In fact, the math of modern physics is not understood. In this chapter I'll just show you a few famous classical mechanics problems which are beatifully solved by Lagrange's approach. We'll also see how expressing the Lagrangian in non-Cartesian coordinates can give us an easy way to derive forces that arise from geometric contraints. Hopefully we can derive the coriolis force in this manner. I also plan to include a problem or two about Maxwell's equations from the variational viewpoint. There must be at least a dozen different ways to phrase Maxwell's equations, one reason I revisit them is to give you a concrete example as to the fact that physics has many formulations.

I am following the typical physics approach to variational calculus. Edwards' last chapter is more natural mathematically but I think the math is a bit much for your first exposure to the subject. The treatment given here is close to that of Arfken and Weber's Mathematical Physics text, however I suspect you can find these calculations in dozens of classical mechanics texts. More or less our approach is that of Lagrange.

8.2 the variational problem

Our goal in what follows here is to maximize or minimize a particular function of functions. Suppose $\mathcal{F}_o$ is a set of functions with some particular property. For now, we may could assume that all the functions in $\mathcal{F}_o$ have graphs that include (x_1, y_1) and (x_2, y_2). Consider a functional $J : \mathcal{F}_o \to \mathcal{F}_o$ which is defined by an integral of some function f which we call the **Lagrangian**,

$$J[y] = \int_{x_1}^{x_2} f(y, y', x)\, dx.$$

We suppose that f is given but y is a variable. Consider that if we are given a function $y^* \in \mathcal{F}_o$ and another function η such that $\eta(x_1) = \eta(x_2) = 0$ then we can reach a whole family of functions indexed by a real variable α as follows (relabel $y^*(x)$ by $y(x, 0)$ so it matches the rest of the family of functions):

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x)$$

Note that $x \mapsto y(x, \alpha)$ gives a function in $\mathcal{F}_o$. We define the **variation** of y to be

$$\boxed{\delta y = \alpha\eta(x)}$$

This means $y(x, \alpha) = y(x, 0) + \delta y$. We may write J as a function of α given the variation we just described:

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x, \alpha), y(x, \alpha)', x)\, dx.$$

It is intuitively obvious that if the function $y^*(x) = y(x, 0)$ is an extremum of the functional then we ought to expect

$$\left[\frac{\partial J(\alpha)}{\partial \alpha}\right]_{\alpha=0} = 0$$

Notice that we can calculate the derivative above using multivariate calculus. Remember that $y(x, \alpha) = y(x, 0) + \alpha\eta(x)$ hence $y(x, \alpha)' = y(x, 0)' + \alpha\eta(x)'$ thus $\frac{\partial y}{\partial \alpha} = \eta$ and $\frac{\partial y'}{\partial \alpha} = \eta' = \frac{d\eta}{dx}$. Consider that:

$$\begin{aligned}
\frac{\partial J(\alpha)}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left[\int_{x_1}^{x_2} f(y(x, \alpha), y(x, \alpha)', x)\, dx \right] \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} \right) dx \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\frac{d\eta}{dx} \right) dx \qquad (8.1)
\end{aligned}$$

Observe that

$$\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\eta\right] = \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}$$

Hence continuing Equation 8.1 in view of the product rule above,

$$\begin{aligned}\frac{\partial J(\alpha)}{\partial \alpha} &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\eta\right] - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta \right) dx \\ &= \frac{\partial f}{\partial y'}\eta\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta \right) dx \\ &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] \right)\eta \, dx\end{aligned} \tag{8.2}$$

Note we used the conditions $\eta(x_1) = \eta(x_2)$ to see that $\frac{\partial f}{\partial y'}\eta\big|_{x_1}^{x_2} = \frac{\partial f}{\partial y'}\eta(x_2) - \frac{\partial f}{\partial y'}\eta(x_1) = 0$. Our goal is to find the extreme values for the functional J. Let me take a few sentences to again restate our set-up. Generally, we take a function y then J maps to a new function $J[y]$. The family of functions indexed by α gives a whole ensemble of functions in $\mathcal{F}_o$ which are near y^* according to the formula,

$$y(x, \alpha) = y^*(x) + \alpha\eta(x)$$

Let's call this set of functions W_η. If we took another function like η, say ζ such that $\zeta(x_1) = \zeta(x_2) = 0$ then we could look at another family of functions:

$$y(x, \alpha) = y^*(x) + \alpha\zeta(x)$$

and we could denote the set of all such functions generated from ζ to be W_ζ. The total variation of y based at y^* should include all possible families of functions in $\mathcal{F}_o$. You could think of W_η and W_ζ be two different subspaces in $\mathcal{F}_o$. If $\eta \neq \zeta$ then these subspaces of $\mathcal{F}_o$ are likely disjoint except for the proposed extremal solution y^*. It is perhaps a bit unsettling to realize there are infinitely many such subspaces because there are infinitely many choices for the function η or ζ. In any event, each possible variation of y^* must satisfy the condition $\left[\frac{\partial J(\alpha)}{\partial \alpha}\right]_{\alpha=0} = 0$ since we **assume** that y^* is an extreme value of the functional J. It follows that the Equation 8.2 holds for all possible η. Therefore, we ought to expect that any extreme value of the functional $J[y] = \int_{x_1}^{x_2} f(y, y', x)\, dx$ must solve the **Euler Lagrange Equations:**

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0 \text{ Euler-Lagrange Equations for } J[y] = \int_{x_1}^{x_2} f(y, y', x)\, dx}$$

8.3 variational derivative

The role that η played in the discussion in the preceding section is somewhat similar to the role that the "h" plays in the definition $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. You might hope we could replace arguments in η with a more direct approach. Physicists have a heuristic way of making such arguments in terms of the variation δ. They would cast the arguments in the last page by just

"taking the variation of J". Let me give you their formal argument,

$$\begin{aligned}
\delta J &= \delta\left[\int_{x_1}^{x_2} f(y,y',x)\,dx \right] \\
&= \left[\int_{x_1}^{x_2} \delta f(y,y',x)\,dx \right] \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta\left(\tfrac{dy}{dx}\right) + \frac{\partial f}{\partial x}\delta x \right) dx \\
&= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y) \right) dx \\
&= \frac{\partial f}{\partial y'}\delta y\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] \right)\delta y\,dx
\end{aligned} \tag{8.3}$$

Therefore, since $\delta y = 0$ at the endpoints of integration, the Euler-Lagrange equations follow from $\delta J = 0$. Now, if you're like me, the argument above is less than satisfying since we never actually defined what it means to "take δ" of something. Also, why could I commute the variational δ and $\frac{d}{dx}$)? That said, the formal method is not without use since it allows the focus to be on the Euler Lagrange equations rather than the technical details of the variation.

Remark 8.3.1.

The more adept reader at this point should realize the hypocrisy of me calling the above calculation formal since even my presentation here was formal. I also used an analogy, I assumed that the theory of extreme values for multivariate calculus extends to function space. But, $\mathcal{F}_o$ is not $\mathbb{R}^n$, it's much bigger. Edwards builds the correct formalism for a rigourous calculation of the variational derivative. To be careful we'd need to develop the norm on function space and prove a number of results about infinite dimensional linear algebra. Take a look at the last chapter in Edwards' text if you're interested. I don't believe I'll have time to go over that material this semester.

8.4 Euler-Lagrange examples

I present a few standard examples in this section. We make use of the calculation in the last section. Also, we will use a result from your homework which states an equivalent form of the Euler-Lagrange equation is

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'} \right] = 0.$$

This form of the Euler Lagrange equation yields better differential equations for certain examples.

8.4.1 shortest distance between two points in plane

If s denotes the arclength in the xy-plane then the pythagorean theorem gives $ds^2 = dx^2 + dy^2$ infinitesimally. Thus, $ds = \sqrt{1 + \frac{dy}{dx}^2}\, dx$ and we may add up all the little distances ds to find the total length between two given points (x_1, y_1) and (x_2, y_2):

$$J[y] = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$$

Identify that we have $f(y, y', x) = \sqrt{1 + (y')^2}$. Calculate then,

$$\frac{\partial f}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}.$$

Euler Lagrange equations yield,

$$\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial y} \qquad \Rightarrow \qquad \frac{d}{dx}\left[\frac{y'}{\sqrt{1 + (y')^2}}\right] = 0 \qquad \Rightarrow \qquad \frac{y'}{\sqrt{1 + (y')^2}} = k$$

where $k \in \mathbb{R}$ is constant with respect to x. Moreover, square both sides to reveal

$$\frac{(y')^2}{1 + (y')^2} = k^2 \qquad \Rightarrow \qquad (y')^2 = \frac{k^2}{1 - k^2} \qquad \Rightarrow \qquad \frac{dy}{dx} = \pm\sqrt{\frac{k^2}{1 - k^2}} = m$$

where I have defined m is defined in the obvious way. We find solutions $y = mx + b$. Finally, we can find m, b to fit the given pair of points (x_1, y_1) and (x_2, y_2) as follows:

$$y_1 = mx_1 + b \qquad \text{and} \qquad y_2 = mx_2 + b \qquad \Rightarrow \qquad y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

provided $x_1 \neq x_2$. If $x_1 \neq x_2$ and $y_1 \neq y_2$ then we could perform the same calculation as above with the roles of x and y interchanged,

$$J[x] = \int_{y_1}^{y_2} \sqrt{1 + (x')^2}\, dy$$

where $x' = dx/dy$ and the Euler Lagrange equations would yield the solution

$$x = x_1 + \frac{x_2 - x_1}{y_2 - y_1}(y - y_1).$$

Finally, if both coordinates are equal then $(x_1, y_1) = (x_2, y_2)$ and the shortest path between these points is the trivial path, the armchair solution. Silly comments aside, we have shown that a straight line provides the curve with the shortest arclength between any two points in the plane.

8.4.2 surface of revolution with minimal area

Suppose we wish to revolve some curve which connects (x_1, y_1) and (x_2, y_2) around the x-axis. A surface constructed in this manner is called a **surface of revolution**. In calculus we learn how to calculate the surface area of such a shape. One can imagine deconstructing the surface into a sequence of ribbons. Each ribbon at position x will have a "radius" of y and a width of dx however, because the shape is tilted the area of the ribbon works out to $dA = 2\pi y ds$ where ds is the arclength.

If we choose x as the parameter this yields $dA = 2\pi y\sqrt{1+(y')^2}\,dx$. To find the surface of minimal surface area we ought to consider the functional:

$$A[y] = \int_{x_1}^{x_2} 2\pi y\sqrt{1+(y')^2}\,dx$$

Identify that $f(y, y', x) = 2\pi y\sqrt{1+(y')^2}$ hence $f_y = 2\pi\sqrt{1+(y')^2}$ and $f_{y'} = 2\pi y y'/\sqrt{1+(y')^2}$. The usual Euler-Lagrange equations are not easy to solve for this problem, it's easier to work with the equations you derived in homework,

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0.$$

Hence,

$$\frac{d}{dx}\left[2\pi y\sqrt{1+(y')^2} - \frac{2\pi y(y')^2}{\sqrt{1+(y')^2}}\right] = 0$$

Dividing by 2π and making a common denominator,

$$\frac{d}{dx}\left[\frac{y}{\sqrt{1+(y')^2}}\right] = 0 \qquad \Rightarrow \qquad \frac{y}{\sqrt{1+(y')^2}} = k$$

where k is a constant with respect to x. Squaring the equation above yields

$$\frac{y^2}{1+(\frac{dy}{dx})^2} = k^2 \qquad \Rightarrow \qquad y^2 - k^2 = k^2(\tfrac{dy}{dx})^2$$

Solve for dx, integrate, assuming the given points are in the first quadrant,

$$x = \int dx = \int \frac{k\,dy}{\sqrt{y^2-k^2}} = k\cosh^{-1}(\tfrac{y}{k}) + c$$

Hence,

$$\boxed{y = k\cosh\left(\frac{x-c}{k}\right)}$$

generates the surface of revolution of least area between two points. These shapes are called **Catenoids** they can be observed in the formation of soap bubble between rings. There is a vast literature on this subject and there are many cases to consider, I simply exhibit a simple solution. For a given pair of points it is not immediately obvious if there exists a solution to the Euler-Lagrange equations which fits the data. (see page 622 of Arfken).

8.4.3 Braichistochrone

Suppose a particle slides freely along some curve from (x_1, y_1) to $(x_2, y_2) = (0, 0)$ under the influence of gravity where we take y to be the vertical direction. **What is the curve of quickest descent?** Notice that if $x_1 = 0$ then the answer is easy to see, however, if $x_1 \neq 0$ then the question is not trivial. To solve this problem we must first offer a functional which accounts for the time of descent. Note that the speed $v = ds/dt$ so we'd clearly like to minimize $J = \int_{(0,0)}^{(x_1,y_1)} \frac{ds}{v}$. Since the object is assumed to fall freely we may assume that energy is conserved in the motion hence

$$\frac{1}{2}mv^2 = mg(y - y_1) \qquad \Rightarrow \qquad v = \sqrt{2g(y_1 - y)}$$

As we've discussed in previous examples, $ds = \sqrt{1 + (y')^2}dt$ so we find

$$J[y] = \int_0^{x_1} \underbrace{\sqrt{\frac{1 + (y')^2}{2g(y_1 - y)}}}_{f(y,y',x)}\, dx$$

Notice that the modified Euler-Lagrange equations $\frac{\partial f}{\partial x} - \frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0$ are convenient since $f_x = 0$. We calculate that

$$\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{\frac{1+(y')^2}{2g(y_1-y)}}}\frac{2y'}{2g(y_1 - y)} = \frac{y'}{\sqrt{2g(y_1 - y)(1 + (y')^2)}}$$

Hence there should exist some constant $1/(k\sqrt{2g})$ such that

$$\sqrt{\frac{1 + (y')^2}{2g(y_1 - y)}} - \frac{(y')^2}{\sqrt{2g(y_1 - y)(1 + (y')^2)}} = \frac{1}{k\sqrt{2g}}$$

It follows that,

$$\frac{1}{\sqrt{(y_1 - y)(1 + (y')^2)}} = \frac{1}{k} \qquad \Rightarrow \qquad \Big(y_1 - y\Big)\left(1 + \left(\frac{dy}{dx}\right)^2\right) = k^2$$

We need to solve for dy/dx,

$$(y_1 - y)\left(\frac{dy}{dx}\right)^2 = k^2 - y_1 + y \qquad \Rightarrow \qquad \left(\frac{dy}{dx}\right)^2 = \frac{y + k^2 - y_1}{y_1 - y}$$

Or, relabeling constants $a = y_1$ and $b = k^2 - y_1$ and we must solve

$$\frac{dy}{dx} = \pm\sqrt{\frac{b+y}{a-y}} \qquad \Rightarrow \qquad x = \pm \int \sqrt{\frac{a-y}{b+y}}\, dy$$

The integral is not trivial. It turns out that the solution is a cycloid (Arfken p. 624):

$$\boxed{x = \frac{a+b}{2}\Big(\theta + \sin(\theta)\Big) - d \qquad y = \frac{a+b}{2}\Big(1 - \cos(\theta)\Big) - b}$$

This is the curve that is traced out by a point on a wheel as it travels. If you take this solution and calculate $J[y_{cycloid}]$ you can show the time of descent is simply

$$T = \frac{\pi}{2}\sqrt{\frac{y_1}{2g}}$$

if the mass begins to descend from (x_2, y_2). But, this point has no connection with (x_1, y_1) except that they both reside on the same cycloid. It follows that the period of a pendulum that follows a cycloidal path is indpendent of the starting point on the path. This is not true for a circular pendulum in general, we need the small angle approximation to derive simple harmonic motion. It turns out that it is possible to make a pendulum follow a cycloidal path if you let the string be guided by a frame which is also cycloidal. The neat thing is that even as it loses energy it still follows a cycloidal path and hence has the same period. The "Brachistochrone" problem was posed by Johann Bernoulli in 1696 and it actually predates the variational calculus of Lagrange by some 50 or so years. This problem and ones like it are what eventually prompted Lagrange and Euler to systematically develop the subject. Apparently Galileo also studied this problem however lacked the mathematics to crack it.

8.5 Euler-Lagrange equations for several dependent variables

We still consider problems with just one independent parameter underlying everything. For problems of classical mechanics this is almost always time t. In anticipation of that application we choose to use the usual physics notation in the section. We suppose that our functional depends on functions $y_1, y_2, \ldots, y_n$ of time t along with their time derivatives $\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n$. We again suppose the functional of interest is an integral of a **Lagrangian** function f from time t_1 to time t_2,

$$J[(y_i)] = \int_{t_1}^{t_2} f(y_i, \dot{y}_i, t)\, dt$$

here we use (y_i) as shorthand for $(y_1, y_2, \ldots, y_n)$ and $(\dot{y}_i)$ as shorthand for $(\dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n)$. We suppose that n-conditions are given for each of the endpoints in this problem; $y_i(t_1) = y_{i1}$ and $y_i(t_2) = y_{i2}$. Moreover, we define $\mathcal{F}_o$ to be the set of paths from $\mathbb{R}$ to $\mathbb{R}^n$ subject to the conditions just stated. We now set out to find necessary conditions on a proposed solution to the extreme value problem for the functional J above. As before let's assume that an extremal solution $y* \in \mathcal{F}_o$ exists. Moreover, imagine varying the solution by some variational function $\eta = (\eta_i)$ which has $\eta(t_1) = (0, 0, \ldots, 0)$ and $\eta(t_2) = (0, 0, \ldots, 0)$. Consequently the family of paths defined below are all in $\mathcal{F}_o$,

$$y(t, \alpha) = y^*(t) + \alpha\eta(t)$$

Thus $y(t, 0) = y^*$. In terms of component functions we have that

$$y_i(t, \alpha) = y_i^*(t) + \alpha\eta_i(t).$$

You can identify that $\delta y_i = y_i(t, \alpha) - y_i^*(t) = \alpha\eta_i(t)$. Since y^* is an extreme solution we should expect that $\left(\frac{\partial J}{\partial \alpha}\right)_{\alpha=0} = 0$. Differentiate the functional with respect to α and make use of the chain rule for f which is a function of some $2n + 1$ variables,

$$\begin{aligned}
\frac{\partial J(\alpha)}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\left[\int_{t_1}^{t_2} f(y_i(t, \alpha), \dot{y}_i(t, \alpha), t)\, dt \right] \\
&= \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}_j}\frac{\partial \dot{y}_j}{\partial \alpha} \right) dt \\
&= \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j}\eta_j + \frac{\partial f}{\partial \dot{y}_j}\frac{d\eta_j}{dt} \right) dt \qquad (8.4) \\
&= \sum_{j=1}^{n} \frac{\partial f}{\partial \dot{y}_j}\eta \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial y_j} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}_j} \right)\eta_j\, dt
\end{aligned}$$

Since $\eta(t_1) = \eta(t_2) = 0$ the first term vanishes. Moreover, since we may repeat this calculation for all possible variations about the optimal solution y^* it follows that we obtain a set of Euler-Lagrange equations for each component function of the solution:

$$\boxed{\frac{\partial f}{\partial y_j} - \frac{d}{dt}\left[\frac{\partial f}{\partial \dot{y}_j}\right] = 0 \quad j = 1, 2, \ldots n \quad \text{Euler-Lagrange Eqns. for} \quad J[(y_i)] = \int_{t_1}^{t_2} f(y_i, \dot{y}_i, t)\, dt}$$

Often we simply use $y_1 = x$, $y_2 = y$ and $y_3 = z$ which denote the position of particle or perhaps just the component functions of a path which gives the geodesic on some surface. In either case we should have 3 sets of Euler-Lagrange equations, one for each coordinate. We will also use non-Cartesian coordinates to describe certain Lagrangians. We develop many useful results for set-up of Lagrangians in non-Cartesian coordinates in the next section.

8.5.1 free particle Lagrangian

For a particle of mass m the kinetic energy K is given in terms of the time derivatives of the coordinate functions x, y, z as follows:

$$K = \tfrac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Construct a functional by integrating the kinetic energy over time t,

$$S = \int_{t_1}^{t_2} \tfrac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)\, dt$$

The Euler-Lagrange equations for this functional are

$$\frac{\partial K}{\partial x} = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{x}}\right] \qquad \frac{\partial K}{\partial y} = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{y}}\right] \qquad \frac{\partial K}{\partial z} = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{z}}\right]$$

Since $\frac{\partial K}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial K}{\partial \dot{y}} = m\dot{y}$ and $\frac{\partial K}{\partial \dot{z}} = m\dot{z}$ it follows that

$$\boxed{0 = m\ddot{x} \qquad 0 = m\ddot{y} \qquad 0 = m\ddot{z}.}$$

You should recognize these as Newton's equation for a particle with no force applied. The solution is $(x(t), y(t), z(t)) = (x_o + tv_x, y_o + tv_y, z_o + tv_z)$ which is uniform rectilinear motion at constant velocity (v_x, v_y, v_z). The solution to Newton's equation minimizes the integral of the Kinetic energy. Generally the quantity S is called the **action** and Hamilton's Principle states that the laws of physics all arise from minimizing the action of the physical phenomena. We'll return to this discussion in a later section.

8.5.2 geodesics in $\mathbb{R}^3$

A **geodesic** is the path of minimal length between a pair of points on some manifold. Note we already proved that geodesics in the plane are just lines. In general, for $\mathbb{R}^3$, the square of the infinitesimal arclength element is $ds^2 = dx^2 + dy^2 + dz^2$. The arclength integral from $p = 0$ to $q = (q_x, q_y, q_z)$ in $\mathbb{R}^3$ is most naturally given from the parametric viewpoint:

$$S = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

We assume $(x(0), y(0), z(0)) = (0, 0, 0)$ and $(x(1), y(1), z(1)) = q$ and it should be clear that the integral above calculates the arclength. The Euler-Lagrange equations for x, y, z are

$$\frac{d}{dt}\left[\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right] = 0, \qquad \frac{d}{dt}\left[\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right] = 0, \qquad \frac{d}{dt}\left[\frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right] = 0.$$

It follows that there exist constants, say a, b and c, such that

$$a = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}, \qquad b = \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}, \qquad c = \frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}.$$

These equations are said to be **coupled** since each involves derivatives of the others. We usually need a way to uncouple the equations if we are to be successful in solving the system. We can calculate, and equate each with the constant 1:

$$1 = \frac{\dot{x}}{a\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \frac{\dot{y}}{b\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \frac{\dot{z}}{c\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}.$$

But, multiplying by the denominator reveals an interesting identity

$$\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} = \frac{\dot{x}}{a} = \frac{\dot{y}}{b} = \frac{\dot{z}}{c}$$

The solution has the form, $x(t) = tq_x$, $y(t) = tq_y$ and $z(t) = tq_z$. Therefore,

$$(x(t), y(t), z(t)) = t(q_x, q_y, q_z) = tq.$$

for $0 \leq t \leq 1$. These are the parametric equations for the line segment from the origin to q.

8.6 the Euclidean metric

The square root in the functional of the last subsection certainly complicated the calculation. It is intuitively clear that if we add up squared line elements ds^2 to give a minimum then that ought to correspond to the minimum for the sum of the positive square roots ds of those elements. Let's check if my conjecture works for $\mathbb{R}^3$:

$$S = \int_0^1 \Big(\underbrace{\dot{x}^2+\dot{y}^2+\dot{z}^2}_{f(x,y,z,\dot{x},\dot{y},\dot{z})} \Big)\, dt$$

This gives us the Euler Lagrange equations below:

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \qquad \ddot{z} = 0$$

The solution of these equations is clearly a line. In this formalism the equations were uncoupled from the outset.

Definition 8.6.1.

The Euclidean metric is $ds^2 = dx^2 + dy^2 + dz^2$. Generally, for orthogonal curvelinear coordinates u, v, w we calculate $ds^2 = \frac{1}{||\nabla u||^2}du^2 + \frac{1}{||\nabla v||^2}dv^2 + \frac{1}{||\nabla w||^2}dw^2$. We use this as a guide for constructing functionals which calculate arclength or speed

The beauty of the metric is that it allows us to calculate in other coordinates, consider

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

For which we have implicit inverse coordinate transformations $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$. From these inverse formulas we calculate:

$$\nabla r = < x/r, y/r > \qquad \nabla\theta = < -y/r^2, x/r^2 >$$

Thus, $||\nabla r|| = 1$ whereas $||\nabla\theta|| = 1/r$. We find that the metric in polar coordinates takes the form:

$$\boxed{ds^2 = dr^2 + r^2 d\theta^2}$$

Physicists and engineers tend to like to think of these as arising from calculating the length of infinitesimal displacements in the r or θ directions. Generically, for u, v, w coordinates

$$dl_u = \frac{1}{||\nabla u||}du \qquad dl_v = \frac{1}{||\nabla v||}dv \qquad dl_w = \frac{1}{||\nabla w||}dw$$

and $ds^2 = dl_u^2 + dl_v^2 + dl_w^2$. So in that notation we just found $dl_r = dr$ and $dl_\theta = rd\theta$. Notice then that cylindircal coordinates have the metric,

$$\boxed{ds^2 = dr^2 + r^2 d\theta^2 + dz^2.}$$

For spherical coordinates $x = r\cos(\phi)\sin(\theta)$, $y = r\sin(\phi)\sin(\theta)$ and $z = r\cos(\theta)$ (here $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \pi$, physics notation). Calculation of the metric follows from the line elements,

$$dl_r = dr \qquad dl_\phi = r\sin(\theta)d\phi \qquad dl_\theta = rd\theta$$

Thus,

$$\boxed{ds^2 = dr^2 + r^2\sin^2(\theta)d\phi^2 + r^2 d\theta^2.}$$

We now have all the tools we need for examples in spherical or cylindrical coordinates. What about other cases? In general, given some p-manifold in $\mathbb{R}^n$ how does one find the metric on that manifold? If we are to follow the approach of this section we'll need to find coordinates on $\mathbb{R}^n$ such that the manifold S is described by setting all but p of the coordinates to a constant. For example, in $\mathbb{R}^4$ we have generalized cylindircal coordinates (r, ϕ, z, t) defined implicitly by the equations below

$$x = r\cos(\phi), \qquad y = r\sin(\phi), \qquad z = z, \qquad t = t$$

On the hyper-cylinder $r = R$ we have the metric $ds^2 = R^2 d\theta^2 + dz^2 + dw^2$. There are mathematicians/physicists whose careers are founded upon the discovery of a metric for some manifold. This is generally a difficult task.

8.7 geodesics

A **geodesic** is a path of smallest distance on some manifold. In general relativity, it turns out that the solutions to Eistein's field equations are geodesics in 4-dimensional curved spacetime. Particles that fall freely are following geodesics, for example projectiles or planets in the absense of other frictional/non-gravitational forces. We don't follow a geodesic in our daily life because the earth pushes back up with a normal force. Also, do be honest, the idea of length in general relativity is a bit more abstract that the geometric length studied in this section. The metric of general relativity is non-Euclidean. General relativity is based on semi-Riemannian geometry whereas this section is all Riemannian geometry. The metric in Riemannian geometry is positive definite. The metric in semi-Riemannian geometry can be written as a quadratic form with both positive and negative eigenvalues. In any event, if you want to know more I know some books you might like.

8.7.1 geodesic on cylinder

The equation of a cylinder of radius R is most easily framed in cylindrical coordinates (r, θ, z); the equation is merely $r = R$ hence the metric reads

$$ds^2 = R^2 d\theta^2 + dz^2$$

Therefore, we ought to minimize the following functional in order to locate the parametric equations of a geodesic on the cylinder: note $ds^2 = \left(R^2 \frac{d\theta^2}{dt^2} + \frac{dz^2}{dt^2}\right) dt^2$ thus:

$$S = \int (R^2 \dot{\theta}^2 + \dot{z}^2)\, dt$$

Euler-Lagrange equations for the dependent variables θ and z are simply:

$$\ddot{\theta} = 0 \qquad \ddot{z} = 0.$$

We can integrate twice to find solutions

$$\boxed{\theta(t) = \theta_o + At \qquad z(t) = z_o + Bt}$$

Therefore, the geodesic on a cylinder is simply the line connecting two points in the plane which is curved to assemble the cylinder. Simple cases that are easy to understand:

1. Geodesic from $(R\cos(\theta_o), R\sin(\theta_o), z_1)$ to $(R\cos(\theta_o), R\sin(\theta_o), z_2)$ is parametrized by $\theta(t) = \theta_o$ and $z(t) = z_1 + t(z_2 - z_1)$ for $0 \leq t \leq 1$. Technically, there is some ambiguity here since I never declared over what range the t is to range. Could pick other intervals, we could use z at the parameter is we wished then $\theta(z) = \theta_o$ and $z = z$ for $z_1 \leq z \leq z_2$

2. Geodesic from $(R\cos(\theta_1), R\sin(\theta_1), z_o)$ to $(R\cos(\theta_2), R\sin(\theta_2), z_o)$ is parametrized by $\theta(t) = \theta_1 + t(\theta_2 - \theta_1)$ and $z(t) = z_o$ for $0 \leq t \leq 1$.

3. Geodesic from $(R\cos(\theta_1), R\sin(\theta_1), z_1)$ to $(R\cos(\theta_2), R\sin(\theta_2), z_2)$ is parametrized by

   $$\theta(t) = \theta_1 + t(\theta_2 - \theta_1) \qquad z(t) = z_1 + t(z_2 - z_1)$$

   You can eliminate t and find the equation $z = \frac{z_2 - z_1}{\theta_2 - \theta_1}(\theta - \theta_1)$ which again just goes to show you this is a line in the curved coordinates.

8.7.2 geodesic on sphere

The equation of a sphere of radius R is most easily framed in spherical coordinates (r, ϕ, θ); the equation is merely $r = R$ hence the metric reads

$$ds^2 = R^2 \sin^2(\theta) d\phi^2 + R^2 d\theta^2.$$

Therefore, we ought to minimize the following functional in order to locate the parametric equations of a geodesic on the sphere: note $ds^2 = \left(R^2 \sin^2(\theta) \frac{d\phi^2}{dt^2} + R^2 \frac{d\theta^2}{dt^2}\right) dt^2$ thus:

$$S = \int (\underbrace{R^2 \sin^2(\theta)\dot{\phi}^2 + R^2\dot{\theta}^2}_{f(\theta,\phi,\dot{\theta},\dot{\phi})}) \, dt$$

Euler-Lagrange equations for the dependent variables ϕ and θ are simply: $f_\theta = \frac{d}{dt}(f_{\dot{\theta}})$ and $f_\phi = \frac{d}{dt}(f_{\dot{\phi}})$ which yield:

$$2R^2 \sin(\theta)\cos(\theta)\dot{\phi}^2 = \tfrac{d}{dt}(2R^2\dot{\theta}) \qquad 0 = \frac{d}{dt}\Big(2R^2 \sin^2(\theta)\dot{\phi}\Big).$$

We find a **constant of motion** $L = 2R^2 \sin^2(\theta)\dot{\phi}$ inserting this in the equation for the azmuthial angle θ yields:

$$2R^2 \sin(\theta)\cos(\theta)\dot{\phi}^2 = \tfrac{d}{dt}(2R^2\dot{\theta}) \qquad 0 = \frac{d}{dt}\Big(2R^2 \sin^2(\theta)\dot{\phi}\Big).$$

If you can solve these and demonstrate through some reasonable argument that the solutions are great circles then I will give you points. I have some solutions but nothing looks too pretty.

8.8 Lagrangian mechanics

8.8.1 basic equations of classical mechanics summarized

Classical mechanics is the study of massive particles at relatively low velocities. Let me refresh your memory about the basics equations of Newtonian mechanics. Our goal in this section will be to rephrase Newtonian mechanics in the variational langauge and then to solve problems with the Euler-Lagrange equations. Newton's equations tell us how a particle of mass m evolves through time according to the net-force impressed on m. In particular,

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

If m is not constant then you may recall that it is better to use momentum $\vec{P} = m\vec{v} = m\frac{d\vec{r}}{dt}$ to set-up Newton's 2nd Law:

$$\frac{d\vec{P}}{dt} = \vec{F}$$

In terms of components we have a system of differential equations with indpendent variable time t. If we use position as the dependent variable then Newton's 2nd Law gives three second order ODEs,

$$m\ddot{x} = F_x \qquad m\ddot{y} = F_y \qquad m\ddot{z} = F_z$$

where $\vec{r} = (x, y, z)$ and the dots denote time-derivatives. Moreover, $\vec{F} =< F_x, F_y, F_z >$ is the sum of the forces that act on m. In contrast, if you work with momentum then you would want to solve six first order ODEs,

$$\dot{P}_x = F_x \qquad \dot{P}_y = F_y \qquad \dot{P}_z = F_z$$

and $P_x = m\dot{x}$, $P_y = m\dot{y}$ and $P_z = m\dot{z}$. These equations are easiest to solve when the force is not a function of velocity or time. In particular, if the force $\vec{F}$ is conservative then there exists a potential energy function $U : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = -\nabla U$. We can prove that in the case the force is conservative the total energy is conserved.

8.8.2 kinetic and potential energy, formulating the Lagrangian

Recall the kinetic energy is $T = \frac{1}{2}m||\vec{v}||^2$, in Cartesian coordinates this gives us the formula:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

If $\vec{F}$ is a conservative force then it is independent of path so we may construct the potential energy function as follows:

$$U(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{F} \cdot d\vec{r}$$

Here $\mathcal{O}$ is the origin for the potential and we can prove that the potential energy constructed in this manner has $\vec{F} = -\nabla U$. We can prove that the total (mechanical) energy $E = T + U$ for a conservative system is a constant; $dE/dt = 0$. Hopefully these comments are at least vaguely familiar from some physics course in your distant memory. If not relax, calculationally this chapter is self-contained, read onward.

We already calculated that if we use T as the Lagrangian then the Euler-Lagrange equations produce Newton's equations in the case that the force is zero (see 8.5.1). Suppose that we define the Lagrangian to be $L = T - U$ for a system governed by a conservative force with potential energy function U. We seek to prove the Euler-Lagrange equations are precisely Newton's equations for this conservative system[1] Generically we have a Lagrangian of the form

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z).$$

[1] don't mistake this example as an admission that Lagrangian mechanics is limited to conservative systems. Quite the contrary, Lagrangian mechanics is actually more general than the orginal framework of Newton!

We wish to find extrema for the functional $S = \int L(t)\,dt$. This yields three sets of Euler-Lagrange equations, one for each dependent variable x, y or z

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = \frac{\partial L}{\partial x} \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}}\right] = \frac{\partial L}{\partial y} \qquad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}}\right] = \frac{\partial L}{\partial z}.$$

Note that $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial L}{\partial \dot{y}} = m\dot{y}$ and $\frac{\partial L}{\partial \dot{z}} = m\dot{z}$. Also note that $\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x$, $\frac{\partial L}{\partial y} = -\frac{\partial U}{\partial y} = F_y$ and $\frac{\partial L}{\partial z} = -\frac{\partial U}{\partial z} = F_z$. It follows that

$$\boxed{m\ddot{x} = F_x \qquad m\ddot{y} = F_y \qquad m\ddot{z} = F_z.}$$

Of course this is precisely $m\vec{a} = \vec{F}$ for a net-force $\vec{F} =< F_x, F_y, F_z >$. We have shown that **Hamilton's principle** reproduces Newton's Second Law for conservative forces. Let me take a moment to state it.

Definition 8.8.1. Hamilton's Principle:

If a physical system has generalized coordinates q_j with velocities $\dot{q}_j$ and Lagrangian $L = T - U$ then the solutions of physics will minimize the action S defined below:

$$\boxed{S = \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt}$$

Mathematically, this means the variation $\delta S = 0$ for physical trajectories.

This is a necessary condition for solutions of the equations of physics. Sufficient conditions are known, you can look in any good variational calculus text. You'll find analogues to the second derivative test for variational differentiation. As far as I can tell physicists don't care about this logical gap, probably because the solutions to the Euler-Lagrange equations are the ones for which they are looking.

8.8.3 easy physics examples

Now, you might just see this whole exercise as some needless multiplication of notation and formalism. After all, I just told you we just get Newton's equations back from the Euler-Lagrange equations. To my taste the impressive thing about Lagrangian mechanics is that you get to start the problem with energy. Moreover, the Lagrangian formalism handles non-Cartesian coordinates with ease. If you search your memory from classical mechanics you'll notice that you either do constant acceleration, circular motion or motion along a line. What if you had a particle constrained to move in some frictionless ellipsoidal bowl. Or what if you had a pendulum hanging off another pendulum? How would you even write Newtons' equations for such systems? In contrast, the problem is at least easy to set-up in the Lagrangian approach. Of course, solutions may be less easy to obtain.

Example 8.8.2. Projectile motion: *take z as the vertical direction and suppose a bullet is fired with initial velocity $v_o =< v_{ox}, v_{oy}, v_{oz} >$. The potential energy due to gravity is simply $U = mgz$ and kinetic energy is given by $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Thus,*

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Euler-Lagrange equations are simply:

$$\frac{d}{dt}\left[m\dot{x}\right] = 0 \qquad \frac{d}{dt}\left[m\dot{y}\right] = 0 \qquad \frac{d}{dt}\left[m\dot{z}\right] = \frac{\partial}{\partial z}(-mgz) = -mg.$$

Integrating twice and applying initial conditions gives us the (possibly familiar) equations

$$x(t) = x_o + v_{ox}t, \qquad y(t) = y_o + v_{oy}t, \qquad z(t) = z_o + v_{oz}t - \tfrac{1}{2}gt^2.$$

Example 8.8.3. Simple Pendulum: *let θ denote angle measured off the vertical for a simple pendulum of mass m and length l. Trigonmetry tells us that*

$$x = l\sin(\theta) \qquad y = l\cos(\theta) \qquad \Rightarrow \qquad \dot{x} = l\cos(\theta)\dot{\theta} \qquad y = -l\sin(\theta)\dot{\theta}$$

Thus $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\theta}^2$. Also, the potential energy due to gravity is $U = -mgl\cos(\theta)$ which gives us

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos(\theta)$$

Then, the Euler-Lagrange equation in θ is simply:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = \frac{\partial L}{\partial \theta} \qquad \Rightarrow \qquad \frac{d}{dt}(ml^2\dot{\theta}) = -mgl\sin(\theta) \qquad \Rightarrow \qquad \ddot{\theta} + \frac{g}{l}\sin(\theta) = 0.$$

In the small angle approximation, $\sin(\theta) = \theta$ then we have the solution $\theta(t) = \theta_o\cos(\omega t + \phi_o)$ for angular frequency $\omega = \sqrt{g/l}$

Chapter 9

multilinear algebra

The principle aim of this chapter is to introduce how to calculate with $\otimes$ and $\wedge$. We take a very concrete approach where the tensor and wedge product are understood in terms of multilinear mappings for which they form a basis. That said, there is a univerisal algebraic approach to construct the tensor and wedge products, I encourage the reader to study Dummit and Foote's *Abstract Algebra* Part III on *Modules and Vector Spaces* where these constructions are explained in a much broader algebraic context.

Beyond the basic definitions, we also study how wedge products capture determinants and give a natural language to ask certain questions of linear dependence. We also study metrics with particular attention to four dimensional Minkowski space with signature $(-1, 1, 1, 1)$. Hodge duality is detailed for three dimensional Euclidean space and four dimensional Minkowski space. However, there is sufficient detail that one ought to be able to extrapolate to euclidean spaces of other dimension. Moreover, the Hodge duality is reduced to a few tables for computational convenience. I encourage the reader to see David Bleeker's text for a broader discussion of Hodge duality with a physical bent.

When I lecture this material I'll probably just give examples to drive home the computational aspects. Also, it should be noted this Chapter can be studied without delving deeply into Sections 9.4 and 9.6. However, Chapter 11 requires some understanding of both of those sections.

9.1 dual space

Definition 9.1.1.

Suppose V is a vector space over $\mathbb{R}$. We define the **dual space** to V to be the set of all linear functions from V to $\mathbb{R}$. In particular, we denote:

$$V^* = \{f : V \to \mathbb{R} \mid f(x+y) = f(x) + f(y) \text{ and } f(cx) = cf(x) \ \ \forall x, y \in V \text{ and } c \in \mathbb{R}\}$$

If $\alpha \in V^*$ then we say α is a **dual vector**.

I offer several abstract examples to begin, however the majority of this section concerns $\mathbb{R}^n$.

Example 9.1.2. *Suppose $\mathcal{F}$ denotes the set of continuous functions on $\mathbb{R}$. Define $\alpha(f) = \int_0^1 f(t)\,dt$. The mapping $\alpha : \mathcal{F} \to \mathbb{R}$ is linear by properties of definite integrals therefore we identify the definite integral defines a dual-vector to the vector space of continuous functions.*

Example 9.1.3. *Suppose $V = \mathcal{F}(W, \mathbb{R})$ denotes a set of functions from a vector space W to $\mathbb{R}$. Note that V is a vector space with respect to point-wise defined addition and scalar multiplication of functions. Let $w_o \in W$ and define $\alpha(f) = f(w_o)$. The mapping $\alpha : V \to \mathbb{R}$ is linear since $\alpha(cf + g) = (cf + g)(w_o) = cf(w_o) + g(w_o) = c\alpha(f) + \alpha(g)$ for all $f, g \in V$ and $c \in \mathbb{R}$. We find that the* **evaluation** *map defines a dual-vector $\alpha \in V^*$.*

Example 9.1.4. *The determinant is a mapping from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$ but it does not define a dual-vector to the vector space of square matrices since $det(A + B) \neq det(A) + det(B)$.*

Example 9.1.5. *Suppose $\alpha(x) = x \cdot v$ for a particular vector $v \in \mathbb{R}^n$. We argue $\alpha \in V^*$ where we recall $V = \mathbb{R}^n$ is a vector space. Additivity follows from a property of the dot-product on $\mathbb{R}^n$,*

$$\alpha(x + y) = (x + y) \cdot v = x \cdot v + y \cdot v = \alpha(x) + \alpha(y)$$

for all $x, y \in \mathbb{R}^n$. Likewise, homogeneity follows from another property of the dot-product: observe

$$\alpha(cx) = (cx) \cdot v = c(x \cdot v) = c\alpha(x)$$

for all $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Example 9.1.6. *Let $\alpha(x, y) = 2x + 5y$ define a function $\alpha : \mathbb{R}^2 \to \mathbb{R}$. Note that*

$$\alpha(x, y) = (x, y) \cdot (2, 5)$$

hence by the preceding example we find $\alpha \in (\mathbb{R}^2)^*$.

The preceding example is no accident. It turns out there is a one-one correspondance between row vectors and dual vectors on $\mathbb{R}^n$. Let $v \in \mathbb{R}^n$ then we define $\alpha_v(x) = x \cdot v$. We proved in Example 9.1.5 that $\alpha_v \in (\mathbb{R}^n)^*$. Suppose $\alpha \in (\mathbb{R}^n)^*$ we see to find $v \in \mathbb{R}^n$ such that $\alpha = \alpha_v$. Recall that a linear function is uniquely defined by its values on a basis; the values of α on the standard basis will show us how to choose v. This is a standard technique. Consider: $v \in \mathbb{R}^n$ with[1] $v = \sum_{j=1}^n v^j e_j$

$$\alpha(x) = \underbrace{\alpha(\sum_{j=1}^n x^j e_j) = \sum_{j=1}^n}_{additivity} \underbrace{\alpha(x^j e_j) = \sum_{j=1}^n x^j \alpha(e_j)}_{homogeneity} = x \cdot v$$

where we define $v = (\alpha(e_1), \alpha(e_2), \ldots, \alpha(e_n)) \in \mathbb{R}^n$. The vector which corresponds naturally[2] to α is simply the vector of of the values of α on the standard basis.

[1] the super-index is not a power in this context, it is just a notation to emphasize v^j is the component of a vector.

[2] some authors will say $\mathbb{R}^{n\times 1}$ is dual to $\mathbb{R}^{1\times n}$ since $\alpha_v(x) = v^T x$ and v^T is a row vector, I will avoid that langauge in these notes.

The dual space to $\mathbb{R}^n$ is a vector space and the correspondance $v \to \alpha_v$ gives an isomorphism of $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$. The image of a basis under an isomorphism is once more a basis. Define $\Phi : \mathbb{R}^n \to (\mathbb{R})^*$ by $\Phi(v) = \alpha_v$ to give the correspondance an explicit label. The image of the standard basis under Φ is called the **standard dual basis** for $(\mathbb{R}^n)^*$. Consider $\Phi(e_j)$, let $x \in \mathbb{R}^n$ and calculate

$$\Phi(e_j)(x) = \alpha_{e_j}(x) = x \cdot e_j$$

In particular, notice that when $x = e_i$ then $\Phi(e_j)(e_i) = e_i \cdot e_j = \delta_{ij}$. Dual vectors are linear transformations therefore we can define the dual basis by its values on the standard basis.

Definition 9.1.7.

The **standard dual basis** of $(\mathbb{R}^n)^*$ is denoted $\{e^1, e^2, \ldots, e^n\}$ where we define $e^j : \mathbb{R}^n \to \mathbb{R}$ to be the linear transformation such that $e^j(e_i) = \delta_{ij}$ for all $i, j \in \mathbb{N}_n$. Generally, given a vector space V with basis $\beta = \{f_1, f_2, \ldots, f_m\}$ we say the basis $\beta^* = \{f^1, f^2, \ldots, f^n\}$ is dual to β iff $f^j(f_i) = \delta_{ij}$ for all $i, j \in \mathbb{N}_n$.

The term *basis* indicates that $\{e^1, e^2, \ldots, e^n\}$ is linearly independent[3] and $span\{e^1, e^2, \ldots, e^n\} = (\mathbb{R}^n)^*$. The following calculation is often useful: if $x \in \mathbb{R}^n$ with $x = \sum_{j=1}^n x^j e_j$ then

$$e^i(x) = e^i\left(\sum_{j=1}^n x^j e_j\right) = \sum_{j=1}^n x^j e^i(e_j) = \sum_{j=1}^n x^j \delta_{ij} = x^i \quad \Rightarrow \quad \boxed{e^i(x) = x^i.}$$

The calculation above is a prototype for many that follow in this chapter. Next, suppose $\alpha \in (\mathbb{R}^n)^*$ and suppose $x \in \mathbb{R}^n$ with $x = \sum_{j=1}^n x^j e_j$. Calculate,

$$\alpha(x) = \alpha\left(\sum_{i=1}^n x^i e_i\right) = \sum_{i=1}^n \alpha(e_i) e^i(x) \quad \Rightarrow \quad \boxed{\alpha = \sum_{i=1}^n \alpha(e_i) e^i}$$

this shows every dual vector is in the span of the dual basis $\{e^j\}_{j=1}^n$.

9.2 multilinearity and the tensor product

A multilinear mapping is a function of a Cartesian product of vector spaces which is linear with respect to each "slot". The goal of this section is to explain what that means. It turns out the set of all multilinear mappings on a particular set of vector spaces forms a vector space and we'll show how the tensor product can be used to construct an explicit basis by tensoring a bases which are dual to the bases in the domain. We also examine the concepts of symmetric and antisymmetric multilinear mappings, these form interesting subspaces of the set of all multilinear mappings. Our approach in this section is to treat the case of bilinearity in depth then transition to the case of multilinearity. Naturally this whole discussion demands a familarity with the preceding section.

[3] direct proof of LI is left to the reader

9.2.1 bilinear maps

Definition 9.2.1.

Suppose V_1, V_2 are vector spaces then $b : V_1 \times V_2 \to \mathbb{R}$ is a **binear mapping** on $V_1 \times V_2$ iff for all $x, y \in V_1$, $z, w \in V_2$ and $c \in \mathbb{R}$:

$$\begin{array}{lrcll} (1.) & b(cx+y,z) & = & cb(x,z)+b(y,z) & \text{(linearity in the first slot)} \\ (2.) & b(x,cz+w) & = & cb(x,z)+b(x,w) & \text{(linearity in the second slot).} \end{array}$$

bilinear maps on $V \times V$

When $V_1 = V_2 = V$ we simply say that $b : V \times V \to \mathbb{R}$ is a **bilinear mapping on V**. The **set of all bilinear maps of V is denoted $T^2_0 V$**. You can show that $T^2_0 V$ forms a vector space under the usual point-wise defined operations of function addition and scalar multiplication[4]. Hopefully you are familar with the example below.

Example 9.2.2. *Define $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $b(x, y) = x \cdot y$ for all $x, y \in \mathbb{R}^n$. Linearity in each slot follows easily from properties of dot-products:*

$$b(cx + y, z) = (cx + y) \cdot z = cx \cdot z + y \cdot z = cb(x, z) + b(y, z)$$

$$b(x, cy + z) = x \cdot (cy + z) = cx \cdot y + x \cdot z = cb(x, y) + b(x, z).$$

We can use matrix multiplication to generate a large class of examples with ease.

Example 9.2.3. *Suppose $A \in \mathbb{R}^{n\times n}$ and define $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $b(x, y) = x^T Ay$ for all $x, y \in \mathbb{R}^n$. Observe that, by properties of matrix multiplication,*

$$b(cx + y, z) = (cx + y)^T Az = (cx^T + y^T)Az = cx^T Az + y^T Az = cb(x, z) + b(y, z)$$

$$b(x, cy + z) = x^T A(cy + z) = cx^T Ay + x^T Az = cb(x, y) + b(x, z)$$

for all $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. It follows that b is bilinear on $\mathbb{R}^n$.

Suppose $b : V \times V \to \mathbb{R}$ is bilinear and suppose $\beta = \{e_1, e_2, \ldots, e_n\}$ is a basis for V whereas

[4] sounds like homework

$\beta^* = \{e^1, e^2, \ldots, e^n\}$ is a basis of V^* with $e^j(e_i) = \delta_{ij}$

$$\begin{aligned} b(x,y) &= b\Big(\sum_{i=1}^n x^i e_i,\ \sum_{j=1}^n y^j e_j \Big) \\ &= \sum_{i,j=1}^n b(x^i e_i, y^j e_j) \\ &= \sum_{i,j=1}^n x^i y^j b(e_i, e_j) \\ &= \sum_{i,j=1}^n b(e_i, e_j) e^i(x)^j(y) \end{aligned} \tag{9.1}$$

Therefore, if we define $b_{ij} = b(e_i, e_j)$ then we may compute $b(x,y) = \sum_{i,j=1}^n b_{ij} x^i y^j$. The calculation above also indicates that b is a linear combination of certain basic bilinear mappings. In particular, b can be written a linear combination of a tensor product of dual vectors on V.

Definition 9.2.4.

Suppose V is a vector space with dual space V^*. If $\alpha, \beta \in V^*$ then we define $\alpha \otimes \beta : V \times V \to \mathbb{R}$ by $(\alpha \otimes \beta)(x,y) = \alpha(x)\beta(y)$ for all $x, y \in V$.

Given the notation[5] preceding this definition, we note $(e^i \otimes e^j)(x,y) = e^i(x)e^j(y)$ hence for all $x, y \in V$ we find:

$$b(x,y) = \sum_{i,j=1}^n b(e_i, e_j)(e^i \otimes e^j)(x,y) \quad \text{therefore,} \quad \boxed{b = \sum_{i,j=1}^n b(e_i, e_j) e^i \otimes e^j}$$

We find[6] that $T^2_0 V = span\{e^i \otimes e^j\}_{i,j=1}^n$. Moreover, it can be argued[7] that $\{e^i \otimes e^j\}_{i,j=1}^n$ is a linearly independent set, therefore $\{e^i \otimes e^j\}_{i,j=1}^n$ forms a basis for $T^2_0 V$. We can count there are n^2 vectors in $\{e^i \otimes e^j\}_{i,j=1}^n$ hence $dim(\, T^2_0 V\,) = n^2$.

If $V = \mathbb{R}^n$ and if $\{e^i\}_{i=1}^n$ denotes the standard dual basis, then there is a standard notation for the set of coefficients found in the summation for b. In particular, we denote $B = [b]$ where $B_{ij} = b(e_i, e_j)$ hence, following Equation 9.1,

$$b(x,y) = \sum_{i,j=1}^n x^i y^j b(e_i, e_j) = \sum_{i=1}^n \sum_{j=1}^n x^i B_{ij} y^j = x^T B y$$

[5] perhaps you would rather write $(e^i \otimes e^j)(x,y)$ as $e^i \otimes e^j(x,y)$, that is also fine.
[6] with the help of your homework where you will show $\{e^i \otimes e^j\}_{i,j=1}^n \subseteq T^2_0 V$
[7] yes, again, in your homework

Definition 9.2.5.

Suppose $b : V \times V \to \mathbb{R}$ is a bilinear mapping then we say:

1. b is **symmetric** iff $b(x, y) = b(y, x)$ for all $x, y \in V$
2. b is **antisymmetric** iff $b(x, y) = -b(y, x)$ for all $x, y \in V$

Any bilinear mapping on V can be written as the sum of a symmetric and antisymmetric bilinear mapping, this claim follows easily from the calculation below:

$$b(x,y) = \underbrace{\frac{1}{2}\Big(b(x,y) + b(y,x)\Big)}_{symmetric} + \underbrace{\frac{1}{2}\Big(b(x,y) - b(y,x)\Big)}_{antisymmetric}.$$

We say S_{ij} is **symmetric** in i, j iff $S_{ij} = S_{ji}$ for all i, j. Likewise, we say A_{ij} is **antisymmetric** in i, j iff $A_{ij} = -A_{ji}$ for all i, j. If S is a symmetric bilinear mapping and A is an antisymmetric bilinear mapping then the components of S are symmetric and the components of A are antisymmetric. Why? Simply note:

$$S(e_i, e_j) = S(e_j, e_i) \quad \Rightarrow \quad S_{ij} = S_{ji}$$

and

$$A(e_i, e_j) = -A(e_j, e_i) \quad \Rightarrow \quad A_{ij} = -A_{ji}.$$

You can prove that the sum or scalar multiple of an (anti)symmetric bilinear mapping is once more (anti)symmetric therefore the set of antisymmetric bilinear maps $\Lambda^2(V)$ and the set of symmetric bilinear maps ST^0_2V are subspaces of T^0_2V. The notation $\Lambda^2(V)$ is part of a larger discussion on the wedge product, we will return to it in a later section.

Finally, if we consider the special case of $V = \mathbb{R}^n$ once more we find that a bilinear mapping $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ has a symmetric matrix $[b]^T = [b]$ iff b is symmetric whereas it has an antisymmetric matric $[b]^T = -[b]$ iff b is antisymmetric.

bilinear maps on $V^* \times V^*$

Suppose $h : V^* \times V^* \to \mathbb{R}$ is bilinear then we say $h \in T^2_0 V$. In addition, suppose $\beta = \{e_1, e_2, \ldots, e_n\}$ is a basis for V whereas $\beta^* = \{e^1, e^2, \ldots, e^n\}$ is a basis of V^* with $e^j(e_i) = \delta_{ij}$. Let $\alpha, \beta \in V^*$

$$\begin{aligned} h(\alpha, \beta) &= h\Big(\sum_{i=1}^n \alpha_i e^i,\ \sum_{j=1}^n \beta_j e^j \Big) \\ &= \sum_{i,j=1}^n h(\alpha_i e^i, \beta_j e^j) \\ &= \sum_{i,j=1}^n \alpha_i \beta_j h(e^i, e^j) \\ &= \sum_{i,j=1}^n h(e^i, e^j)\alpha(e_i)\beta(e_j) \end{aligned} \tag{9.2}$$

Therefore, if we define $h^{ij} = h(e^i, e^j)$ then we find the nice formula $h(\alpha, \beta) = \sum_{i,j=1}^n h^{ij}\alpha_i\beta_j$. To further refine the formula above we need a new concept.

The dual of the dual is called the double-dual and it is denoted V^{**}. For a finite dimensional vector space there is a cannonical isomorphism of V and V^{**}. In particular, $\Phi : V \to V^{**}$ is defined by $\Phi(v)(\alpha) = \alpha(v)$ for all $\alpha \in V^*$. It is customary to replace V with V^{**} wherever the context allows. For example, to define the tensor product of two vectors $x, y \in V$ as follows:

Definition 9.2.6.

Suppose V is a vector space with dual space V^*. We define the tensor product of vectors x, y as the mapping $x \otimes y : V^* \times V^* \to \mathbb{R}$ by $(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y)$ for all $x, y \in V$.

We could just as well have defined $x \otimes y = \Phi(x) \otimes \Phi(y)$ where Φ is once more the cannonical isomorphism of V and V^{**}. It's called *cannonical* because it has no particular dependendence on the coordinates used on V. In contrast, the isomorphism of $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ was built around the dot-product and the standard basis.

All of this said, note that $\alpha(e_i)\beta(e_j) = e_i \otimes e_j(\alpha, \beta)$ thus,

$$h(\alpha, \beta) = \sum_{i,j=1}^n h(e^i, e^j) e_i \otimes e_j(\alpha, \beta) \quad \Rightarrow \quad \boxed{h = \sum_{i,j=1}^n h(e^i, e^j) e_i \otimes e_j}$$

We argue that $\{e_i \otimes e_j\}_{i,j=1}^n$ is a basis[8]

Definition 9.2.7.

[8] $T^2_0 V$ is a vector space and we've shown $T^2_0(V) \subseteq span\{e_i \otimes e_j\}_{i,j=1}^n$ but we should also show $e_i \otimes e_j \in T^2_0$ and check for LI of $\{e_i \otimes e_j\}_{i,j=1}^n$.

Suppose $h : V^* \times V^* \to \mathbb{R}$ is a bilinear mapping then we say:

1. h is **symmetric** iff $h(\alpha, \beta) = h(\beta, \alpha)$ for all $\alpha, \beta \in V^*$
2. h is **antisymmetric** iff $h(\alpha, \beta) = -h(\beta, \alpha)$ for all $\alpha, \beta \in V^*$

The discussion of the preceding subsection transfers to this context, we simply have to switch some vectors to dual vectors and move some indices up or down. I leave this to the reader.

bilinear maps on $V \times V^*$

Suppose $H : V \times V^* \to \mathbb{R}$ is bilinear, we say $H \in T^1_1 V$ (or, if the context demands this detail $H \in T_1{}^1 V$). We define $\alpha \otimes x \in T_1{}^1(V)$ by the natural rule; $(\alpha \otimes x)(y, \beta) = \alpha(x)\beta(x)$ for all $(y, \beta) \in V \times V^*$. We find, by calculations similar to those already given in this section,

$$H(y, \beta) = \sum_{i,j=1}^{n} H_i{}^j y^i \beta_j \qquad \text{and} \qquad H = \sum_{i,j=1}^{n} H_i{}^j e^i \otimes e_j$$

where we defined $H_i{}^j = H(e_i, e^j)$.

bilinear maps on $V^* \times V$

Suppose $G : V^* \times V \to \mathbb{R}$ is bilinear, we say $G \in T^1_1 V$ (or, if the context demands this detail $G \in T^1{}_1 V$). We define $x \otimes \alpha \in T^1{}_1 V$ by the natural rule; $(x \otimes \alpha)(\beta, y) = \beta(x)\alpha(y)$ for all $(\beta, y) \in V^* \times V$. We find, by calculations similar to those already given in this section,

$$G(\beta, y) = \sum_{i,j=1}^{n} G^i{}_j \beta_i y^j \qquad \text{and} \qquad G = \sum_{i,j=1}^{n} G^i{}_j e_i \otimes e^j$$

where we defined $G^i{}_j = G(e^i, e_j)$.

9.2.2 trilinear maps

Definition 9.2.8.

Suppose V_1, V_2, V_3 are vector spaces then $T : V_1 \times V_2 \times V_3 \to \mathbb{R}$ is a **trilinear mapping** on $V_1 \times V_2 \times V_3$ iff for all $u, v \in V_1$, $w, x \in V_2$. $y, z \in V_3$ and $c \in \mathbb{R}$:

$$\begin{aligned}
&(1.)\ T(cu + v, w, y) &&= cT(u, w, y) + T(v, w, y) && \text{(linearity in the first slot)} \\
&(2.)\ T(u, cw + x, y) &&= cT(u, w, y) + T(u, x, y) && \text{(linearity in the second slot).} \\
&(3.)\ T(u, w, cy + z) &&= cT(u, w, y) + T(u, w, z) && \text{(linearity in the third slot).}
\end{aligned}$$

If $T : V \times V \times V \to \mathbb{R}$ is trilinear on $V \times V \times V$ then we say T **is a trilinear mapping on** V and we denote the set of all such mappings $T^0_3 V$. The tensor product of three dual vectors is defined much in the same way as it was for two,

$$(\alpha \otimes \beta \otimes \gamma)(x, y, z) = \alpha(x)\beta(y)\gamma(z)$$

Let $\{e_i\}_{i=1}^n$ is a basis for V with dual basis $\{e^i\}_{i=1}^n$ for V^*. If T is trilinear on V it follows

$$T(x, y, z) = \sum_{i,j,k=1}^{n} T_{ijk} x^i y^j z^k \qquad \text{and} \qquad T = \sum_{i,j,k=1}^{n} T_{ijk} e^i \otimes e^j \otimes e^k$$

where we defined $T_{ijk} = T(e_i, e_j, e_k)$ for all $i, j, k \in \mathbb{N}_n$.

Generally suppose that V_1, V_2, V_3 are possibly distinct vector spaces. Moreover, suppose V_1 has basis $\{e_i\}_{i=1}^{n_1}$, V_2 has basis $\{f_j\}_{j=1}^{n_2}$ and V_3 has basis $\{g_k\}_{k=1}^{n_3}$. Denote the dual bases for V_1^*, V_2^*, V_3^* in the usual fashion: $\{e^i\}_{i=1}^{n_1}$, $\{f^j\}_{j=1}^{n_1}$, $\{g^k\}_{k=1}^{n_1}$. With this notation, we can write a trilinear mapping on $V_1 \times V_2 \times V_3$ as follows: (where we define $T_{ijk} = T(e_i, f_j, g_k)$)

$$T(x, y, z) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} T_{ijk} x^i y^j z^k \qquad \text{and} \qquad T = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} T_{ijk} e^i \otimes f^j \otimes g^k$$

However, if V_1, V_2, V_3 happen to be related by duality then it is customary to use up/down indices. For example, if $T : V \times V \times V^* \to \mathbb{R}$ is trilinear then we write[9]

$$T = \sum_{i,j,k=1}^{n} T_{ij}{}^{k} e^i \otimes e^j \otimes e_k$$

and say $T \in T_2\ {}^1V$. On the other hand, if $S : V^* \times V^* \times V$ is trilinear then we'd write

$$T = \sum_{i,j,k=1}^{n} S^{ij}{}_{k} e_i \otimes e_j \otimes e^k$$

and say $T \in T^2{}_1 V$. I'm not sure that I've ever seen this notation elsewhere, but perhaps it could be useful to denote the set of trinlinear maps $T : V \times V^* \times V \to \mathbb{R}$ as $T_1{}^1{}_1 V$. Hopefully we will not need such silly notation in what we consider this semester.

There was a natural correspondance between bilinear maps on $\mathbb{R}^n$ and square matrices. For a trilinear map we would need a three-dimensional array of components. In some sense you could picture $T : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as multiplication by a cube of numbers. Don't think too hard about these silly comments, we actually already wrote the useful formulae for dealing with trilinear objects. Let's stop to look at an example.

[9] we identify e_k with its double-dual hence this tensor product is already defined, but to be safe let me write it out in this context $e^i \otimes e^j \otimes e_k(x, y, \alpha) = e^i(x)e^j(y)\alpha(e_k)$.

Example 9.2.9. *Define $T : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $T(x, y, z) = det(x|y|z)$. You may not have learned this in your linear algebra course[10] but a nice formula[11] for the determinant is given by the Levi-Civita symbol,*

$$det(A) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} A_{i1} A_{j2} A_{k3}$$

note that $col_1(A) = [A_{i1}], col_2(A) = [A_{i2}]$ and $col_3(A) = [A_{i3}]$. It follows that

$$T(x, y, z) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} x^i y^j z^k$$

Multilinearity follows easily from this formula. For example, linearity in the third slot:

$$\begin{aligned} T(x, y, cz + w) &= \sum_{i,j,k=1}^{3} \epsilon_{ijk} x^i y^j (cz + w)^k && (9.3)\\ &= \sum_{i,j,k=1}^{3} \epsilon_{ijk} x^i y^j (cz^k + w^k) && (9.4)\\ &= c \sum_{i,j,k=1}^{3} \epsilon_{ijk} x^i y^j z^k + \sum_{i,j,k=1}^{3} \epsilon_{ijk} x^i y^j w^k && (9.5)\\ &= cT(x, y, z) + T(x, y, w). && (9.6) \end{aligned}$$

Observe that by properties of determinants, or the Levi-Civita symbol if you prefer, swapping a pair of inputs generates a minus sign, hence:

$$T(x, y, z) = -T(y, x, z) = T(y, z, x) = -T(z, y, x) = T(z, x, y) = -T(x, z, y).$$

If $T : V \times V \times V \to \mathbb{R}$ is a trilinear mapping such that

$$T(x, y, z) = -T(y, x, z) = T(y, z, x) = -T(z, y, x) = T(z, x, y) = -T(x, z, y)$$

for all $x, y, z \in V$ then we say T **is antisymmetric**. Likewise, if $S : V \times V \times V \to \mathbb{R}$ is a trilinear mapping such that

$$S(x, y, z) = -S(y, x, z) = S(y, z, x) = -S(z, y, x) = S(z, x, y) = -S(x, z, y).$$

for all $x, y, z \in V$ then we say T **is symmetric**. Clearly the mapping defined by the determinant is antisymmetric. In fact, many authors define the determinant of an $n \times n$ matrix as the antisymmetric n-linear mapping which sends the identity matrix to 1. It turns out these criteria unquely

[10] maybe you haven't even taken linear yet!

[11] actually, I take this as the definition in linear algebra, it does take considerable effort to recover the expansion by minors formula which I use for concrete examples

define the determinant. That is the motivation behind my Levi-Civita symbol definition. That formula is just the nuts and bolts of complete antisymmetry.

You might wonder, can every trilinear mapping can be written as a the sum of a symmetric and antisymmetric mapping? The answer is no. Consider $T : V \times V \times V \to \mathbb{R}$ defined by $T = e^1 \otimes e^2 \otimes e^3$. Is it possible to find constants a, b such that:

$$e^1 \otimes e^2 \otimes e^3 = ae^{[1} \otimes e^2 \otimes e^{3]} + be^{(1} \otimes e^2 \otimes e^{3)}$$

where $[\dots]$ denotes complete antisymmetrization of $1, 2, 3$ and $(\dots)$ complete symmetrization:

$$e^{[1} \otimes e^2 \otimes e^{3]} = \frac{1}{6}\left[e^{123} + e^{231} + e^{312} - e^{321} - e^{213} - e^{132}\right]$$

For the symmetrization we also have to include all possible permutations of $(1, 2, 3)$ but all with $+$:

$$e^{(1} \otimes e^2 \otimes e^{3)} = \frac{1}{6}\left[e^{123} + e^{231} + e^{312} + e^{321} + e^{213} + e^{132}\right]$$

As you can see:

$$ae^{[1} \otimes e^2 \otimes e^{3]} + be^{(1} \otimes e^2 \otimes e^{3)} = \frac{a+b}{6}(e^{123} + e^{231} + e^{312}) + \frac{b-a}{6}(e^{321} + e^{213} + e^{132})$$

There is no way for these to give back only $e^1 \otimes e^2 \otimes e^3$. I leave it to the reader to fill the gaps in this argument. Generally, the decomposition of a multilinear mapping into more basic types is a problem which requires much more thought than we intend here. Representation theory does address this problem: how can we decompose a tensor product into irreducible pieces. Their idea of tensor product is not precisely the same as ours, however algebraically the problems are quite intertwined. I'll leave it at that unless you'd like to do an independent study on representation theory. Ideally you'd already have linear algebra and abstract algebra complete before you attempt that study.

9.2.3 multilinear maps

Definition 9.2.10.

Suppose $V_1, V_2, \dots V_k$ are vector spaces then $T : V_1 \times V_2 \times \cdots \times V_k \to \mathbb{R}$ is a **k-multilinear mapping** on $V_1 \times V_2 \times \cdots \times V_k$ iff for each $c \in \mathbb{R}$ and $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, $\dots$, $x_k, y_k \in V_k$

$$T(x_1, \dots, cx_j + y_j, \dots, x_k) = cT(x_1, \dots, x_j, \dots, x_k) + T(x_1, \dots, y_j, \dots, x_k)$$

for $j = 1, 2, \dots, k$. In other words, we assume T is linear in each of its k-slots. If T is multilinear on $V^r \times (V^*)^s$ then we say that $T \in T^s_r V$ and we say T **is a type** (r, s) **tensor on** V.

The definition above makes a dual vector a type $(1, 0)$ tensor whereas a double dual of a vector a type $(0, 1)$ tensor, a bilinear mapping on V is a type $(2, 0)$ tensor and a bilinear mapping on V^* is

a type $(0,2)$ tensor with respect to V.

We are free to define tensor products in this context in the same manner as we have previously. Suppose $\alpha_1 \in V_1^*, \alpha_2 \in V_2^*, \ldots, \alpha_k \in V_k^*$ and $v_1 \in V_1, v_2 \in V_2, \ldots, v_k \in V_k$ then

$$\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_k(v_1, v_2, \ldots, v_k) = \alpha_1(v_1)\alpha_2(v_2)\cdots\alpha_k(v_k)$$

It is easy to show the tensor produce of k-dual vectors as defined above is indeed a k-multilinear mapping. Moreover, the set of all k-multilinear mappings on $V_1 \times V_2 \times \cdots \times V_k$ clearly forms a vector space of dimension $dim(V_1)dim(V_2)\cdots dim(V_k)$ since it naturally takes the tensor product of the dual bases for $V_1^*, V_2^*, \ldots, V_k^*$ as its basis. In particular, suppose for $j = 1, 2, \ldots, k$ that V_j has basis $\{E_{ji}\}_{i=1}^{n_j}$ which is dual to $\{E_j^i\}_{i=1}^{n_j}$ the basis for V_j^*. Then we can derive that a k-multilinear mapping can be written as

$$T = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_k=1}^{n_k} T_{i_1 i_2 \ldots i_k} E_1^{i_1} \otimes E_2^{i_2} \otimes E_k^{i_k}$$

If T is a type (r,s) tensor on V then there is no need for the ugly double indexing on the basis since we need only tensor a basis $\{e_i\}_{i=1}^n$ for V and its dual $\{e^i\}_{i=1}^n$ for V^* in what follows:

$$T = \sum_{i_1,\ldots,i_r=1}^{n} \sum_{j_1,\ldots,j_s=1}^{n} T_{i_1 i_2 \ldots i_r}^{j_1 j_2 \ldots j_s} e^{i_1} \otimes e^{i_2} \otimes \cdots \otimes e^{i_r} \otimes e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_s}.$$

permutations

Before I define symmetric and antisymmetric for k-linear mappings on V I think it is best to discuss briefly some ideas from the theory of permutations.

Definition 9.2.11.

A permutation on $\{1, 2, \ldots p\}$ is a bijection on $\{1, 2, \ldots p\}$. We define the set of permutations on $\{1, 2, \ldots p\}$ to be Σ_p. Further, define the sign of a permutation to be $sgn(\sigma) = 1$ if σ is the product of an even number of transpositions whereas $sgn(\sigma) = -1$ if σ is the product of a odd number transpositions.

Let us consider the set of permutations on $\{1, 2, 3, \ldots n\}$, this is called S_n the symmetric group, its order is $n!$ if you were wondering. Let me remind[12] you how the cycle notation works since it allows us to explicitly present the number of transpositions contained in a permutation,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{pmatrix} \quad \Longleftrightarrow \quad \sigma = (12)(356) = (12)(36)(35) \tag{9.7}$$

recall the cycle notation is to be read right to left. If we think about inputing 5 we can read from the matrix notation that we ought to find $5 \mapsto 6$. Clearly that is the case for the first version of

[12] or perhaps, more likely, introduce you to this notation

σ written in cycle notation; (356) indicates that $5 \mapsto 6$ and nothing else messes with 6 after that. Then consider feeding 5 into the version of σ written with just two-cycles (a.k.a. transpositions), first we note (35) indicates $5 \mapsto 3$, then that 3 hits (36) which means $3 \mapsto 6$, finally the cycle (12) doesn't care about 6 so we again have that $\sigma(5) = 6$. Finally we note that $sgn(\sigma) = -1$ since it is made of 3 transpositions.

It is always possible to write any permutation as a product of transpositions, such a decomposition is not unique. However, if the number of transpositions is even then it will remain so no matter how we rewrite the permutation. Likewise if the permutation is an product of an odd number of transpositions then any other decomposition into transpositions is also comprised of an odd number of transpositions. This is why we can **define** an **even** permutation is a permutation comprised by an even number of transpositions and an **odd** permutation is one comprised of an odd number of transpositions.

Example 9.2.12. Sample cycle calculations: *we rewrite as product of transpositions to determin if the given permutation is even or odd,*

$$\sigma = (12)(134)(152) = (12)(14)(13)(12)(15) \quad \Longrightarrow \quad sgn(\sigma) = -1$$

$$\lambda = (1243)(3521) = (13)(14)(12)(31)(32)(35) \quad \Longrightarrow \quad sgn(\lambda) = 1$$

$$\gamma = (123)(45678) = (13)(12)(48)(47)(46)(45) \quad \Longrightarrow \quad sgn(\gamma) = 1$$

We will not actually write down permutations in the calculations the follow this part of the notes. I merely include this material as to give a logically complete account of antisymmetry. In practice, if you understood the terms as the apply to the bilinear and trilinear case it will usually suffice for concrete examples. Now we are ready to define symmetric and antisymmetric.

Definition 9.2.13.

A k-linear mapping $L : V \times V \times \cdots \times V \to \mathbb{R}$ is **completely symmetric** if

$$L(x_1, \ldots, x, \ldots, y, \ldots, x_k) = L(x_1, \ldots, y, \ldots, x, \ldots, x_k)$$

for all possible $x, y \in V$. Conversely, if a k-linear mapping on V has

$$L(x_1, \ldots, x, \ldots, y, \ldots, x_p) = -L(x_1, \ldots, y, \ldots, x, \ldots, x_p)$$

for all possible pairs $x, y \in V$ then it is said to be **completely antisymmetric or alternating**. Equivalently a k-linear mapping L is alternating if for all $\pi \in \Sigma_k$

$$L(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_k}) = sgn(\pi) L(x_1, x_2, \ldots, x_k)$$

The set of alternating multilinear mappings on V is denoted ΛV, the set of k-linear alternating maps on V is denoted $\Lambda^k V$. Often an alternating k-linear map is called a **k-form**. Moreover, we say the **degree** of a k-form is k.

Similar terminology applies to the components of tensors. We say $T_{i_1 i_2 \ldots i_k}$ is completely symmetric in $i_1, i_2, \ldots, i_k$ iff $T_{i_1 i_2 \ldots i_k} = T_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(k)}}$ for all $\sigma \in \Sigma_k$. On the other hand, $T_{i_1 i_2 \ldots i_k}$ is completely antisymmetric in $i_1, i_2, \ldots, i_k$ iff $T_{i_1 i_2 \ldots i_k} = sgn(\sigma) T_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(k)}}$ for all $\sigma \in \Sigma_k$. It is a simple exercise to show that a completely (anti)symmetric tensor[13] has completely (anti)symmetric components.

The tensor product is an interesting construction to discuss at length. To summarize, it is associative and distributive across addition. Scalars factor out and it is not generally commutative. For a given vector space V we can in principle generate by tensor products multilinear mappings of arbitrarily high order. This tensor algebra is infinite dimensional. In contrast, the space ΛV of forms on V is a finite-dimensional subspace of the tensor algebra. We discuss this next.

9.3 wedge product

We assume V is a vector space with basis $\{e_i\}_{i=1}^n$ throughout this section. The dual basis is denoted $\{e^i\}_{i=1}^n$ as is our usual custom. Our goal is to find a basis for the alternating maps on V and explore the structure implicit within its construction. This will lead us to call ΛV the **exterior algebra** of V after the discussion below is complete.

9.3.1 wedge product of dual basis generates basis for ΛV

Suppose $b : V \times V \to \mathbb{R}$ is antisymmetric and $b = \sum_{i.j=1}^n b_{ij} e^i \otimes e^j$, it follows that $b_{ij} = -b_{ji}$ for all $i, j \in \mathbb{N}_n$. Notice this implies that $b_{ii} = 0$ for $i = 1, 2, \ldots, n$. For a given pair of indices i, j either $i < j$ or $j < i$ or $i = j$ hence,

$$\begin{aligned} b &= \sum_{i<j} b_{ij} e^i \otimes e^j + \sum_{j<i} b_{ij} e^i \otimes e^j + \sum_{i=j} b_{ij} e^i \otimes e^j \\ &= \sum_{i<j} b_{ij} e^i \otimes e^j + \sum_{j<i} b_{ij} e^i \otimes e^j \\ &= \sum_{i<j} b_{ij} e^i \otimes e^j - \sum_{j<i} b_{ji} e^i \otimes e^j \\ &= \sum_{k<l} b_{kl} e^k \otimes e^l - \sum_{k<l} b_{kl} e^l \otimes e^k \\ &= \sum_{k<l} b_{kl} (e^k \otimes e^l - e^l \otimes e^k). \end{aligned} \tag{9.8}$$

Therefore, $\{e^k \otimes e^l - e^l \otimes e^k | l, k \in \mathbb{N}_n \text{ and } l < k\}$ spans the set of antisymmetric bilinear maps on V. Moreover, you can show this set is linearly independent hence it is a basis fo $\Lambda^2 V$. We define

[13] in this context a tensor is simply a multilinear mapping, in physics there is more attached to the term

the wedge product of $e^k \wedge e^l = e^k \otimes e^l - e^l \otimes e^k$. With this notation we find that the alternating bilinear form b can be written as

$$\boxed{b = \sum_{k<l} b_{kl} e^k \wedge e^l = \sum_{i,j=1}^{n} \frac{1}{2} b_{ij} e^i \wedge e^j}$$

where the summation on the r.h.s. is over all indices[14]. Notice that $e^i \wedge e^j$ is an antisymmetric bilinear mapping because $e^i \wedge e^j(x,y) = -e^i \wedge e^j(y,x)$, however, there is more structure here than just that. It is also true that $e^i \wedge e^j = -e^j \wedge e^i$. This is a conceptually different antisymmetry, it is the antisymmetry of the wedge produce $\wedge$.

Suppose $b : V \times V \times V \to \mathbb{R}$ is antisymmetric and $b = \sum_{i,j,k=1}^{n} b_{ijk} e^i \otimes e^j \otimes e^k$, it follows that $b_{ijk} = b_{jki} = b_{kij}$ and $b_{ijk} = -b_{kji} = -b_{jik} = b_{ikj}$ for all $i, j, k \in \mathbb{N}_n$. Notice this implies that $b_{iii} = 0$ for $i = 1, 2, \ldots, n$. A calculation similar to the one just offered for the case of a bilinear map reveals that we can write b as follows:

$$b = \sum_{i<j<k} b_{ijk} \Big(e^i \otimes e^j \otimes e^k + e^j \otimes e^k \otimes e^i + e^k \otimes e^i \otimes e^j \\ - e^k \otimes e^j \otimes e^i - e^j \otimes e^i \otimes e^k - e^i \otimes e^k \otimes e^j \Big) \tag{9.9}$$

Define $e^i \wedge e^j \wedge e^k = e^i \otimes e^j \otimes e^k + e^j \otimes e^k \otimes e^i + e^k \otimes e^i \otimes e^j - e^k \otimes e^j \otimes e^i - e^j \otimes e^i \otimes e^k - e^i \otimes e^k \otimes e^j$ thus

$$\boxed{b = \sum_{i<j<k} b_{ijk} e^i \wedge e^j \wedge e^k = \sum_{i,j,k=1}^{n} \frac{1}{3!} b_{ijk} e^i \wedge e^j \wedge e^k} \tag{9.10}$$

and it is clear that $\{e^i \wedge e^j \wedge e^k \mid i, j, k \in \mathbb{N}_n \text{ and } i < j < k\}$ forms a basis for the set of alternating trilinear maps on V.

Following the patterns above, we define the wedge product of p dual basis vectors,

$$e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} = \sum\nolimits_{\pi \in \Sigma_p} sgn(\pi) e^{i_{\pi(1)}} \otimes e^{i_{\pi(2)}} \otimes \cdots \otimes e^{i_{\pi(p)}} \tag{9.11}$$

If $x, y \in V$ we would like to show that

$$e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}(\ldots, x, \ldots, y, \ldots) = -e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}(\ldots, y, \ldots, x, \ldots) \tag{9.12}$$

follows from the complete antisymmetrization in the definition of the wedge product. Before we give the general argument, let's see how this works in the trilinear case. Consider, $e^i \wedge e^j \wedge e^k =$

$$= e^i \otimes e^j \otimes e^k + e^j \otimes e^k \otimes e^i + e^k \otimes e^i \otimes e^j - e^k \otimes e^j \otimes e^i - e^j \otimes e^i \otimes e^k - e^i \otimes e^k \otimes e^j.$$

[14] yes there is something to work out here, probably in your homework

Calculate, noting that $e^i \otimes e^j \otimes e^k(x,y,z) = e^i(x)e^j(y)e^k(z) = x^i y^j z^k$ hence

$$e^i \wedge e^j \wedge e^k(x,y,z) = x^i y^j z^k + x^j y^k z^i + x^k y^i z^j - x^k y^j z^i - x^j y^i z^k - x^i y^k z^j$$

Thus,

$$e^i \wedge e^j \wedge e^k(x,z,y) = x^i z^j y^k + x^j z^k y^i + x^k z^i y^j - x^k z^j y^i - x^j z^i y^k - x^i z^k y^j$$

and you can check that $e^i \wedge e^j \wedge e^k(x,y,z) = -e^i \wedge e^j \wedge e^k(x,z,y)$. Similar tedious calculations prove antisymmetry of the the interchange of the first and second or the first and third slots. Therefore, $e^i \wedge e^j \wedge e^k$ is an alternating trilinear map as it is clearly trilinear since it is built from the sum of tensor products which we know are likewise trilinear.

The multilinear case follows essentially the same argument, note

$$e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}(\ldots, x_j, \ldots, x_k, \ldots) = \sum\nolimits_{\pi \in \Sigma_p} sgn(\pi) x_1^{i_{\pi(1)}} \cdots x_j^{i_{\pi(j)}} \cdots x_k^{i_{\pi(k)}} \cdots x_p^{i_{\pi(p)}} \tag{9.13}$$

whereas,

$$e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}(\ldots, x_k, \ldots, x_j, \ldots) = \sum\nolimits_{\sigma \in \Sigma_p} sgn(\sigma) x_1^{i_{\sigma(1)}} \cdots x_k^{i_{\sigma(k)}} \cdots x_j^{i_{\sigma(j)}} \cdots x_p^{i_{\sigma(p)}}. \tag{9.14}$$

Suppose we take each permutation σ and subsitute $\delta \in \Sigma_p$ such that $\sigma(j) = \delta(k)$ and $\sigma(k) = \delta(j)$ and otherwise δ and σ agree. In cycle notation, $\delta(jk) = \sigma$. Substitution δ into Equation 9.14:

$$\begin{aligned} e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}&(\ldots, x_k, \ldots, x_j, \ldots) \\ &= \sum\nolimits_{\delta \in \Sigma_p} sgn(\delta(jk)) x_1^{i_{\delta(1)}} \cdots x_k^{i_{\delta(j)}} \cdots x_j^{i_{\delta(k)}} \cdots x_p^{i_{\delta(p)}} \\ &= -\sum\nolimits_{\delta \in \Sigma_p} sgn(\delta) x_1^{i_{\delta(1)}} \cdots x_j^{i_{\delta(k)}} \cdots x_k^{i_{\delta(j)}} \cdots x_p^{i_{\delta(p)}} \\ &= -e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}(\ldots, x_j, \ldots, x_k, \ldots) \end{aligned} \tag{9.15}$$

Here the sgn of a permutation σ is $(-1)^N$ where N is the number of cycles in σ. We observed that $\delta(jk)$ has one more cycle than δ hence $sgn(\delta(jk)) = -sgn(\delta)$. Therefore, we have shown that $e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} \in \Lambda^p V$.

Recall that $e^i \wedge e^j = -e^j \wedge e^i$ in the $p = 2$ case. There is a generalization of that result to the $p > 2$ case. In words, the wedge product is antisymetric with respect the interchange of any two dual vectors. For $p = 3$ we have the following identities for the wedge product:

$$e^i \wedge e^j \wedge e^k = -\underbrace{e^j \wedge e^i}_{swapped} \wedge e^k = e^j \wedge \underbrace{e^k \wedge e^i}_{swapped} = -\underbrace{e^k \wedge e^j}_{swapped} \wedge e^i = e^k \wedge \underbrace{e^i \wedge e^j}_{swapped} = -\underbrace{e^i \wedge e^k}_{swapped} \wedge e^j$$

I've indicated how these signs are consistent with the $p = 2$ antisymmetry. Any permutation of the dual vectors can be thought of as a combination of several transpositions. In any event, it is sometimes useful to just know that the wedge product of three elements is invariant under **cyclic** permutations of the dual vectors,

$$e^i \wedge e^j \wedge e^k = e^j \wedge e^k \wedge e^i = e^k \wedge e^i \wedge e^j$$

and changes by a sign for **anticyclic** permutations of the given object,

$$e^i \wedge e^j \wedge e^k = -e^j \wedge e^i \wedge e^k = -e^k \wedge e^j \wedge e^i = -e^i \wedge e^k \wedge e^j$$

Generally we can argue that, for any permutation $\pi \in \Sigma_p$:

$$\boxed{e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} = sgn(\pi) e^{i_{\pi(1)}} \wedge e^{i_{\pi(2)}} \wedge \cdots \wedge e^{i_{\pi(p)}}}$$

This is just a slick formula which says the wedge product generates a minus whenever you flip two dual vectors which are wedged.

9.3.2 the exterior algebra

The careful reader will realize we have yet to define wedge products of anything except for the dual basis. But, naturally you must wonder if we can take the wedge product of other dual vectors or morer generally alternating tensors. The answer is yes. Let us define the general wedge product:

Definition 9.3.1. *Suppose $\alpha \in \Lambda^p V$ and $\beta \in \Lambda^q V$. We define $\mathcal{I}_p$ to be the set of all increasing lists of p-indices, this set can be empty if $dim(V)$ is not sufficiently large. Moreover, if $I = (i_1, i_2, \ldots, i_p)$ then introduce notation $e^I = e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}$ hence:*

$$\alpha = \sum_{i_1, i_2, \ldots, i_p = 1}^{n} \frac{1}{p!} \alpha_{i_1 i_2 \ldots i_p} e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} = \sum_I \frac{1}{p!} \alpha_I e^I = \sum_{I \in \mathcal{I}_p} \alpha_I e^I$$

and

$$\beta = \sum_{j_1, j_2, \ldots, j_q = 1}^{n} \frac{1}{q!} \beta_{j_1 j_2 \ldots j_q} e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_q} = \sum_J \frac{1}{q!} \beta_J e^J = \sum_{J \in \mathcal{I}_q} \beta_J e^J$$

Naturally, $e^I \wedge e^J = e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} \wedge e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_q}$ and we defined this carefully in the preceding subsection. Define $\alpha \wedge \beta \in \Lambda^{p+q} V$ as follows:

$$\alpha \wedge \beta = \sum_I \sum_J \frac{1}{p!q!} \alpha_I \beta_J e^I \wedge e^J.$$

Again, but with less slick notation:

$$\alpha \wedge \beta = \sum_{i_1, i_2, \ldots, i_p = 1}^{n} \sum_{j_1, j_2, \ldots, j_q = 1}^{n} \frac{1}{p!q!} \alpha_{i_1 i_2 \ldots i_p} \beta_{j_1 j_2 \ldots j_q} e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} \wedge e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_q}$$

All the definition above really says is that we extend the wedge product on the basis to distribute over the addition of dual vectors. What this means calculationally is that the wedge product obeys the usual laws of addition and scalar multiplication. The one feature that is perhaps foreign is the antisymmetry of the wedge product. We must take care to maintain the order of expressions since the wedge product is not generally commutative.

Proposition 9.3.2.

Let α, β, γ be forms on V and $c \in \mathbb{R}$ then

(i)	$(\alpha+\beta)\wedge\gamma = \alpha\wedge\gamma + \beta\wedge\gamma$	distributes across vector addition
(ii)	$\alpha\wedge(\beta+\gamma) = \alpha\wedge\beta + \alpha\wedge\gamma$	distributes across vector addition
(iii)	$(c\alpha)\wedge\beta = \alpha\wedge(c\beta) = c(\alpha\wedge\beta)$	scalars factor out
(iv)	$\alpha\wedge(\beta\wedge\gamma) = (\alpha\wedge\beta)\wedge\gamma$	associativity

I leave the proof of this proposition to the reader.

Proposition 9.3.3. *graded commutivity of homogeneous forms.*

Let α, β be forms on V of degree p and q respectively then

$$\alpha\wedge\beta = -(-1)^{pq}\beta\wedge\alpha$$

Proof: suppose $\alpha = \sum_I \frac{1}{p!}e^I$ is a p-form on V and $\beta = \sum_J \frac{1}{q!}e^J$ is a q-form on V. Calculate:

$$\begin{aligned}
\alpha\wedge\beta &= \sum_I\sum_J \frac{1}{p!q!}\alpha_I\beta_J e^I\wedge e^J && \text{by defn. of } \wedge,\\
&= \sum_I\sum_J \frac{1}{p!q!}\beta_J\alpha_I e^I\wedge e^J && \text{coefficients are scalars,}\\
&= (-1)^{pq}\sum_I\sum_J \frac{1}{p!q!}\beta_J\alpha_I e^J\wedge e^I && \text{(details on sign given below)}\\
&= (-1)^{pq}\beta\wedge\alpha
\end{aligned}$$

Let's expand in detail why $e^J\wedge e^I = (-1)^{pq}e^I\wedge e^J$. Suppose $I = (i_1, i_2, \ldots, i_p)$ and $J = (j_1, j_2, \ldots, j_q)$, the key is that every swap of dual vectors generates a sign:

$$\begin{aligned}
e^I\wedge e^J &= e^{i_1}\wedge e^{i_2}\wedge\cdots\wedge e^{i_p}\wedge e^{j_1}\wedge e^{j_2}\wedge\cdots\wedge e^{j_q}\\
&= (-1)^q e^{i_1}\wedge e^{i_2}\wedge\cdots\wedge e^{i_{p-1}}\wedge e^{j_1}\wedge e^{j_2}\wedge\cdots\wedge e^{j_q}\wedge e^{i_p}\\
&= (-1)^q(-1)^q e^{i_1}\wedge e^{i_2}\wedge\cdots\wedge e^{i_{p-2}}\wedge e^{j_1}\wedge e^{j_2}\wedge\cdots\wedge e^{j_q}\wedge e^{i_{p-1}}\wedge e^{i_p}\\
&\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&= \underbrace{(-1)^q(-1)^q\cdots(-1)^q}_{p-factors} e^{j_1}\wedge e^{j_2}\wedge\cdots\wedge e^{j_q}\wedge e^{i_1}\wedge e^{j_2}\wedge\cdots\wedge e^{i_p}\\
&= (-1)^{pq}e^J\wedge e^I. \qquad\qquad \square
\end{aligned}$$

Example 9.3.4. *Let α be a 2-form defined by*

$$\alpha = ae^1\wedge e^2 + be^2\wedge e^3$$

And let β be a 1-form defined by

$$\beta = 3e^1$$

Consider then,

$$\begin{aligned} \alpha \wedge \beta &= (ae^1 \wedge e^2 + be^2 \wedge e^3) \wedge (3e^1) \\ &= (3ae^1 \wedge e^2 \wedge e^1 + 3be^2 \wedge e^3 \wedge e^1 \\ &= 3be^1 \wedge e^2 \wedge e^3. \end{aligned} \tag{9.16}$$

whereas,

$$\begin{aligned} \beta \wedge \alpha &= 3e^1 \wedge (ae^1 \wedge e^2 + be^2 \wedge e^3) \\ &= (3ae^1 \wedge e^1 \wedge e^2 + 3be^1 \wedge e^2 \wedge e^3 \\ &= 3be^1 \wedge e^2 \wedge e^3. \end{aligned} \tag{9.17}$$

so this agrees with the proposition, $(-1)^{pq} = (-1)^2 = 1$ so we should have found that $\alpha \wedge \beta = \beta \wedge \alpha$. This illustrates that although the wedge product is antisymmetric on the basis, it is not always antisymmetric, in particular it is commutative for even forms.

The graded commutivity rule $\alpha \wedge \beta = -(-1)^{pq}\beta \wedge \alpha$ has some suprising implications. This rule is ultimately the reason ΛV is finite dimensional. Let's see how that happens.

Proposition 9.3.5. *linear dependent one-forms wedge to zero:*

If $\alpha, \beta \in V^*$ and $\alpha = c\beta$ for some $c \in \mathbb{R}$ then $\alpha \wedge \beta = 0$.

Proof: to begin, note that $\beta \wedge \beta = -\beta \wedge \beta$ hence $2\beta \wedge \beta = 0$ and it follows that $\beta \wedge \beta = 0$. Note:

$$\alpha \wedge \beta = c\beta \wedge \beta = c(0) = 0$$

therefore the proposition is true. □

Proposition 9.3.6.

Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_p$ are linearly dependent 1-forms then

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p = 0.$$

Proof: by assumption of linear dependence there exist constants $c_1, c_2, \ldots, c_p$ not all zero such that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots c_p\alpha_p = 0.$$

Suppose that c_k is a nonzero constant in the sum above, then we may divide by it and consequently we can write α_k in terms of all the other 1-forms,

$$\alpha_k = \frac{-1}{c_k}\Big(c_1\alpha_1 + \cdots + c_{k-1}\alpha_{k-1} + c_{k+1}\alpha_{k+1} + \cdots + c_p\alpha_p\Big)$$

Insert this sum into the wedge product in question,

$$\begin{aligned}\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_p &= \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k \wedge \cdots \wedge \alpha_p \\ &= (-c_1/c_k)\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_1 \wedge \cdots \wedge \alpha_p \\ &\quad +(-c_2/c_k)\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_2 \wedge \cdots \wedge \alpha_p + \cdots \\ &\quad +(-c_{k-1}/c_k)\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{k-1} \wedge \cdots \wedge \alpha_p \\ &\quad +(-c_{k+1}/c_k)\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{k+1} \wedge \cdots \wedge \alpha_p + \cdots \\ &\quad +(-c_p/c_k)\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p \wedge \cdots \wedge \alpha_p \\ &= 0.\end{aligned} \tag{9.18}$$

We know all the wedge products are zero in the above because in each there is at least one 1-form repeated, we simply permute the wedge products till they are adjacent and by the previous proposition the term vanishes. The proposition follows. □

Let us pause to reflect on the meaning of the proposition above for a n-dimensional vector space V. The dual space V^* is likewise n-dimensional, this is a general result which applies to all finite-dimensional vector spaces[15]. Thus, any set of more than n dual vectors is necessarily linearly dependent. Consquently, using the proposition above, we find the wedge product of more than n one-forms is trivial. Therefore, while it is possible to construct $\Lambda^k V$ for $k > n$ we should understand that this space only contains zero. The highest degree of a nontrivial form over a vector space of dimension n is an n-form.

Moreover, we can use the proposition to deduce the dimension of a basis for $\Lambda^p V$, it must consist of the wedge product of distinct linearly independent one-forms. The number of ways to choose p distinct objects from a list of n distinct objects is precisely "n choose p",

$$\binom{n}{p} = \frac{n!}{(n-p)!p!} \qquad \text{for } 0 \leq p \leq n. \tag{9.19}$$

Proposition 9.3.7.

If V is an n-dimensional vector space then $\Lambda^k V$ is an $\binom{n}{p}$-dimensional vector space of p-forms. Moreover, the direct sum of all forms over V has the structure

$$\Lambda V = \mathbb{R} \oplus \Lambda^1 V \oplus \cdots \Lambda^{n-1} V \oplus \Lambda^n V$$

and is a vector space of dimension 2^n

Proof: define $\Lambda^0 V = \mathbb{R}$ then it is clear $\Lambda^k V$ forms a vector space for $k = 0, 1, \ldots, n$. Moreover, $\Lambda^j V \cap \Lambda^k V = \{0\}$ for $j \neq k$ hence the term "direct sum" is appropriate. It remains to show $dim(\Lambda V) = 2^n$ where $dim(V) = n$. A natural basis β for ΛV is found from taking the union of the bases for each subspace of k-forms,

$$\beta = \{1, e^{i_1}, e^{i_1} \wedge e^{i_2}, \ldots, e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_n} \mid 1 \leq i_1 < i_2 < \cdots < i_n \leq n\}$$

[15] however, in infinite dimensions, the story is not so simple

But, we can count the number of vectors N in the set above as follows:

$$N = 1 + n + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

Recall the binomial theorem states

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = a^n + na^{n-1}b + \cdots + nab^{n-1} + b^n.$$

Recognize that $N = (1+1)^n$ and the proposition follows. □

We should note that in the basis above the space of n-forms is one-dimensional because there is only one way to choose a strictly increasing list of n integers in $\mathbb{N}_n$. In particular, it is useful to note $\Lambda^n V = span\{e^1 \wedge e^2 \wedge \cdots \wedge e^n\}$. The form $e^1 \wedge e^2 \wedge \cdots \wedge e^n$ is sometimes called the the *top-form*[16].

Example 9.3.8. exterior algebra of $\mathbb{R}^2$ *Let us begin with the standard dual basis $\{e^1, e^2\}$. By definition we take the $p = 0$ case to be the field itself; $\Lambda^0 V \equiv \mathbb{R}$, it has basis 1. Next, $\Lambda^1 V = span(e^1, e^2) = V^*$ and $\Lambda^2 V = span(e^1 \wedge e^2)$ is all we can do here. This makes ΛV a $2^2 = 4$-dimensional vector space with basis*

$$\{1, e^1, e^2, e^1 \wedge e^2\}.$$

Example 9.3.9. exterior algebra of $\mathbb{R}^3$ *Let us begin with the standard dual basis $\{e^1, e^2, e^3\}$. By definition we take the $p = 0$ case to be the field itself; $\Lambda^0 V \equiv \mathbb{R}$, it has basis 1. Next, $\Lambda^1 V = span(e^1, e^2, e^3) = V^*$. Now for something a little more interesting,*

$$\Lambda^2 V = span(e^1 \wedge e^2, e^1 \wedge e^3, e^2 \wedge e^3)$$

and finally,

$$\Lambda^3 V = span(e^1 \wedge e^2 \wedge e^3).$$

This makes ΛV a $2^3 = 8$-dimensional vector space with basis

$$\{1, e^1, e^2, e^3, e^1 \wedge e^2, e^1 \wedge e^3, e^2 \wedge e^3, e^1 \wedge e^2 \wedge e^3\}$$

it is curious that the number of independent one-forms and 2-forms are equal.

Example 9.3.10. exterior algebra of $\mathbb{R}^4$ *Let us begin with the standard dual basis $\{e^1, e^2, e^3, e^4\}$. By definition we take the $p = 0$ case to be the field itself; $\Lambda^0 V \equiv \mathbb{R}$, it has basis 1. Next, $\Lambda^1 V = span(e^1, e^2, e^3, e^4) = V^*$. Now for something a little more interesting,*

$$\Lambda^2 V = span(e^1 \wedge e^2, e^1 \wedge e^3, e^1 \wedge e^4, e^2 \wedge e^3, e^2 \wedge e^4, e^3 \wedge e^4)$$

and three forms,

$$\Lambda^3 V = span(e^1 \wedge e^2 \wedge e^3, e^1 \wedge e^2 \wedge e^4, e^1 \wedge e^3 \wedge e^4, e^2 \wedge e^3 \wedge e^4).$$

and $\Lambda^3 V = span(e^1 \wedge e^2 \wedge e^3)$. Thus ΛV a $2^4 = 16$-dimensional vector space. Note that, in contrast to $\mathbb{R}^3$, we do not have the same number of independent one-forms and two-forms over $\mathbb{R}^4$.

[16] or *volume form* for reasons we will explain later, other authors begin the discussion of forms from the consideration of volume, see Chapter 4 in Bernard Schutz' *Geometrical methods of mathematical physics*

Let's explore how this algebra fits with calculations we already know about determinants.

Example 9.3.11. *Suppose $A = [A_1|A_2]$. I propose the determinant of A is given by the top-form on $\mathbb{R}^2$ via the formula $det(A) = (e^1 \wedge e^2)(A_1, A_2)$. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A_1 = (a, c)$ and $A_2 = (b, d)$. Thus,*

$$\begin{aligned} det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= (e^1 \wedge e^2)(A_1, A_2) \\ &= (e^1 \otimes e^2 - e^2 \otimes e^1)((a, c), (b, d)) \\ &= e^1(a, c)e^2(b, d) - e^2(a, c)e^1(b, d) \\ &= ad - bc. \end{aligned}$$

I hope this is not surprising!

Example 9.3.12. *Suppose $A = [A_1|A_2|A_3]$. I propose the determinant of A is given by the top-form on $\mathbb{R}^3$ via the formula $det(A) = (e^1 \wedge e^2 \wedge e^3)(A_1, A_2, A_3)$. Let's see if we can find the expansion by cofactors. By the definition we have $e^1 \wedge e^2 \wedge e^3 =$*

$$\begin{aligned} &= e^1 \otimes e^2 \otimes e^3 + e^2 \otimes e^3 \otimes e^1 + e^3 \otimes e^1 \otimes e^2 - e^3 \otimes e^2 \otimes e^1 - e^2 \otimes e^1 \otimes e^3 - e^1 \otimes e^3 \otimes e^2 \\ &= e^1 \otimes (e^2 \otimes e^3 - e^3 \otimes e^2) - e^2 \otimes (e^1 \otimes e^3 - e^3 \otimes e^1) + e^3 \otimes (e^1 \otimes e^2 - e^2 \otimes e^1) \\ &= e^1 \otimes (e^2 \wedge e^3) - e^2 \otimes (e^1 \wedge e^3) + e^3 \otimes (e^1 \wedge e^2). \end{aligned}$$

I submit to the reader that this is precisely the cofactor expansion formula with respect to the first column of A. Suppose $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $A_1 = (a, d, g)$, $A_2 = (b, e, h)$ and $A_3 = (c, f, i)$. Calculate,

$$\begin{aligned} det(A) &= e^1(A_1)(e^2 \wedge e^3)(A_2, A_3) - e^2(A_1)(e^1 \wedge e^3)(A_2, A_3) + e^3(A_1)(e^1 \wedge w^2)(A_2, A_3) \\ &= a(e^2 \wedge e^3)(A_2, A_3) - d(e^1 \wedge e^3)(A_2, A_3) + g(e^1 \wedge w^2)(A_2, A_3) \\ &= a(ei - fh) - d(bi - ch) + g(bf - ce) \end{aligned}$$

which is precisely my claim.

9.3.3 connecting vectors and forms in $\mathbb{R}^3$

There are a couple ways to connect vectors and forms in $\mathbb{R}^3$. Mainly we need the following maps:

Definition 9.3.13.

Given $v =< a, b, c > \in \mathbb{R}^3$ we can construct a corresponding one-form $\omega_v = ae^1 + be^2 + ce^3$ or we can construct a corresponding two-form $\Phi_v = ae^2 \wedge e^3 + be^3 \wedge e^1 + ce^1 \wedge e^2$

Recall that $dim(\Lambda^1\mathbb{R}^3) = dim(\Lambda^2\mathbb{R}^3) = 3$ hence the space of vectors, one-forms, and also two-forms are isomorphic as vector spaces. It is not difficult to show that $\omega_{v_1+cv_2} = \omega_{v_1} + c\omega_{v_2}$ and $\Phi_{v_1+cv_2} = \Phi_{v_1} + c\Phi_{v_2}$ for all $v_1, v_2 \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Moreover, $\omega_v = 0$ iff $v = 0$ and $\Phi_v = 0$ iff $v = 0$ hence $ker(\omega) = \{0\}$ and $ker(\Phi) = \{0\}$ but this means that ω and Φ are injective and since the dimensions of the domain and codomain are 3 and these are linear transformations[17] it follows ω and Φ are isomorphisms.

It appears we have two ways to represent vectors with forms in $\mathbb{R}^3$. We'll see why this is important as we study integration of forms. It turns out the two-forms go with surfaces whereas the one-forms attach to curves. This corresponds to the fact in calculus III we have two ways to integrate a vector-field, we can either calculate flux or work. Partly for this reason the mapping ω is called the **work-form correspondence** and Φ is called the **flux-form correspondence**. Integration has to wait a bit, for now we focus on algebra.

Example 9.3.14. *Suppose* $v =< 2, 0, 3 >$ *and* $w =< 0, 1, 2 >$ *then* $\omega_v = 2e^1+3e^3$ *and* $\omega_w = e^2+2e^3$. *Calculate the wedge product,*

$$\begin{aligned}
\omega_v \wedge \omega_w &= (2e^1 + 3e^3) \wedge (e^2 + 2e^3) \\
&= 2e^1 \wedge (e^2 + 2e^3) + 3e^3 \wedge (e^2 + 2e^3) \\
&= 2e^1 \wedge e^2 + 4e^1 \wedge e^3 + 3e^3 \wedge e^2 + 6e^3 \wedge e^3 \\
&= -3e^2 \wedge e^3 - 4e^3 \wedge e^1 + 2e^1 \wedge e^2 \\
&= \Phi_{<-3,-4,2>} \\
&= \Phi_{v\times w} \qquad (9.20)
\end{aligned}$$

Coincidence? Nope.

Proposition 9.3.15.

Suppose $v, w \in \mathbb{R}^3$ then $\omega_v \wedge \omega_w = \Phi_{v\times w}$ where $v \times w$ denotes the cross-product which is defined by $v \times w = \sum_{i,j,k=1}^3 \epsilon_{ijk} v_i w_j e_k$.

Proof: Suppose $v = \sum_{i=1}^3 v_i e_i$ and $w = \sum_{j=1}^3 w_j e_j$ then $\omega_v = \sum_{i=1}^3 v_i e^i$ and $\omega_w = \sum_{j=1}^3 w_j e^j$. Calculate,

$$\omega_v \wedge \omega_w = \left(\sum_{i=1}^3 v_i e^i\right) \wedge \left(\sum_{j=1}^3 w_j e^j\right) = \sum_{i=1}^3 \sum_{j=1}^3 v_i w_j e^i \wedge e^j$$

In invite the reader to show $e^i \wedge e^j = \Phi(\sum_{k=1}^3 \epsilon_{ijk} e_k)$ where I'm using $\Phi_v = \Phi(v)$ to make the argument of the flux-form mapping easier to read, hence,

$$\omega_v \wedge \omega_w = \underbrace{\sum_{i=1}^3 \sum_{j=1}^3 v_i w_j \Phi(\sum_{k=1}^3 \epsilon_{ijk} e_k) = \Phi(\sum_{i,j,k=1}^3 v_i w_j \epsilon_{ijk} e_k)}_{linearity\ of\ \Phi} = \Phi(v \times w) = \Phi_{v\times w}$$

[17] this is not generally true, note $f(x) = x^2$ has $f(x) = 0$ iff $x = 0$ and yet f is not injective. The linearity is key.

Of course, if you don't like my proof you could just work it out like the example that precedes this proposition. I gave the proof to show off the mappings a bit more. □

Is the wedge product just the cross-product generalized? Well, not really. I think they're quite different animals. The wedge product is an associative product which makes sense in any vector space. The cross-product only matches the wedge product after we interpret it through a pair of isomorphisms (ω and ϕ) which are special to $\mathbb{R}^3$. However, there is debate, largely the question comes down to what you think makes the cross-product the cross-product. If you think it must pick a unique perpendicular direction to a pair of given directions then that is only going to work in $\mathbb{R}^3$ since even in $\mathbb{R}^4$ there is a whole plane of perpendicular vectors to a given pair. On the other hand, if you think the cross-product in $\mathbb{R}^4$ should be pick the unique perpendicular to a given triple of vectors then you could set something up. You could define $v \times w \times x = \omega^{-1}(\psi(\omega_v \wedge \omega_w \wedge \omega_x))$ where $\psi : \Lambda^3\mathbb{R}^4 \to \Lambda^1\mathbb{R}^4$ is an isomorphism we'll describe in a upcoming section. But, you see it's no longer a product of two vectors, it's not a binary operation, it's a tertiary operation. In any event, you can read a lot more on this if you wish. We have all the tools we need for this course. The wedge product provides the natural antisymmetric algebra for n-dimensiona and the work and flux-form maps naturally connect us to the special world of three-dimensional mathematics.

There is more algebra for forms on $\mathbb{R}^3$ however we defer it to a later section where we have a few more tools. Chief among those is the Hodge dual. But, before we can discuss Hodge duality we need to generalize our idea of a dot-product just a little.

9.4 bilinear forms and geometry, metric duality

The concept of a metric goes beyond the familar case of the dot-product. If you want a more strict generalization of the dot-product then you should think about an *inner-product*. In contrast to the definition below, the inner-product replaces non-degeneracy with the stricter condition of positive-definite which would read $g(x, x) > 0$ for $x \neq 0$. I included a discussion of inner products at the end of this section for the interested reader although we are probably not going to need all of that material.

9.4.1 metric geometry

A **geometry** is a vector space paired with a metric. For example, if we pair $\mathbb{R}^n$ with the dot-product you get Euclidean space. However, if we pair $\mathbb{R}^4$ with the Minkowski metric then we obtain Minkowski space.

Definition 9.4.1.

If V is a vector space and $g : V \times V \to \mathbb{R}$ is

1. bilinear: $g \in T^0_2 V$,
2. **symmetric**: $g(x, y) = g(y, x)$ for all $x, y \in V$,
3. **nondegenerate**: $g(x, y) = 0$ for all $x \in V$ implies $y = 0$.

the we call g a **metric** on V.

If $V = \mathbb{R}^n$ then we can write $g(x, y) = x^T G y$ where $[g] = G$. Moreover, $g(x, y) = g(y, x)$ implies $G^T = G$. Nondegenerate means that $g(x, y) = 0$ for all $y \in \mathbb{R}^n$ iff $x = 0$. It follows that $Gy = 0$ has no non-trivial solutions hence G^{-1} exists.

Example 9.4.2. *Suppose $g(x, y) = x^T y$ for all $x, y \in \mathbb{R}^n$. This defines a metric for $\mathbb{R}^n$, it is just the dot-product. Note that $g(x, y) = x^T y = x^T I y$ hence we see $[g] = I$ where I denotes the identity matrix in $\mathbb{R}^{n\times n}$.*

Example 9.4.3. *Suppose $v = (v^0, v^1, v^2, v^3), w = (w^0, w^1, w^2, w^3) \in \mathbb{R}^4$ then define the* **Minkowski product** *of v and w as follows:*

$$g(v, w) = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$

It is useful to write the Minkowski product in terms of a matrix multiplication. Observe that for $x, y \in \mathbb{R}^4$,

$$g(x, y) = -x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3 = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \end{pmatrix} \equiv x^t \eta y$$

where we have introduced η the matrix of the Minkowski product. Notice that $\eta^T = \eta$ and $det(\eta) = -1 \neq 0$ hence $g(x, y) = x^t \eta y$ makes g a symmetric, nondegenerate bilinear form on $\mathbb{R}^4$. The formula is clearly related to the dot-product. Suppose $\bar{v} = (v^0, \vec{v})$ and $\bar{w} = (w^0, \vec{w})$ then note

$$g(v, w) = -v^0 w^0 + \vec{v} \cdot \vec{w}$$

For vectors with zero in the zeroth slot this Minkowski product reduces to the dot-product. However, for vectors which have nonzero entries in both the zeroth and later slots much differs. Recall that any vector's dot-product with itself gives the square of the vectors length. Of course this means that $\vec{x} \cdot \vec{x} = 0$ iff $\vec{x} = 0$. Contrast that with the following: if $v = (1, 1, 0, 0)$ then

$$g(v, v) = -1 + 1 = 0$$

Yet $v \neq 0$. Why study such a strange generalization of length? The answer lies in physics. I'll give you a brief account by defining a few terms: Let $v = (v^0, v^1, v^2, v^3) \in \mathbb{R}^4$ then we say

1. *v is a timelike vector if $< v, v > < 0$*
2. *v is a lightlike vector if $< v, v > = 0$*
3. *v is a spacelike vector if $< v, v > > 0$*

If we consider the trajectory of a massive particle in $\mathbb{R}^4$ that begins at the origin then at any later time the trajectory will be located at a timelike vector. If we consider a light beam emitted from the origin then at any future time it will located at the tip of a lightlike vector. Finally, spacelike vectors point to points in $\mathbb{R}^4$ which cannot be reached by the motion of physical particles that pass throughout the origin. We say that massive particles are confined within their light cones, this means that they are always located at timelike vectors relative to their current position in space time. If you'd like to know more I can reccomend a few books.

At this point you might wonder if there are other types of metrics beyond these two examples. Surprisingly, in a certain sense, no. A rather old theorem of linear algebra due to Sylvester states that we can change coordinates so that the metric more or less resembles either the dot-product or something like it with some sign-flips. We'll return to this in a later section.

9.4.2 metric duality for tensors

Throughout this section we consider a vector space V paired with a metric $g : V \times V \to \mathbb{R}$. Moreover, the vector space V has basis $\{e_i\}_{i=1}^n$ which has a g-dual basis $\{e^i\}_{i=1}^n$. Up to this point we always have used a g-dual basis where the duality was offered by the dot-product. In the context of Minkowski geometry that sort of duality is no longer natural. Instead we must follow the definition below:

Definition 9.4.4.

If V is a vector space with metric g and basis $\{e_i\}_{i=1}^n$ then we say the basis $\{e^i\}_{i=1}^n$ is g-dual iff

Suppose $e^i(e_j) = \delta_{ij}$ and consider $g = \sum_{i,j=1}^n g_{ij} e^i \otimes e^j$. Furthermore, suppose g^{ij} are the components of the inverse matrix to (g_{ij}) this means that $\sum_{k=1}^n g_{ik} g^{kj} = \delta_{ij}$. We use the components of the metric and its inverse to raise and lower indices on tensors. Here are the basic conventions: given an object A^j which has the *contravariant* index j we can lower it to be covariant by contracting against the metric components as follows:

$$A_i = \sum_j g_{ij} A^j$$

On the other hand, given an object B_j which has a *covariant* index j we can raise it to be contravariant by contracting against the inverse components of the metric:

$$\boxed{B^i = \sum_j g^{ij} B_j}$$

I like to think of this as some sort of conservation of indices. Strict adherence to the notation drives us to write things such as $\sum_{k=1}^n g_{ik} g^{kj} = \delta_i^j$ just to keep up the up/down index pattern. I should mention that these formulas are much more beautiful in the physics literature, you can look at my old Math 430 notes from NCSU if you'd like a healthy dose of that notation[18]. I use Einstein's implicit summation notation throughout those notes and I discuss this index calculation more in the way a physicist typically approaches it. Here I am trying to be careful enough that these equations are useful to mathematicians. Let me show you some examples:

Example 9.4.5. *Specialize for this example to $V = \mathbb{R}^4$ with $g(x,y) = x^T \eta y$. Suppose $x = \sum_{\mu=0}^4 x^\mu e_\mu$ the components x^μ are called* **contravariant components**. *The metric allows us to define* **covariant components** *by*

$$x_\nu = \sum_{\mu=0}^{4} \eta_{\nu\mu} x^\mu.$$

For the minkowski metric this just adjoins a minus to the zeroth component: if $(x^\mu) = (a,b,c,d)$ then $x_\mu = (-a,b,c,d)$.

Example 9.4.6. *Suppose we are working on $\mathbb{R}^n$ with the Euclidean metric $g_{ij} = \delta_{ij}$ and it follows that $g^{ij} = \delta_{ij}$ or to be a purist for a moment $\sum_k g_{ik} g^{kj} = \delta_i^j$. In this case $v^i = \sum_j g^{ij} v_j = \sum_j \delta_{ij} v_j = v_i$. The covariant and contravariant components are the same. This is why is was ok to ignore up/down indices when we work with a dot-product exclusively.*

What if we raise an index and the lower it back down once more? Do we really get back where we started? Given x^μ we lower the index by $x_\nu = \sum_\mu g_{\nu\mu} x^\mu$ then we raise it once more by

$$x^\alpha = \sum_\nu g^{\alpha\nu} x_\nu = \sum_\nu g^{\alpha\nu} \sum_\mu g_{\nu\mu} x^\mu = \sum_{\mu,\nu} g^{\alpha\nu} g_{\nu\mu} x^\mu = \sum_\mu \delta_\mu^\alpha x^\mu$$

and the last summation squishes down to x^α once more. It would seem this procedure of raising and lowering indices is at least consistent.

Example 9.4.7. *Suppose we raise the index on the basis $\{e_i\}$ and formally obtain $\{e^j = \sum_k g^{jk} e_k\}$ on the other hand suppose we lower the index on the dual basis $\{e^l\}$ to formally obtain $\{e_m = \sum_l g_{ml} e^l\}$. I'm curious, are these consistent? We should get $e^j(e_m) = \delta_m^j$, I'll be nice an look at $e_m(e^j)$ in the following sense:*

$$\sum_l g_{ml} e^l \left(\sum_k g^{jk} e_k \right) = \sum_{l,k} g_{ml} g^{jk} e^l(e_k) = \sum_{l,k} g_{ml} g^{jk} \delta_k^l = \sum_k g_{mk} g^{jk} = \delta_m^j$$

Interesting, but what does it mean?

[18] just a taste: $v_\mu = \eta_{\mu\nu} v^\nu$ or $v^\mu = \eta^{\mu\nu} v_\nu$ or $v^\mu v_\mu = \eta^{\mu\nu} v_\nu v_\mu = \eta_{\mu\nu} v^\mu v^\nu$

I used the term *formal* in the preceding example to mean that the example makes sense in as much as you accept the equations which are written. If you think harder about it then you'll find it was rather meaningless. That said, this index notation is rather forgiving.

Ok, but what are we doing? Recall that I insisted on using lower indices for forms and upper indices for vectors? The index conventions I'm toying with above are the reason for this strange notation. When we lower an index we might be changing a vector to a dual vector, or vice-versa when we raise an index we might be changing a dual vector into a vector. Let me be explicit.

1. given $v \in V$ we create $\alpha_v \in V^*$ by the rule $\alpha_v(x) = g(x, v)$.

2. given $\alpha \in V^*$ we create $v_\alpha \in V^{**}$ by the rule $v_\alpha(\beta) = g^{-1}(\alpha, \beta)$ where $g^{-1}(\alpha, \beta) = \sum_{ij} \alpha_i \beta_j g^{ij}$.

Recall we at times identify V and V^{**}. Let's work out the component structure of α_v and see how it relates to v,

$$\alpha_v(e_i) = g(v, e_i) = g(\sum_j v^j e_j, e_i) = \sum_j v^j g(e_j, e_i) = \sum_j v^j g_{ji}$$

Thus, $\alpha_v = \sum_i v_i e^i$ where $v_i = \sum_j v^j g_{ji}$. When we lower the index we're actually using an isomorphism which is provided by the metric to map vectors to forms. The process of raising the index is just the inverse of this isomorphism.

$$v_\alpha(e^i) = g^{-1}(\alpha, e^i) = g^{-1}(\sum_j \alpha_j e^j, e^i) = \sum_j \alpha_j g^{ji}$$

thus $v_\alpha = \sum_i \alpha^i e_i$ where $\alpha^i = \sum_j \alpha_j g^{ji}$.

Suppose we want to change a type $(0, 2)$ tensor to a type $(2, 0)$ tensor. We're given $T : V^* \times V^*$ where $T = \sum_{ij} T^{ij} e_i \otimes e_j$. Define $\tilde{T} : V \times V \to \mathbb{R}$ as follows:

$$\tilde{T}(v, w) = T(\alpha_v, \alpha_w)$$

What does this look like in components? Note $\alpha_{e_i}(e_j) = g(e_i, e_j) = g_{ij}$ hence $\alpha_{e_i} = \sum_j g_{ij} e^j$ and

$$\tilde{T}(e_i, e_j) = T(\alpha_{e_i}, \alpha_{e_j}) = T\left(\sum_k g_{ik} e^k,\ \sum_l g_{jl} e^l\right) = \sum_{k,l} g_{ki} g_{lj} T(e^k, e^l) = \sum_{k,l} g_{ki} g_{lj} T^{kl}$$

Or, as is often customary, we could write $T_{ij} = \sum_{k,l} g_{ik} g_{jl} T^{kl}$. However, this is an abuse of notation since T_{ij} are not technically components for T. If we have a metric we can recover either T from $\tilde{T}$ or vice-versa. Generally, if we are given two tensors, say T_1 of rank (r, s) and the T_2 of rank (r', s'), then these might be equilvalent if $r + s = r' + s'$. It may be that through raising and lowering indices (a.k.a. appropriately composing with the vector$\leftrightarrow$dual vector isomorphisms) we can convert T_1 to T_2. If you read *Gravitation* by Misner, Thorne and Wheeler you'll find many more thoughts

on this equivalence. Challenge: can you find the explicit formulas like $\tilde{T}(v,w) = T(\alpha_v, \alpha_w)$ which back up the index calculations below?

$$T_{ij}{}^{k} = \sum_{a,b} g_{ia} g_{jb} T^{abk} \qquad \text{or} \qquad S^{ij} = \sum_{a,b} g^{ia} g^{jb} S_{ab}$$

I hope I've given you enough to chew on in this section to put these together.

9.4.3 inner products and induced norm

There are generalized dot-products on many abstract vector spaces, we call them **inner-products**.

Definition 9.4.8.

Suppose V is a vector space. If $<,>: V \times V \to \mathbb{R}$ is a function such that for all $x, y, z \in V$ and $c \in \mathbb{R}$:

1. $< x, y >=< y, x >$ (symmetric)
2. $< x + y, z >=< x, z > + < y, z >$ (additive in the first slot)
3. $< cx, y >= c < x, y >$ (homogeneity in the first slot)
4. $< x, x > \geq 0$ and $< x, x >= 0$ iff $x = 0$

then we say $(V, <,>)$ is an **inner-product space** with inner product $<,>$.

Given an inner-product space $(V, <,>)$ we can easily induce a norm for V by the formula $||x|| = \sqrt{< x, x >}$ for all $x \in V$. Properties (1.), (3.) and (4.) in the definition of the norm are fairly obvious for the induced norm. Let's think throught the triangle inequality for the induced norm:

$$\begin{aligned}
||x+y||^2 &= < x+y, x+y > && \text{def. of induced norm} \\
&= < x, x+y > + < y, x+y > && \text{additive prop. of inner prod.} \\
&= < x+y, x > + < x+y, y > && \text{symmetric prop. of inner prod.} \\
&= < x, x > + < y, x > + < x, y > + < y, y > && \text{additive prop. of inner prod.} \\
&= ||x||^2 + 2 < x, y > + ||y||^2
\end{aligned}$$

At this point we're stuck. A nontrivial identity[19] called the **Cauchy-Schwarz** identity helps us proceed; $< x, y > \leq ||x||||y||$. It follows that $||x+y||^2 \leq ||x||^2 + 2||x||||y|| + ||y||^2 = (||x|| + ||y||)^2$. However, the induced norm is clearly positive[20] so we find $||x+y|| \leq ||x|| + ||y||$.

Most linear algebra texts have a whole chapter on inner-products and their applications, you can look at my notes for a start if you're curious. That said, this is a bit of a digression for this course.

[19] I prove this for the dot-product in my linear notes, however, the proof is written in such a way it equally well applies to a general inner-product

[20] note: if you have $(-5)^2 < (-7)^2$ it does not follow that $-5 < -7$, in order to take the squareroot of the inequality we need positive terms squared

9.5 hodge duality

We can prove that $\binom{n}{p} = \binom{n}{n-p}$. This follows from explicit computation of the formula for $\binom{n}{p}$ or from the symmetry of Pascal's triangle if you prefer. In any event, this equality suggests there is some isomorphism between p and $(n-p)$-forms. When we are given a metric g on a vector space V (and the notation of the preceding section) it is fairly simple to construct the isomorphism. Suppose we are given $\alpha \in \Lambda^p V$ and following our usual notation:

$$\alpha = \sum_{i_1,i_2,\ldots,i_p=1}^{n} \frac{1}{p!}\alpha_{i_1 i_2 \ldots i_p} e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}$$

Then, define $*\alpha$ the **hodge dual** to be the $(n-p)$-form given below:

$$\boxed{*\alpha = \sum_{i_1,i_2,\ldots,i_n=1}^{n} \frac{1}{p!(n-p)!}\alpha^{i_1 i_2 \ldots i_p}\epsilon_{i_1 i_2 \ldots i_p j_1 j_2 \ldots j_{n-p}} e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_{n-p}}}$$

I should admit, to prove this is a reasonable definition we'd need to do some work. It's clearly a linear transformation, but bijectivity and coordinate invariance of this definition might take a little work. I intend to omit those details and instead focus on how this works for $\mathbb{R}^3$ or $\mathbb{R}^4$. My advisor taught a course on fiber bundles and there is a much more general and elegant presentation of the hodge dual over a manifold. Ask if interested, I think I have a pdf.

9.5.1 hodge duality in euclidean space $\mathbb{R}^3$

To begin, consider a scalar 1, this is a 0-form so we expect the hodge dual to give a 3-form:

$$*1 = \sum_{i,j,k} \frac{1}{0!3!}\epsilon_{ijk} e^i \wedge e^j \wedge e^k = e^1 \wedge e^2 \wedge e^3$$

Interesting, the hodge dual of 1 is the top-form on $\mathbb{R}^3$. Conversely, calculate the dual of the top-form, note $e^1 \wedge e^2 \wedge e^3 = \sum_{ijk} \frac{1}{6}\epsilon_{ijk} e^i \wedge e^j \wedge e^k$ reveals the components of the top-form are precisely ϵ_{ijk} thus:

$$*(e^1 \wedge e^2 \wedge e^3) = \sum_{i,j,k=1}^{3} \frac{1}{3!(3-3)!}\epsilon^{ijk}\epsilon_{ijk} = \frac{1}{6}(1+1+1+(-1)^2+(-1)^2+(-1)^2) = 1.$$

Next, consider e^1, note that $e^1 = \sum_k \delta_{k1} e^k$ hence the components are δ_{k1}. Thus,

$$*e^1 = \sum_{i,j,k} \frac{1}{1!2!}\epsilon_{ijk}\delta^{k1} e^i \wedge e^j = \frac{1}{2}\sum_{i,j} \epsilon_{ij1} e^i \wedge e^j = \frac{1}{2}(\epsilon_{231} e^2 \wedge e^3 + \epsilon_{321} e^3 \wedge e^2) = e^2 \wedge e^3$$

Similar calculations reveal $*e^2 = e^3 \wedge e^1$ and $*e^3 = e^1 \wedge e^2$. What about the duals of the two-forms? Begin with $\alpha = e^1 \wedge e^2$ note that $e^1 \wedge e^2 = e^1 \otimes e^2 - e^2 \otimes e^1$ thus we can see the components are

$\alpha_{ij} = \delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}$. Thus,

$$*(e^1 \wedge e^2) = \sum_{i,j,k} \frac{1}{2!1!}\epsilon_{ijk}(\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1})e^k = \frac{1}{2}\left(\sum_k \epsilon_{12k}e^k - \sum_k \epsilon_{21k}e^k\right) = \frac{1}{2}(e^3 - (-e^3)) = e^3.$$

Similar calculations show that $*(e^2 \wedge e^3) = e^1$ and $*(e^3 \wedge e^1) = e^2$. Put all of this together and we find that

$$*(ae^1 + be^2 + ce^3) = ae^2 \wedge e^3 + be^3 \wedge e^1 + ce^1 \wedge e^2$$

and

$$*(ae^2 \wedge e^3 + be^3 \wedge e^1 + ce^1 \wedge e^2) = ae^1 + be^2 + ce^3$$

Which means that $\boxed{*\omega_v = \Phi_v}$ and $\boxed{*\Phi_v = \omega_v}$. Hodge duality links the two different form-representations of vectors in a natural manner. Moveover, for $\mathbb{R}^3$ we should also note that $**\alpha = \alpha$ for all $\alpha \in \Lambda\mathbb{R}^3$. In general, for other metrics, we can have a change of signs which depends on the degree of α.

We can summarize hodge duality for three-dimensional Euclidean space as follows:

$^*1 = e^1 \wedge e^2 \wedge e^3$	$^*(e^1 \wedge e^2 \wedge e^3) = 1$
$^*e^1 = e^2 \wedge e^3$	$^*(e^2 \wedge e^3) = e^1$
$^*e^2 = e^3 \wedge e^1$	$^*(e^3 \wedge e^1) = e^2$
$^*e^3 = e^1 \wedge e^2$	$^*(e^1 \wedge e^2) = e^3$

A simple rule to calculate the hodge dual of a basis form is as follows

1. begin with the top-form $e^1 \wedge e^2 \wedge e^3$
2. permute the forms until the basis form you wish to hodge dual is to the left of the expression, whatever remains to the right is the hodge dual.

For example, to calculate the dual of $e^2 \wedge e^3$ note

$$e^1 \wedge e^2 \wedge e^3 = \underbrace{e^2 \wedge e^3}_{\text{to be dualed}} \wedge \underbrace{e^1}_{\text{the dual}} \quad \Rightarrow \quad *(e^2 \wedge e^3) = e^1.$$

Consider what happens if we calculate $**\alpha$, since the dual is a linear operation it suffices to think about the basis forms. Let me sketch the process of $**e^I$ where I is a multi-index:

1. begin with $e^1 \wedge e^2 \wedge e^3$
2. write $e^1 \wedge e^2 \wedge e^3 = (-1)^N e^I \wedge e^J$ and identify $*e^I = (-1)^N e^J$.
3. then to calculate the second dual once more begin with $e^1 \wedge e^2 \wedge e^3$ and note

$$e^1 \wedge e^2 \wedge e^3 = (-1)^N e^J \wedge e^I$$

since the same N transpositions are required to push e^I to the left or e^J to the right.

4. It follows that $**e^I = e^I$ for any multi-index hence $\boxed{**\alpha = \alpha}$ for all $\alpha \in \Lambda\mathbb{R}^3$.

I hope that once you get past the index calculation you can see the hodge dual is not a terribly complicated construction. Some of the index calculation in this section was probably gratutious, but I would like you to be aware of such techniques. Brute-force calculation has it's place, but a well-thought index notation can bring far more insight with much less effort.

9.5.2 hodge duality in minkowski space $\mathbb{R}^4$

The logic here follows fairly close to the last section, however the wrinkle is that the metric here demands more attention. We must take care to raise the indices on the forms when we Hodge dual them. First let's list the basis forms, we have to add time to the mix (again $c = 1$ so $x^0 = ct = t$ if you worried about it) Remember that the Greek indices are defined to range over $0, 1, 2, 3$. Here

Name	Degree	Typical Element	Basis for $\Lambda^p\mathbb{R}^4$
function	$p = 0$	f	1
one-form	$p = 1$	$\alpha = \sum_\mu \alpha_\mu e^\mu$	e^0, e^1, e^2, e^3
two-form	$p = 2$	$\beta = \sum_{\mu,\nu} \frac{1}{2}\beta_{\mu\nu} e^\mu \wedge e^\nu$	$e^2 \wedge e^3,\ e^3 \wedge e^1,\ e^1 \wedge e^2$ $e^0 \wedge e^1,\ e^0 \wedge e^2,\ e^0 \wedge e^3$
three-form	$p = 3$	$\gamma = \sum_{\mu,\nu,\alpha} \frac{1}{3!}\gamma_{\mu\nu\alpha} e^\mu \wedge e^\nu e^\alpha$	$e^1 \wedge e^2 \wedge e^3, e^0 \wedge e^2 \wedge e^3$ $e^0 \wedge e^1 \wedge e^3, e^0 \wedge e^1 \wedge e^2$
four-form	$p = 4$	$ge^0 \wedge e^1 \wedge e^2 \wedge e^3$	$e^0 \wedge e^1 \wedge e^2 \wedge e^3$

the top form is degree four since in four dimensions we can have at most four dual-basis vectors without a repeat. Wedge products work the same as they have before, just now we have e^0 to play with. Hodge duality may offer some surprises though.

Definition 9.5.1. *The antisymmetric symbol in* **flat** $\mathbb{R}^4$ *is denoted* $\epsilon_{\mu\nu\alpha\beta}$ *and it is defined by the value*

$$\epsilon_{0123} = 1$$

plus the demand that it be completely antisymmetric.

We must not assume that this symbol is invariant under a cyclic exhange of indices. Consider,

$$\begin{aligned} \epsilon_{0123} &= -\epsilon_{1023} && \text{flipped (01)} \\ &= +\epsilon_{1203} && \text{flipped (02)} \\ &= -\epsilon_{1230} && \text{flipped (03).} \end{aligned} \tag{9.21}$$

In four dimensions we'll use antisymmetry directly and forego the cyclicity shortcut. Its not a big deal if you notice it before it confuses you.

Example 9.5.2. *Find the Hodge dual of* $\gamma = e^1$ *with respect to the Minkowski metric* $\eta_{\mu\nu}$*, to begin notice that dx has components* $\gamma_\mu = \delta^1_\mu$ *as is readily verified by the equation* $e^1 = \sum_\mu \delta^1_\mu e^\mu$*. Lets*

raise the index using η as we learned previously,

$$\gamma^\mu = \sum_\nu \eta^{\mu\nu}\gamma_\nu = \sum_\nu \eta^{\mu\nu}\delta^1_\nu = \eta^{1\mu} = \delta^{1\mu}$$

Starting with the definition of Hodge duality we calculate

$$\begin{aligned}
{}^*(e^1) &= \textstyle\sum_{\alpha,\beta,\mu,\nu} \frac{1}{p!}\frac{1}{(n-p)!}\gamma^\mu \epsilon_{\mu\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\
&= \textstyle\sum_{\alpha,\beta,\mu,\nu} (1/6)\delta^{1\mu}\epsilon_{\mu\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\
&= \textstyle\sum_{\alpha,\beta,\nu} (1/6)\epsilon_{1\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\
&= (1/6)[\epsilon_{1023} e^0 \wedge e^2 \wedge e^3 + \epsilon_{1230} e^2 \wedge e^3 \wedge e^0 + \epsilon_{1302} e^3 \wedge e^0 \wedge e^2 \\
&\qquad + \epsilon_{1320} e^3 \wedge e^2 \wedge e^0 + \epsilon_{1203} e^2 \wedge e^0 \wedge e^3 + \epsilon_{1032} e^0 \wedge e^3 \wedge e^2] \\
&= (1/6)[-e^0 \wedge e^2 \wedge e^3 - e^2 \wedge e^3 \wedge e^0 - e^3 \wedge e^0 \wedge e^2 \\
&\qquad + e^3 \wedge e^2 \wedge e^0 + e^2 \wedge e^0 \wedge e^3 + e^0 \wedge e^3 \wedge e^2] \\
&= -e^2 \wedge e^3 \wedge e^0 = -e^0 \wedge e^2 \wedge e^3.
\end{aligned} \tag{9.22}$$

the difference between the three and four dimensional Hodge dual arises from two sources, for one we are using the Minkowski metric so indices up or down makes a difference, and second the antisymmetric symbol has more possibilities than before because the Greek indices take four values.

I suspect we can calculate the hodge dual by the following pattern: suppose we wish to find the dual of α where α is a basis form for $\Lambda\mathbb{R}^4$ with the Minkowski metric

1. begin with the top-form $e^0 \wedge e^1 \wedge e^2 \wedge e^3$
2. permute factors as needed to place α to the left,
3. the form which remains to the right will be the hodge dual of α if no e^0 is in α otherwise the form to the right multiplied by -1 is $*\alpha$.

Note this works for the previous example as follows:

1. begin with $e^0 \wedge e^1 \wedge e^2 \wedge e^3$
2. note $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = -e^1 \wedge e^0 \wedge e^2 \wedge e^3 = e^1 \wedge (-e^0 \wedge e^2 \wedge e^3)$
3. identify $*e^1 = -e^0 \wedge e^2 \wedge e^3$ (no extra sign since no e^0 appears in e^1)

Follow the algorithm for finding the dual of e^0,

1. begin with $e^0 \wedge e^1 \wedge e^2 \wedge e^3$
2. note $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = e^0 \wedge (e^1 \wedge e^2 \wedge e^3)$

3. identify $*e^0 = -e^1 \wedge e^2 \wedge e^3$ (added sign since e^0 appears in form being hodge dualed)

Let's check from the definition if my algorithm worked out right.

Example 9.5.3. *Find the Hodge dual of $\gamma = e^0$ with respect to the Minkowski metric $\eta_{\mu\nu}$, to begin notice that e^0 has components $\gamma_\mu = \delta^0_\mu$ as is readily verified by the equation $e^0 = \sum_\mu \delta^0_\mu e^\mu$. Lets raise the index using η as we learned previously,*

$$\gamma^\mu = \sum_\nu \eta^{\mu\nu}\gamma_\nu = \sum_\nu \eta^{\mu\nu}\delta^0_\nu = \eta^{\mu 0} = -\delta^{0\mu}$$

the minus sign is due to the Minkowski metric. Starting with the definition of Hodge duality we calculate

$$\begin{aligned}
*(e^0) &= \textstyle\sum_{\alpha,\beta,\mu,\nu} \frac{1}{p!}\frac{1}{(n-p)!}\gamma^\mu \epsilon_{\mu\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\
&= \textstyle\sum_{\alpha,\beta,\mu,\nu} -(1/6)\delta^{0\mu}\epsilon_{\mu\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \\
&= \textstyle\sum_{\alpha,\beta,\nu} -(1/6)\epsilon_{0\nu\alpha\beta} e^\nu \wedge e^\alpha \wedge e^\beta \qquad (9.23) \\
&= \textstyle\sum_{i,j,k} -(1/6)\epsilon_{0ijk} e^i \wedge e^j \wedge e^k \\
&= \textstyle\sum_{i,j,k} -(1/6)\epsilon_{ijk}\epsilon_{ijk} e^1 \wedge e^2 \wedge e^3 \qquad \leftarrow \textit{sneaky step} \\
&= -e^1 \wedge e^2 \wedge e^3.
\end{aligned}$$

Notice I am using the convention that Greek indices sum over $0,1,2,3$ whereas Latin indices sum over $1,2,3$.

Example 9.5.4. *Find the Hodge dual of $\gamma = e^0 \wedge e^1$ with respect to the Minkowski metric $\eta_{\mu\nu}$, to begin notice the following identity, it will help us find the components of γ*

$$e^0 \wedge e^1 = \sum_{\mu,\nu} \frac{1}{2} 2\delta^0_\mu \delta^1_\nu e^\mu \wedge e^\nu$$

now we antisymmetrize to get the components of the form,

$$e^0 \wedge e^1 = \sum_{\mu,\nu} \frac{1}{2}\delta^0_{[\mu}\delta^1_{\nu]} dx^\mu \wedge dx^\nu$$

where $\delta^0_{[\mu}\delta^1_{\nu]} = \delta^0_\mu\delta^1_\nu - \delta^0_\nu\delta^1_\mu$ and the factor of two is used up in the antisymmetrization. Lets raise the index using η as we learned previously,

$$\gamma^{\alpha\beta} = \sum_{\mu,\nu} \eta^{\alpha\mu}\eta^{\beta\nu}\gamma_{\mu\nu} = \sum_{\mu,\nu} \eta^{\alpha\mu}\eta^{\beta\nu}\delta^0_{[\mu}\delta^1_{\nu]} = -\eta^{\alpha 0}\eta^{\beta 1} + \eta^{\beta 0}\eta^{\alpha 1} = -\delta^{[\alpha 0}\delta^{\beta]1}$$

the minus sign is due to the Minkowski metric. Starting with the definition of Hodge duality we

calculate

$$\begin{aligned} *(e^0 \wedge e^1) &= \tfrac{1}{p!}\tfrac{1}{(n-p)!}\gamma^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu}e^\mu \wedge e^\nu \\ &= (1/4)(-\delta^{[\alpha 0}\delta^{\beta]1})\epsilon_{\alpha\beta\mu\nu}e^\mu \wedge e^\nu \\ &= -(1/4)(\epsilon_{01\mu\nu}e^\mu \wedge e^\nu - \epsilon_{10\mu\nu}e^\mu \wedge e^\nu) \\ &= -(1/2)\epsilon_{01\mu\nu}e^\mu \wedge e^\nu \\ &= -(1/2)[\epsilon_{0123}e^2 \wedge e^3 + \epsilon_{0132}e^3 \wedge e^2] \\ &= -e^2 \wedge e^3 \end{aligned} \tag{9.24}$$

Note, the algorithm works out the same,

$$e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \underbrace{e^0 \wedge e^1}_{has\ e^0} \wedge (e^2 \wedge e^3) \quad \Rightarrow \quad *(e^0 \wedge e^1) = -e^2 \wedge e^3$$

The other Hodge duals of the basic two-forms calculate by almost the same calculation. Let us make a table of all the basic Hodge dualities in Minkowski space, I have grouped the terms to emphasize

$*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$	$*(e^0 \wedge e^1 \wedge e^2 \wedge e^3) = -1$
$*(e^1 \wedge e^2 \wedge e^3) = -e^0$	$*e^0 = -e^1 \wedge e^2 \wedge e^3$
$*(e^0 \wedge e^2 \wedge e^3) = -e^1$	$*e^1 = -e^2 \wedge e^3 \wedge e^0$
$*(e^0 \wedge e^3 \wedge e^1) = -e^2$	$*e^2 = -e^3 \wedge e^1 \wedge e^0$
$*(e^0 \wedge e^1 \wedge e^2) = -e^3$	$*e^3 = -e^1 \wedge e^2 \wedge e^0$
$*(e^3 \wedge e^0) = e^1 \wedge e^2$	$*(e^1 \wedge e^2) = -e^3 \wedge e^0$
$*(e^1 \wedge e^0) = e^2 \wedge e^3$	$*(e^2 \wedge e^3) = -e^1 \wedge e^0$
$*(e^2 \wedge e^0) = e^3 \wedge e^1$	$*(e^3 \wedge e^1) = -e^2 \wedge e^0$

the isomorphisms between the one-dimensional $\Lambda^0\mathbb{R}^4$ and $\Lambda^4\mathbb{R}^4$, the four-dimensional $\Lambda^1\mathbb{R}^4$ and $\Lambda^3\mathbb{R}^4$, the six-dimensional $\Lambda^2\mathbb{R}^4$ and itself. Notice that the dimension of $\Lambda\mathbb{R}^4$ is 16 which we have explained in depth in the previous section. Finally, it is useful to point out the three-dimensional work and flux form mappings to provide some useful identities in this $1+3$-dimensional setting.

$$\boxed{*\omega_{\vec{v}} = -e^0 \wedge \Phi_{\vec{v}} \qquad * \Phi_{\vec{v}} = e^0 \wedge \omega_{\vec{v}} \qquad * \left(e^0 \wedge \Phi_{\vec{v}}\right) = \omega_{\vec{v}}}$$

I leave verification of these formulas to the reader (use the table). Finally let us analyze the process of taking two hodge duals in succession. In the context of $\mathbb{R}^3$ we found that $**\alpha = \alpha$, we seek to discern if a similar formula is available in the context of $\mathbb{R}^4$ with the minkowksi metric. We can calculate one type of example with the identities above:

$$*\omega_{\vec{v}} = -e^0 \wedge \Phi_{\vec{v}} \quad \Rightarrow \quad **\omega_{\vec{v}} = -*\left(e^0 \wedge \Phi_{\vec{v}}\right) = -\omega_{\vec{v}} \quad \Rightarrow \quad **\omega_{\vec{v}} = -\omega_{\vec{v}}$$

Perhaps this is true in general?

If we accept my algorithm then it's not too hard to sort through using multi-index notation: since hodge duality is linear it suffices to consider a basis element e^I where I is a multi-index,

1. transpose dual vectors so that $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = (-1)^N e^I \wedge e^J$
2. if $0 \notin I$ then $*e^I = (-1)^N e^J$ and $0 \in J$ since $I \cup J = \{0,1,2,3\}$. Take a second dual by writing $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = (-1)^N e^J \wedge e^I$ but note $*((-1)^N e^J) = -e^I$ since $0 \in J$. We find $**e^I = -e^I$ for all I not containing the 0-index.
3. if $0 \in I$ then $*e^I = -(-1)^N e^J$ and $0 \notin J$ since $I \cup J = \{0,1,2,3\}$. Take a second dual by writing $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = -(-1)^N e^J \wedge (-e^I)$ and hence $*(-(-1)^N e^J) = -e^I$ since $0 \notin J$. We find $**e^I = -e^I$ for all I containing the 0-index.
4. it follows that $\boxed{**\alpha = -\alpha}$ for all $\alpha \in \Lambda\mathbb{R}^4$ with the minkowski metric.

To conclude, I would warn the reader that the results in this section pertain to our choice of notation for $\mathbb{R}^4$. Some other texts use a metric which is $-\eta$ relative to our notation. This modifies many signs in this section. See Misner, Thorne and Wheeler's *Gravitation* or Bertlmann's *Anomalies in Field Theory* for future reading on Hodge duality and a more systematic explaination of how and when these signs arise from the metric.

9.6 coordinate change

Suppose V has two bases $\bar{\beta} = \{\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n\}$ and $\beta = \{f_1, f_2, \ldots, f_n\}$. If $v \in V$ then we can write v in as a linear combination of the $\bar{\beta}$ basis or the β basis:

$$v = x^1 f_1 + x^2 f_2 + \cdots + x^n f_n \quad \text{and} \quad v = \bar{x}^1 \bar{f}_1 + \bar{x}^2 \bar{f}_2 + \cdots + \bar{x}^n \bar{f}_n$$

given the notation above, we define **coordinate maps** as follows:

$$\Phi_\beta(v) = (x^1, x^2, \ldots, x^n) = x \quad \text{and} \quad \Phi_{\bar{\beta}}(v) = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n) = \bar{x}$$

We sometimes use the notation $\Phi_\beta(v) = [v]_\beta = x$ whereas $\Phi_{\bar{\beta}}(v) = [v]_{\bar{\beta}} = \bar{x}$. A coordinate map takes an abstract vector v and maps it to a particular representative in $\mathbb{R}^n$. A natural question to ask is how do different representatives compare? How do x and $\bar{x}$ compare in our current notation? Because the coordinate maps are isomorphisms it follows that $\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism and given the domain and codomain we can write its formula via matrix multiplication:

$$\Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}(u) = Pu \quad \Rightarrow \quad \Phi_\beta \circ \Phi_{\bar{\beta}}^{-1}(\bar{x}) = P\bar{x}$$

However, $\Phi_{\bar{\beta}}^{-1}(\bar{x}) = v$ hence $\Phi_\beta(v) = P\bar{x}$ and consequently, $\boxed{x = P\bar{x}}$. Conversely, to switch to barred coordinates we multiply the coordinate vectors by P^{-1}; $\boxed{\bar{x} = P^{-1}x}$.

Continuing this discussion we turn to the dual space. Suppose $\bar{\beta}^* = \{\bar{f}^j\}_{j=1}^n$ is dual to $\bar{\beta} = \{\bar{f}_j\}_{j=1}^n$ and $\beta^* = \{f^j\}_{j=1}^n$ is dual to $\beta = \{f_j\}_{j=1}^n$. By definition we are given that $f^j(f_i) = \delta_{ij}$ and $\bar{f}^j(\bar{f}_i) = \delta_{ij}$ for all $i, j \in \mathbb{N}_n$. Suppose $\alpha \in V^*$ is a dual vector with components α_j with respect to the β^* basis and components $\bar{\alpha}_j$ with respect to the $\bar{\beta}^*$ basis. In particular this means we can either write $\alpha = \sum_{j=1}^n \alpha_j f^j$ or $\alpha = \sum_{j=1}^n \bar{\alpha}_j \bar{f}^j$. Likewise, given a vector $v \in V$ we can either write $v = \sum_{i=1}^n x^i f_i$ or $v = \sum_{i=1}^n \bar{x}^i \bar{f}_i$. With these notations in mind calculate:

$$\alpha(v) = \Big(\sum_{j=1}^n \alpha_j f^j\Big)(\sum_{i=1}^n x^i f_i) = \sum_{i,j=1}^n \alpha_j x^i f^j(f_i) = \sum_{i=1}^n \sum_{j=1}^n \alpha_j x^i \delta_{ij} = \sum_{i=1}^n \alpha_i x^i$$

and by the same calculation in the barred coordinates we find, $\alpha(v) = \sum_{i=1}^n \bar{\alpha}_i \bar{x}^i$. Therefore,

$$\sum_{i=1}^n \alpha_i x^i = \sum_{i=1}^n \bar{\alpha}_i \bar{x}^i.$$

Recall, $x = P\bar{x}$. In components, $x^i = \sum_{k=1}^n P_k^i \bar{x}^k$. Substituting,

$$\sum_{i=1}^n \sum_{k=1}^n \alpha_i P_k^i \bar{x}^k = \sum_{i=1}^n \bar{\alpha}_i \bar{x}^i.$$

But, this formula holds for all possible vectors v and hence all possible coordinate vectors $\bar{x}$. If we consider $v = \bar{f}_j$ then $\bar{x}^i = \delta_{ij}$ hence $\sum_{i=1}^n \bar{\alpha}_i \bar{x}^i = \sum_{i=1}^n \bar{\alpha}_i \delta_{ij} = \bar{\alpha}_j$. Moreover, $\sum_{i=1}^n \sum_{k=1}^n \alpha_i P_k^i \bar{x}^k = \sum_{i=1}^n \sum_{k=1}^n \alpha_i P_k^i \delta_{kj} = \sum_{i=1}^n \alpha_i P_j^i$. Thus, $\bar{\alpha}_j = \sum_{i=1}^n P_j^i \alpha_i$. Compare how vectors and dual vectors transform:

$$\boxed{\bar{\alpha}_j = \sum_{i=1}^n P_j^i \alpha_i \qquad \text{verses} \qquad \bar{x}^j = \sum_{i=1}^n (P^{-1})_i^j x^i.}$$

It is customary to use lower-indices on the components of dual-vectors and upper-indices on the components of vectors: we say $x = \sum_{i=1}^n x^i e_i \in \mathbb{R}^n$ has **contravariant components** whereas $\alpha = \sum_{j=1}^n \alpha_j e^j \in (\mathbb{R}^n)^*$ has **covariant components**. These terms arise from the coordinate change properties we derived in this section. The convenience of the up/down index notation will be more apparent as we continue our study to more complicated objects. It is interesting to note the basis elements tranform inversely:

$$\boxed{\bar{f}^j = \sum_{i=1}^n (P^{-1})_i^j f^i \qquad \text{verses} \qquad \bar{f}_j = \sum_{i=1}^n P_j^i f_i.}$$

The formulas above can be derived by arguments similar to those we already gave in this section,

however I think it may be more instructive to see how these rules work in concert:

$$\begin{aligned} x = \sum_{i=1}^{n} \bar{x}^i \bar{f}_i &= \sum_{i=1}^{n}\sum_{j=1}^{n} (P^{-1})^i_j x^j \bar{f}_i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} (P^{-1})^i_j x^j \sum_{k=1}^{n} P^k_i f_k \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} (P^{-1})^i_j P^k_i x^j f_k \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \delta^k_j x^j f_k \\ &= \sum_{k=1}^{n} x^k f_k. \end{aligned} \tag{9.25}$$

9.6.1 coordinate change for $T^0_2(V)$

For an abstract vector space, or for $\mathbb{R}^n$ with a nonstandard basis, we have to replace v, w with their coordinate vectors. If V has basis $\bar{\beta} = \{\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n\}$ with dual basis $\beta^* = \{\bar{f}^1, \bar{f}^2, \ldots, \bar{f}^n\}$ and v, w have coordinate vectors $\bar{x}, \bar{y}$ (which means $v = \sum_{i=1}^n \bar{x}^i \bar{f}_i$ and $w = \sum_{i=1}^n \bar{y}^i \bar{f}_i$) then,

$$b(v, w) = \sum_{i,j=1}^{n} \bar{x}^i \bar{y}^j \bar{B}_{ij} = \bar{x}^T \bar{B} \bar{y}$$

where $\bar{B}_{ij} = b(\bar{f}_i, \bar{f}_j)$. If $\beta = \{f_1, f_2, \ldots, f_n\}$ is another basis on V with dual basis β^* then we define $B_{ij} = b(f_i, f_j)$ and we have

$$b(v, w) = \sum_{i,j=1}^{n} x^i y^j B_{ij} = x^T B y.$$

Recall that $\bar{f}_i = \sum_{k=1}^n P^k_i f_k$. With this in mind calculate:

$$\bar{B}_{ij} = b(\bar{f}_i, \bar{f}_j) = b\Big(\sum_{k=1}^{n} P^k_i f_k, \sum_{l=1}^{n} P^l_j f_l\Big) = \sum_{k,l=1}^{n} P^k_i P^l_j b(f_k, f_l) = \sum_{k,l=1}^{n} P^k_i P^l_j B_{kl}$$

We find the components of a bilinear map transform as follows:

$$\boxed{\bar{B}_{ij} = \sum_{k,l=1}^{n} P^k_i P^l_j B_{kl}}$$

Chapter 10

calculus with differential forms

A manifold is an abtract space which allows for local calculus. We discuss how coordinate charts cover the a manifold and how we use them to define smoothness in the abstract. For example, a function is smooth if all its local coordinate representations are smooth. The local coordinate representative is a function from $\mathbb{R}^m$ to $\mathbb{R}^n$ thus we may quantify its smoothness in terms of ordinary partial derivatives of the component functions. On the other hand, while the concept of a coordinate chart is at first glance abstract the usual theorems of advanced calculus all lift naturally to the abstract manifold. For example, we see how partial derivatives with respect to manifold coordinates hold to all the usual linearity, product and chain-rules hold in $\mathbb{R}^n$. We prove a number of these results in considerably more detail than Lecture will bear.

The differential gains a deeper meaning than we found in advanced calculus. In the manifold context, the differential acts on tangent space which is **not** identified as some subset of the manifold itself. So, in a sense we lose the direct approximating concept for the differential. One could always return to the best linear approximation ideal as needed, but the path ahead is quite removed from pure numerical approximation. First step towards this abstract picture of tangent space is the realization that tangent vectors themselves should be identified as **derivations**. We show how partial derivatives give derivations and we sketch a technical result which also provides a converse in the smooth category[1] Once we've settled how to study the tangent space to a manifold we find the natural extension of the differential as the **push-forward** induced from a smooth map. It turns out that you have probably already calculated a few push-forwards in multivariate calculus. We attempt to obtain some intuition for this abstract push-forward. We also pause to note how the push-forward might allow us to create new vectors fields from old (or not).

The cotangent space is the dual space to the tangent space. The basis which is dual to the partial derivative basis is naturally identified with the differentials of the coordinate maps themselves. We then have a basis and dual basis for the tangent and cotangent space at each point on the manifold.

[1] it's actually false for C^k manifolds which have an infinite-dimensional space of derivations. The tangent space to a n-dimensional manifold is an n-dimensional vector space so we need an n-dimensional space of derivations to make the identification.

We use these to build vector fields and differential forms just as we used f_i and f^i in the previous chapter. Multilinear algebra transfers over point by point on the manifold. However, something new happens. Differential forms permit a differentiation called the **exterior derivative**. This natural operation takes a p-form and generates a new $(p+1)$-form. We examine how the exterior derivative recovers **all** the interesting vector-calculus derivatives from vector calculus on $\mathbb{R}^3$. Of course, it goes much deeper as the exterior derivative provides part of the machinery to write cohomology on spaces of arbitrarily high dimension. Ultimately, this theory of cohomology detects topological aspects of the space. A basic example of this is the Poincare Lemma.

To understand the Poincare Lemma as well as a number of other interesting calculations we find it necessary to introduce a dual operation to the push-foward; the **pull-back** gives us a natural method to take differential forms in the range of a mapping and transport them back to the domain of the map. Moreover, this pull-back operation plays nicely with the wedge product and the exterior derivative. Several pages are devotes towards understanding some intuitive method of calculating the pull-back. We are indebted to Harold M. Edwards' *Advanced Calculus: A Differential Form Approach* which encouraged us to search for intuition. I also attempted to translate a differential forms version of the implict mapping theorem from the same text, I'm fairly certain there is some conceptual error in that section as it stands. It is a work in progress.

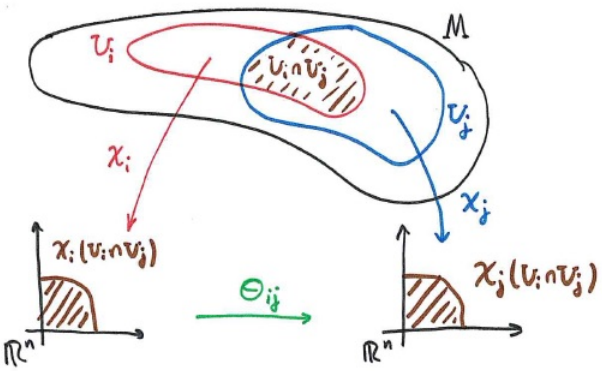
The abstract calculus of forms is interesting in it's own right, but we are happy to find how it reduces to the familar calculus on $\mathbb{R}^3$. We state[2] the Generalized Stokes Theorem and see how the **flux-form** and **work-form** mappings produce the usual theorems of vector calculus as corollaries to the Generalized Stokes Theorem. The definition of integrals of differential forms is accomplished by pulling-back the forms to euclidean space where an ordinary integral quantifies the result. In some sense, this discussion may help answer the question *what is a differential form?*. We spend some effort attempting to understand how the form integration interfaces with ordinary surface or line integration of vector fields.

Finally, the fact that $d^2 = 0$ paired with the nice properties of the pull-back and wedge product proves to give a technique for study of exact differential equations and partial differential equations. This final section opens a door to the vast topic of exterior differential systems. In this study, solutions to PDEs are manifolds and the PDE itself is formulated in terms of wedge products and differential forms. Here I borrow wisdom from *Cartan for Beginners* by Ivey and Landsberg as well as *Equivalence, Invariance, and Symmetry* by Olver. Please keep in mind, we're just dipping are toes in the pond here.

[2] the proof is found in Edwards and many other places

10.1 an informal introduction to manifolds

A manifold $\mathcal{M}$ is space which locally resembles $\mathbb{R}^n$. This requirement is implemented by the existence of coordinate charts (χ, U) which cover $\mathcal{M}$ and allow us to do calculus locally. Let me sketch the idea with a picture:



The charts (χ_j, U_j) have to cover the manifold $\mathcal{M}$ and their **transition functions** θ_{ij} must be smooth mappings on $\mathbb{R}^n$. Rather going on about the proper definition[3], I'll show a few examples.

Example 10.1.1. *Let $\mathcal{M} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. The usual chart on the unit-circle is the angle-chart $\chi = \theta$. Given $(\cos t, \sin t) = p \in \mathcal{M}$ we define $\theta(p) = t$. If $x > 0$ and $x^2 + y^2 = 1$ then $(x, y) \in \mathcal{M}$ and we have $\theta(x, y) = \tan^{-1}(y/x)$.*

Example 10.1.2. *Let $\mathcal{M} = \mathbb{R}^2$. The usual chart is simply the cartesian coordinate system $\chi_1 = (x, y)$ with $U_1 = \mathbb{R}^2$. If $(a, b) \in \mathbb{R}^2$ then $x(a, b) = a$ and $y(a, b) = b$. In practice the symbols x, y are used both as maps and variables so one must pay attention to context. A second coordinate system on $\mathcal{M}$ is given by the polar coordinate chart $\chi_2 = (r, \theta)$ with domain $U_2 = (0, \infty) \times \mathbb{R}$. I'll just take their domain to be the right half-plane for the sake of having a nice formula: $(r, \theta)(a, b) = (\sqrt{a^2 + b^2}, \tan^{-1}(b/a))$. You can extend these to most of the plane, but you have to delete the origin and you must lose a ray since the angle-chart is not injective if we go full-circle. That said, the coordinate systems $(\chi_1 = (x, y), U_1)$ and $(\chi_2 = (r, \theta), U_2)$ are* **compatible** *because they have smooth transition functions. One can calculate $\chi_1 \circ \chi_2^{-1}$ is a smooth mapping on $\mathbb{R}^2$. Explicitly:*

$$\chi_1 \circ \chi_2^{-1}(u, v) = \chi_1(\chi_2^{-1}(u, v)) = \chi_1(u \cos v, u \sin v) = (u \cos(v), u \sin(v))$$

Technically, the use of the term **coordinate system** in calculus III is less strict than the concept which appears in manifold theory. Departure from injectivity in a geometrically tractible setting is manageable, but for the abstract setting, injectivity of the coordinate charts is important to many arguments.

[3] see my 2011 notes, or better yet study Loring Tu's *An Introduction to Manifolds*

Example 10.1.3. *Define* $\chi_{spherical}(x,y,z) = (r,\theta,\phi)$ *implicitly by the coordinate transformations*

$$x = r\cos(\theta)\sin(\phi), \quad y = r\sin(\theta)\sin(\phi), \quad z = r\cos(\phi)$$

These can be inverted,

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\left[\frac{y}{x}\right], \quad \phi = \cos^{-1}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right]$$

To show compatibility with the standard Cartesian coordinates we would need to select a subset of $\mathbb{R}^3$ *for which* $\chi_{spherical}$ *is 1-1 and the since* $\chi_{Cartesian} = Id$ *the transition functions are just* $\chi^{-1}_{spherical}$.

Example 10.1.4. *Define* $\chi_{cylindrical}(x,y,z) = (s,\theta,z)$ *implicitly by the coordinate transformations*

$$x = s\cos(\theta), \quad y = s\sin(\theta), \quad z = z$$

These can be inverted,

$$s = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left[\frac{y}{x}\right], \quad z = z$$

You can take $dom(\chi_{cylindrical}) = \{(x,y,z) \mid 0 < \theta < 2\pi, \} - \{(0,0,0)\}$

Example 10.1.5. *Let* $\mathcal{M} = V$ *where* V *is an* n*-dimensional vector space over* $\mathbb{R}$. *If* β_1 *is a basis for* V *then* $\Phi_{\beta_1} : V \to \mathbb{R}^n$ *gives a global coordinate chart. Moreover, if* β_2 *is another basis for* V *then* $\Phi_{\beta_2} : V \to \mathbb{R}^n$ *also gives a global coordinate chart. The transition function* $\theta_{12} = \Phi_{\beta_2} \circ \Phi^{-1}_{\beta_1}$ *is linear hence it is clearly smooth. In short, an* n*-dimensional vector space is an* n*-dimensional manifold. We could put non-linear charts on* V *if we wished, that freedom is new here, in linear algebra all the coordinate systems considered are linear.*

Example 10.1.6. *Let* $\mathcal{M} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$.

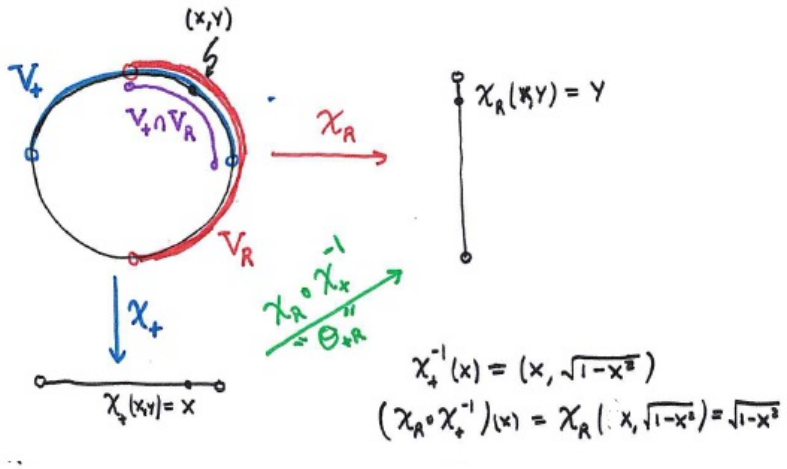
1. *Let* $V_+ = \{(x,y) \in \mathcal{M} \mid y > 0\} = dom(\chi_+)$ *and define* $\chi_+(x,y) = x$
2. *Let* $V_- = \{(x,y) \in \mathcal{M} \mid y < 0\} = dom(\chi_-)$ *and define* $\chi_-(x,y) = x$
3. *Let* $V_R = \{(x,y) \in \mathcal{M} \mid x > 0\} = dom(\chi_R)$ *and define* $\chi_R(x,y) = y$
4. *Let* $V_L = \{(x,y) \in \mathcal{M} \mid x < 0\} = dom(\chi_L)$ *and define* $\chi_L(x,y) = y$

The set of charts $\mathcal{A} = \{(V_+,\chi_+),(V_-,\chi_-),(V_R,\chi_R),(V_L,\chi_L)\}$ *forms an atlas on* $\mathcal{M}$ *which gives the circle a differentiable structure*[4]. *It is not hard to show the transition functions are smooth on the image of the intersection of their respect domains. For example,* $V_+ \cap V_R = W_{+R} = \{(x,y) \in \mathcal{M} \mid x,y > 0\}$, *it's easy to calculate that* $\chi_+^{-1}(x) = (x,\sqrt{1-x^2})$ *hence*

$$(\chi_R \circ \chi_+^{-1})(x) = \chi_R(x,\sqrt{1-x^2}) = \sqrt{1-x^2}$$

for each $x \in \chi_R(W_{+R})$. *Note* $x \in \chi_R(W_{+R})$ *implies* $0 < x < 1$ *hence it is clear the transition function is smooth.*

[4]meaning that if we adjoin the infinity of likewise compatible charts that defines a differentiable structure on $\mathcal{M}$



Similar calculations hold for all the other overlapping charts. This manifold is usually denoted $\mathcal{M} = S_1$.

A cylinder is the Cartesian product of a line and a circle. In other words, we can create a cylinder by gluing a copy of a circle at each point along a line. If all these copies line up and don't twist around then we get a cylinder. The example that follows here illustrates a more general pattern, we can take a given manifold an paste a copy at each point along another manifold by using a Cartesian product.

Example 10.1.7. *Let* $\mathcal{P} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

1. *Let* $V_+ = \{(x, y, z) \in \mathcal{P} \mid y > 0\} = dom(\chi_+)$ *and define* $\chi_+(x, y, z) = (x, z)$
2. *Let* $V_- = \{(x, y, z) \in \mathcal{P} \mid y < 0\} = dom(\chi_-)$ *and define* $\chi_-(x, y, z) = (x, z)$
3. *Let* $V_R = \{(x, y, z) \in \mathcal{P} \mid x > 0\} = dom(\chi_R)$ *and define* $\chi_R(x, y, z) = (y, z)$
4. *Let* $V_L = \{(x, y, z) \in \mathcal{P} \mid x < 0\} = dom(\chi_L)$ *and define* $\chi_L(x, y, z) = (y, z)$

The set of charts $\mathcal{A} = \{(V_+, \chi_+), (V_-, \chi_-), (V_R, \chi_R), (V_L, \chi_L)\}$ *forms an atlas on* $\mathcal{P}$ *which gives the cylinder a differentiable structure. It is not hard to show the transition functions are smooth on the image of the intersection of their respective domains. For example,* $V_+ \cap V_R = W_{+R} = \{(x, y, z) \in \mathcal{P} \mid x, y > 0\}$, *it's easy to calculate that* $\chi_+^{-1}(x, z) = (x, \sqrt{1 - x^2}, z)$ *hence*

$$(\chi_R \circ \chi_+^{-1})(x, z) = \chi_R(x, \sqrt{1 - x^2}, z) = (\sqrt{1 - x^2}, z)$$

for each $(x, z) \in \chi_R(W_{+R})$. *Note* $(x, z) \in \chi_R(W_{+R})$ *implies* $0 < x < 1$ *hence it is clear the transition function is smooth. Similar calculations hold for all the other overlapping charts.*

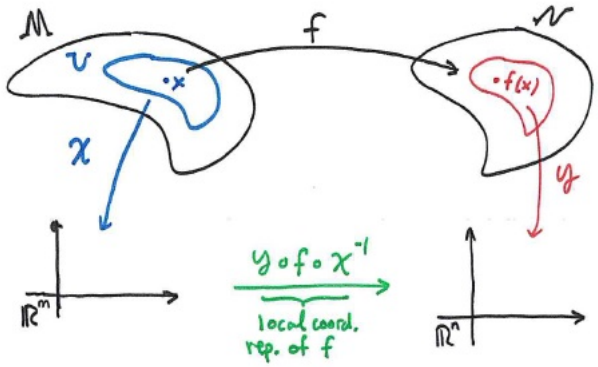
Generally, given two manifolds $\mathcal{M}$ and $\mathcal{N}$ we can construct $\mathcal{M} \times \mathcal{N}$ by taking the Cartesian product of the charts. Suppose $\chi_{\mathcal{M}} : V \subseteq \mathcal{M} \to U \subseteq \mathbb{R}^m$ and $\chi_{\mathcal{N}} : V' \subseteq \mathcal{N} \to U' \subseteq \mathbb{R}^n$ then you can define the product chart $\chi : V \times V' \to U \times U'$ as $\chi = \chi_{\mathcal{M}} \times \chi_{\mathcal{N}}$. The Cartesian product $\mathcal{M} \times \mathcal{N}$ together with all such product charts naturally is given the structure of an $(m + n)$-dimensional manifold. For example, in the preceding example we took $\mathcal{M} = S_1$ and $\mathcal{N} = \mathbb{R}$ to consruct $\mathcal{P} = S_1 \times \mathbb{R}$.

Example 10.1.8. *The 2-torus, or donut, is constructed as $T_2 = S_1 \times S_1$. The n-torus is constructed by taking the product of n-circles:*

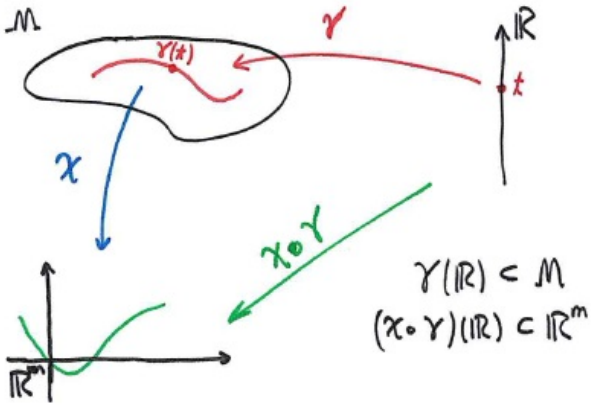
$$T_n = \underbrace{S_1 \times S_1 \times \cdots \times S_1}_{n \text{ copies}}$$

The atlas on this space can be obtained by simply taking the product of the S_1 charts n-times.

Recall from our study of linear algebra that vector space structure is greatly elucidated by the study of linear transformations. In our current context, the analogous objects are smooth maps. These are natural mappings between manifolds. In particular, suppose $\mathcal{M}$ is an m-fold and $\mathcal{N}$ is an n-fold with $f : U \subseteq \mathcal{M} \to \mathcal{N}$ is a function. Then, we say f is **smooth** iff all local coordinate representatives of f are smooth mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$. See the diagram below:



This definition allows us to discuss smooth curves on manifolds. A curve $\gamma : I \subseteq \mathbb{R} \to \mathcal{M}$ is smooth iff $\chi \circ \gamma : I \to \mathbb{R}^n$ is a smooth curve in $\mathbb{R}^n$.



Finally, just for your information, if a bijection between manifolds is smooth with smooth inverse then the manifolds are said to be **diffeomorphic**. One fascinating result of recent mathematics is

that $\mathbb{R}^4$ permits distinct differentiable structures in the sense that there does not exist a diffeomorphism between certain atlases[5]. Curiously, up to diffeomorphism, there is just one differentiable structure on $\mathbb{R}^n$ for $n \neq 4$. Classifying possible differentiable structures for a given point set is an interesting and ongoing problem.

10.2 vectors as derivations

To begin, let us define the set of locally smooth functions at $p \in \mathcal{M}$:

$$C^\infty(p) = \{f : \mathcal{M} \to \mathbb{R} \mid f \text{ is smooth on an open set containing } p\}$$

In particular, we suppose $f \in C^\infty(p)$ to mean there exists a patch $\phi : U \to V \subseteq \mathcal{M}$ such that f is smooth on V. Since we use Cartesian coordinates on $\mathbb{R}$ by convention it follows that $f : V \to \mathbb{R}$ smooth indicates the local coordinate representative $f \circ \phi : U \to \mathbb{R}$ is smooth (it has continuous partial derivatives of all orders).

Definition 10.2.1.

Suppose $X_p : C^\infty(p) \to \mathbb{R}$ is a linear transformation which satisfies the Leibniz rule then we say X_p is a **derivation** on $C^\infty(p)$. Moreover, we denote $X_p \in \mathcal{D}_p\mathcal{M}$ iff $X_p(f + cg) = X_p(f) + cX_p(g)$ and $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$ for all $f, g \in C^\infty(p)$ and $c \in \mathbb{R}$.

Example 10.2.2. *Let $\mathcal{M} = \mathbb{R}$ and consider $X_{t_o} = d/dt|_{t_o}$. Clearly X is a derivation on smooth functions near t_o.*

Example 10.2.3. *Consider $\mathcal{M} = \mathbb{R}^2$. Pick $p = (x_o, y_o)$ and define $X_p = \frac{\partial}{\partial x}\big|_p$ and $Y_p = \frac{\partial}{\partial y}\big|_p$. Once more it is clear that $X_p, Y_p \in \mathcal{D}(p)\mathbb{R}^2$. These derivations action is accomplished by partial differentiation followed by evaluation at p.*

Example 10.2.4. *Suppose $\mathcal{M} = \mathbb{R}^m$. Pick $p \in \mathbb{R}^m$ and define $X = \frac{\partial}{\partial x^j}\big|_p$. Clearly this is a derivation for any $j \in \mathbb{N}_m$.*

Are the other types of derivations? Is the only thing a derivation is is a partial derivative operator? Before we can explore this question we need to define partial differentiation on a manifold. We should hope the definition is consistent with the langauge we already used in multivariate calculus (and the preceding pair of examples) and yet is also general enough to be stated on any abstract smooth manifold.

[5] there is even a whole book devoted to this exotic chapter of mathematics, see the *The Wild World of 4-Manifolds* by Alexandru Scorpan. This is on my list of books I "need" to buy.

Definition 10.2.5.

Let $\mathcal{M}$ be a smooth m-dimensional manifold and let $\phi : U \to V$ be a local parametrization with $p \in V$. The **j-th coordinate function** $x^j : V \to \mathbb{R}$ is the j-component function of $\phi^{-1} : V \to U$. In other words:

$$\phi^{-1}(p) = x(p) = (x^1(p), x^2(p), \ldots, x^m(p))$$

These x^j are **manifold coordinates**. In constrast, we will denote the standard Cartesian coordinates in $U \subseteq \mathbb{R}^m$ via u^j so a typical point has the form $(u^1, u^2, \ldots, u^m)$ and viewed as functions $u^j : \mathbb{R}^m \to \mathbb{R}$ where $u^j(v) = e^j(v) = v^j$. We define the **partial derivative with respect to** x^j at p for $f \in C^\infty(p)$ as follows:

$$\frac{\partial f}{\partial x^j}(p) = \frac{\partial}{\partial u^j}\Big[(f \circ \phi)(u)\Big]\Big|_{u=\phi^{-1}(p)} = \frac{\partial}{\partial u^j}\Big[f \circ x^{-1}\Big]\Big|_{x(p)}.$$

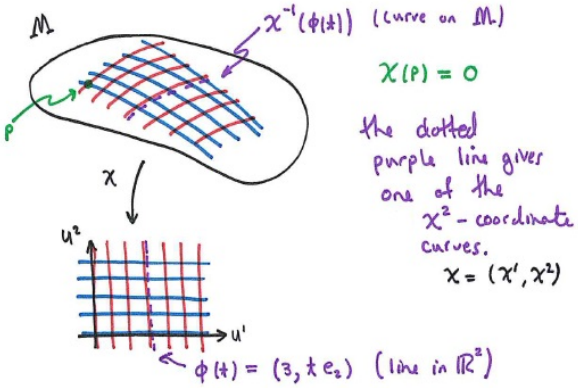
The idea of the definition is simply to take the function f with domain in $\mathcal{M}$ then pull it back to a function $f \circ x^{-1} : U \subseteq \mathbb{R}^m \to V \to \mathbb{R}$ on $\mathbb{R}^m$. Then we can take partial derivatives of $f \circ x^{-1}$ in the same way we did in multivariate calculus. In particular, the partial derivative w.r.t. u^j is calculated by:

$$\frac{\partial f}{\partial x^j}(p) = \frac{d}{dt}\Big[\big(f \circ \phi\big)(x(p) + te_j)\Big]_{t=0}$$

which is precisely the directional derivative of $f \circ x^{-1}$ in the j-direction at $x(p)$. In fact, Note

$$\big(f \circ \phi\big)(x(p) + te_j) = f(x^{-1}(x(p) + te_j)).$$

The curve $t \to x^{-1}(x(p) + te_j)$ is the curve on $\mathcal{M}$ through p where all coordinates are fixed except the j-coordinate. It is a *coordinate curve* on $\mathcal{M}$.



Notice in the case that $\mathcal{M} = \mathbb{R}^m$ is given Cartesian coordinate $\phi = Id$ then $x^{-1} = Id$ as well and the $t \to x^{-1}(x(p) + te_j)$ reduces to $t \to p + te_j$ which is just the j-th coordinate curve through p on

$\mathbb{R}^m$. It follows that the partial derivative defined for manifolds naturally reduces to the ordinary partial derivative in the context of $\mathcal{M} = \mathbb{R}^m$ with Cartesian coordinates. The beautiful thing is that almost everything we know for ordinary partial derivatives equally well transfers to $\frac{\partial}{\partial x^j}\big|_p$.

Theorem 10.2.6. *Partial differentiation on manifolds*

Let $\mathcal{M}$ be a smooth m-dimensional manifold with coordinates $x^1, x^2, \ldots, x^m$ near p. Furthermore, suppose coordinates $y^1, y^2, \ldots, y^m$ are also defined near p. Suppose $f, g \in C^\infty(p)$ and $c \in \mathbb{R}$ then:

1. $\frac{\partial}{\partial x^j}\big|_p[f+g] = \frac{\partial f}{\partial x^j}\big|_p + \frac{\partial g}{\partial x^j}\big|_p$
2. $\frac{\partial}{\partial x^j}\big|_p[cf] = c\frac{\partial f}{\partial x^j}\big|_p$
3. $\frac{\partial}{\partial x^j}\big|_p[fg] = f(p)\frac{\partial g}{\partial x^j}\big|_p + \frac{\partial f}{\partial x^j}\big|_p g(p)$
4. $\frac{\partial x^i}{\partial x^j}\big|_p = \delta_{ij}$
5. $\sum_{k=1}^m \frac{\partial x^k}{\partial y^j}\big|_p \frac{\partial y^i}{\partial x^k}\big|_p = \delta_{ij}$
6. $\frac{\partial f}{\partial y^j}\big|_p = \sum_{k=1}^m \frac{\partial x^k}{\partial y^j}\big|_p \frac{\partial f}{\partial x^k}\big|_p$

Proof: The proof of (1.) and (2.) follows from the calculation below:

$$\begin{aligned}\frac{\partial(f+cg)}{\partial x^j}(p) &= \frac{\partial}{\partial u^j}\Big[(f+cg)\circ x^{-1}\Big]\Big|_{x(p)} \\ &= \frac{\partial}{\partial u^j}\Big[f\circ x^{-1} + cg\circ x^{-1}\Big]\Big|_{x(p)} \\ &= \frac{\partial}{\partial u^j}\Big[f\circ x^{-1}\Big]\Big|_{x(p)} + c\frac{\partial}{\partial u^j}\Big[g\circ x^{-1}\Big]\Big|_{x(p)} \\ &= \frac{\partial f}{\partial x^j}(p) + c\frac{\partial g}{\partial x^j}(p) \end{aligned} \tag{10.1}$$

The key in this argument is that composition $(f+cg)\circ x^{-1} = f\circ x^{-1} + cg\circ x^{-1}$ along side the linearity of the partial derivative. Item (3.) follows from the identity $(fg)\circ x^{-1} = (f\circ x^{-1})(g\circ x^{-1})$ in tandem with the product rule for a partial derivative on $\mathbb{R}^m$. The reader may be asked to complete the argument for (3.) in the homework. Continuing to (4.) we calculate from the definition:

$$\frac{\partial x^i}{\partial x^j}\bigg|_p = \frac{\partial}{\partial u^j}\Big[(x^i\circ x^{-1})(u)\Big]\Big|_{x(p)} = \frac{\partial u^i}{\partial u^j}\bigg|_{x(p)} = \delta_{ij}.$$

where the last equality is known from multivariate calculus. In invite the reader to prove it from the definition if unaware of this fact. Before we prove (5.) it helps to have a picture and a bit

more notation in mind. Near the point p we have two coordinate charts $x : V \to U \subseteq \mathbb{R}^m$ and $y : V \to W \subseteq \mathbb{R}^m$, we take the chart domain V to be small enough so that both charts are defined. Denote Cartesian coordinates on U by $u^1, u^2, \ldots, u^m$ and for W we likewise use Cartesian coordinates $w^1, w^2, \ldots, w^m$. Let us denote patches ϕ, ψ as the inverses of these charts; $\phi^{-1} = x$ and $\psi^{-1} = y$. Transition functions $\psi^{-1} \circ \phi = y \circ x^{-1}$ are mappings from $U \subseteq \mathbb{R}^m$ to $W \subseteq \mathbb{R}^m$ and we note

$$\frac{\partial}{\partial u^j}\left[(y^i \circ x^{-1})(u)\right] = \frac{\partial y^i}{\partial x^j}$$

Likewise, the inverse transition functions $\phi^{-1} \circ \psi = x \circ y^{-1}$ are mappings from $W \subseteq \mathbb{R}^m$ to $U \subseteq \mathbb{R}^m$

$$\frac{\partial}{\partial w^j}\left[(x^i \circ y^{-1})(w)\right] = \frac{\partial x^i}{\partial y^j}$$

Recall that if $F, G : \mathbb{R}^m \to \mathbb{R}^m$ and $F \circ G = Id$ then $F'G' = I$ by the chainrule, hence $(F')^{-1} = G'$. Apply this general fact to the transition functions, we find their derivative matrices are inverses. Item (5.) follows. In matrix notation we item (5.) reads $\frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = I$. Item (6.) follows from:

$$\begin{aligned}
\left.\frac{\partial f}{\partial y^j}\right|_p &= \left.\frac{\partial}{\partial w^j}\left[(f \circ y^{-1})(w)\right]\right|_{y(p)} \\
&= \left.\frac{\partial}{\partial w^j}\left[(f \circ x^{-1} \circ x \circ y^{-1})(w)\right]\right|_{y(p)} \\
&= \left.\frac{\partial}{\partial w^j}\left[\big(f \circ x^{-1}\big)(u^1(w), \ldots, u^m(w))\right]\right|_{y(p)} \quad : \quad \text{where } u^k(w) = (x \circ y^{-1})^k(w) \\
&= \sum_{k=1}^{m} \left.\frac{\partial (x \circ y^{-1})^k}{\partial w^j}\right|_{y(p)} \left.\frac{\partial (f \circ x^{-1})}{\partial u^k}\right|_{(x \circ y^{-1})(y(p))} \quad : \quad \text{chain rule} \\
&= \sum_{k=1}^{m} \left.\frac{\partial (x^k \circ y^{-1})}{\partial w^j}\right|_{y(p)} \left.\frac{\partial (f \circ x^{-1})}{\partial u^k}\right|_{x(p)} \\
&= \sum_{k=1}^{m} \frac{\partial x^k}{\partial y^j}\Big|_p \frac{\partial f}{\partial x^k}\Big|_p
\end{aligned}$$

The key step was the multivariate chain rule. □

This theorem proves we can lift calculus on $\mathbb{R}^m$ to $\mathcal{M}$ in a natural manner. Moreover, we should note that items (1.), (2.) and (3.) together show $\frac{\partial}{\partial x^i}\big|_p$ is a derivation at p. Item (6.) should remind the reader of the contravariant vector discussion. Removing the f from the equation reveals that

$$\left.\frac{\partial}{\partial y^j}\right|_p = \sum_{k=1}^{m} \left.\frac{\partial x^k}{\partial y^j}\right|_p \left.\frac{\partial}{\partial x^k}\right|_p$$

A notation convenient to the current discussion is that a contravariant transformation is $(p, v_x) \to (p, v_y)$ where $v_y = P v_x$ and $P = (y \circ x^{-1})'(x(p)) = \frac{\partial y}{\partial x}\big|_{x(p)}$. Notice this is the inverse of what we see

in (6.). This suggests that the partial derivatives change coordinates like as a basis for the tangent space. To complete this thought we need a few well-known propositions for derivations.

Proposition 10.2.7. *derivations on constant function gives zero.*

If $f \in C^\infty(p)$ is a constant function and $X_p \in \mathcal{D}_p\mathcal{M}$ then $X_p(f) = 0$.

Proof: Suppose $f(x) = c$ for all $x \in V$, define $g(x) = 1$ for all $x \in V$ and note $f = fg$ on V. Since X_p is a derivation is satisfies the Leibniz rule hence

$$X_p(f) = X_p(fg) = f(p)X_p(g) + X(f)g(p) = cX_p(g) + X_p(f) \quad \Rightarrow \quad cX_p(g) = 0.$$

Moreover, by homogeneity of X_p, note $cX_p(g) = X_p(cg) = X_p(f)$. Thus, $X_p(f) = 0$. □

Proposition 10.2.8.

If $f, g \in C^\infty(p)$ and $f(x) = g(x)$ for all $x \in V$ and $X_p \in \mathcal{D}_p\mathcal{M}$ then $X_p(f) = X_p(g)$.

Proof: Note that $f(x) = g(x)$ implies $h(x) = f(x) - g(x) = 0$ for all $x \in V$. Thus, the previous proposition yields $X_p(h) = 0$. Thus, $X_p(f - g) = 0$ and by linearity $X_p(f) - X_p(g) = 0$. The proposition follows. □

Proposition 10.2.9.

Suppose $X_p \in \mathcal{D}_p\mathcal{M}$ and x is a chart defined near p,

$$X_p = \sum_{j=1}^{m} X_p(x^j) \left.\frac{\partial}{\partial x^j}\right|_p$$

Proof: this is a less trivial proposition. We need a standard lemma before we begin.

Lemma 10.2.10.

Let p be a point in smooth manifold $\mathcal{M}$ and let $f : \mathcal{M} \to \mathbb{R}$ be a smooth function. If $x : V \to U$ is a chart with $p \in V$ and $x(p) = 0$ then there exist smooth functions $g_j : \mathcal{M} \to \mathbb{R}$ whose values at p satisfy $g_j(p) = \frac{\partial f}{\partial x^j}(p)$. In addition, for all q near enough to p we have $f(q) = f(p) + \sum_{k=1}^{m} x^j(q)g_j(q)$

Proof: follows from proving a similar identity on $\mathbb{R}^m$ then lifting to the manifold. I leave this as a nontrivial exercise for the reader. This can be found in many texts, see Burns and Gidea page 29 for one source. It should be noted that the manifold must be smooth for this construction to hold. It turns out the set of derivations on a C^k-manifold forms an infinite-dimensional vector space over $\mathbb{R}$, see Lawrence Conlon's *Differentiable Manifolds* page 49. ▽

Consider $f \in C^\infty(p)$, and use the lemma, we assume $x(p) = 0$ and $g_j(p) = \frac{\partial f}{\partial x^j}(p)$:

$$\begin{aligned} X_p(f) &= X_p\Big(f(p) + \sum_{k=1}^m x^j(q)g_j(q)\Big) \\ &= X_p(f(p)) + \sum_{k=1}^m X_p\big(x^j(q)g_j(q)\big) \\ &= \sum_{k=1}^m \Big[X_p(x^j)g_j(q) + x^j(p)X_p(g_j(q))\Big] \\ &= \sum_{k=1}^m X_p(x^j)\frac{\partial f}{\partial x^j}(p). \end{aligned}$$

The calculation above holds for arbitrary $f \in C^\infty(p)$ hence the proposition follows. □

We've answered the question posed earlier in this section. It is true that every derivation of a manifold is simply a linear combination of partial derivatives. We can say more. The set of derivations at p naturally forms a vector space under the usual addition and scalar multiplication of operators: if $X_p, Y_p \in \mathcal{D}_p\mathcal{M}$ then we define $X_p + Y_p$ by $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$ and cX_p by $(cX_p)(f) = cX_p(f)$ for all $f, g \in C^\infty(p)$ and $c \in \mathbb{R}$. It is easy to show $\mathcal{D}_p\mathcal{M}$ is a vector space under these operations. Moreover, the preceding proposition shows that $\mathcal{D}_p\mathcal{M} = span\{\frac{\partial f}{\partial x^j}\big|_p\}_{j=1}^m$ hence $\mathcal{D}_p\mathcal{M}$ is an m-dimensional vector space[6].

Finally, let's examine coordinate change for derivations. Given two coordinate charts x, y at $p \in \mathcal{M}$ we have two ways to write the derivation X_p:

$$X_p = \sum_{j=1}^m X_p(x^j)\frac{\partial}{\partial x^j}\Big|_p \qquad \text{or} \qquad X_p = \sum_{k=1}^m X_p(y^k)\frac{\partial}{\partial y^k}\Big|_p$$

It is simple to connect these formulas. Whereas, for y-coordinates,

$$X_p(y^k) = \sum_{j=1}^m X_p(x^j)\frac{\partial y^k}{\partial x^j}\Big|_p \tag{10.2}$$

This is the contravariant transformation rule. In contrast, recall $\frac{\partial}{\partial y^j}\big|_p = \sum_{k=1}^m \frac{\partial x^k}{\partial y^j}\big|_p \frac{\partial}{\partial x^k}\big|_p$. We should have anticipated this pattern since from the outset it is clear there is no coordinate dependence in the definition of a derivation.

Definition 10.2.11. *tangent space*

We denote $T_p\mathcal{M} = derT_p\mathcal{M}$.

[6] technically, we should show the coordinate derivations $\frac{\partial}{\partial x^j}\big|_p$ are linearly independent to make this conclusion. I don't suppose we've done that directly at this juncture. You might find this as a homework

10.2.1 concerning the geometry of derivations

An obvious question we should ask:

How are derivations related to tangent vectors geometrically?

To give a proper answer, we focus our attention to $\mathbb{R}^3$ and I follow Barret Oneil's phenomenal text on *Elementary Differential Geometry*, second edition. Consider a function $f : \mathbb{R}^3 \to \mathbb{R}$ and calculate the directional derivative in the $v = \langle a, b, c\rangle$-direction at the point p. We make no assumption that $||v|| = 1$, this is the same directional derivative for which we discussed the relation with the Frechet derivative in an earlier chapter. If f is smooth,

$$Df(p)(v) = v \bullet (\nabla f)(p) = a\frac{\partial f}{\partial x}(p) + b\frac{\partial f}{\partial y}(p) + \frac{\partial f}{\partial z}(p) = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right)\Big|_p f$$

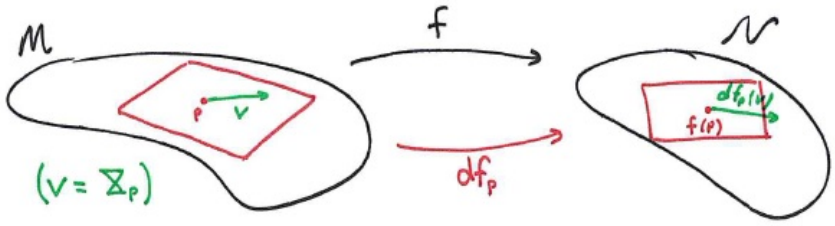
Therefore, we find the following interpretation of the derivation $X_p = (a\partial_x + b\partial_y + c\partial_z)\,|_p$:

When a derivation $X_p = (a\partial_x + b\partial_y + c\partial_z)\,|_p$ acts on a smooth function f it describes the rate at which f changes at p in the $\langle a, b, c\rangle$ direction. In other words, a derivation at p generates directional derivatives of functions at p.

Therefore, our work over the last few pages can be interpreted as abstracting the directional derivative to manifolds. In particular, the partial derivatives with respect to manifold coordinate x^j measure the rate of change of functions along the curve on the manifold which allows x^j to vary while all the other coordinates are held fixed.

10.3 differential for manifolds, the push-forward

In this section we generalize the concept of the differential to the context of manifolds. Recall that for $F : U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^n$ the differential $d_pF : \mathbb{R}^m \to \mathbb{R}^n$ was a linear transformation which best approximated the change in F near p. Notice that while the domain of F could be a mere subset of $\mathbb{R}^m$ the differential always took all of $\mathbb{R}^m$ as its domain. This suggests we should really think of the differential as a mapping which transports tangent vectors to U to tangent vectors at V. I find the following picture helpful at times, here I picture the tangent space as if it is attached to the point p on the manifold. Keep in mind this is an abstract picture:



Often d_pf is called the **push-forward** by f at p because it pushes tangent vectors in the same direction as the mapping transports points.

Definition 10.3.1. *differential for manifolds.*

Suppose $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds of dimension m and n respective. Furthermore, suppose $f : \mathcal{M} \to \mathcal{N}$ is a smooth mapping. We define $d_pf : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ as follows: for each $X_p \in T_p\mathcal{M}$ and $g \in C^\infty(f(p))$

$$d_pf(X_p)(g) = X_p(g \circ f).$$

Notice that $g : dom(g) \subseteq \mathcal{N} \to \mathbb{R}$ and consequently $g \circ f : \mathcal{M} \to \mathcal{N} \to \mathbb{R}$ and it follows $g \circ f \in C^\infty(p)$ and it is natural to find $g \circ f$ in the domain of X_p. In addition, it is not hard to show $d_pf(X_p) \in \mathcal{D}_{f(p)}\mathcal{N}$. Observe:

1. $d_pf(X_p)(g + h) = X_p((g + h) \circ f) = X_p(g \circ f + h \circ f) = X_p(g \circ f) + X_p(h \circ f)$
2. $d_pf(X_p)(cg) = X_p((cg) \circ f) = X_p(cg \circ f)) = cX_p(g \circ f)) = cd_pf(X_p)(g)$

The proof of the Leibniz rule is similar. In this section we generalize the concept of the differential to the context of manifolds. Recall that for $F : U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^n$ the differential $d_pF : \mathbb{R}^m \to \mathbb{R}^n$ was a linear transformation which best approximated the change in F near p. Notice that while the domain of F could be a mere subset of $\mathbb{R}^m$ the differential always took all of $\mathbb{R}^m$ as its domain. This suggests we should really think of the differential as a mapping which transports tangent vectors to U to tangent vectors at V. Therefore, d_pf is called the **push-forward**.

Definition 10.3.2. *differential for manifolds.*

Suppose $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds of dimension m and n respective. Furthermore, suppose $f : \mathcal{M} \to \mathcal{N}$ is a smooth mapping. We define $d_pf : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ as follows: for each $X_p \in T_p\mathcal{M}$ and $g \in C^\infty(f(p))$

$$d_pf(X_p)(g) = X_p(g \circ f).$$

Notice that $g : dom(g) \subseteq \mathcal{N} \to \mathbb{R}$ and consequently $g \circ f : \mathcal{M} \to \mathcal{N} \to \mathbb{R}$ and it follows $g \circ f \in C^\infty(p)$ and it is natural to find $g \circ f$ in the domain of X_p. In addition, it is not hard to show $d_pf(X_p) \in \mathcal{D}_{f(p)}\mathcal{N}$. Observe:

1. $d_pf(X_p)(g + h) = X_p((g + h) \circ f) = X_p(g \circ f + h \circ f) = X_p(g \circ f) + X_p(h \circ f)$
2. $d_pf(X_p)(cg) = X_p((cg) \circ f) = X_p(cg \circ f)) = cX_p(g \circ f)) = cd_pf(X_p)(g)$

The proof of the Leibniz rule is similar.

10.3.1 intuition for the push-forward

Perhaps it will help to see how the push-forward appears in calculus III. I'll omit the point-dependence to reduce some clutter. Consider $f : \mathbb{R}^2_{r\theta} \to \mathbb{R}^2_{xy}$ defined by $f(r,\theta) = (r\cos\theta, r\sin\theta)$. Consider, for g a smooth function on $\mathbb{R}^2_{xy}$, using the chain-rule,

$$df\left(\frac{\partial}{\partial r}\right)(g) = \frac{\partial}{\partial r}(g\circ f) = \frac{\partial g}{\partial x}\frac{\partial (x\circ f)}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial (y\circ f)}{\partial r}$$

Note $x\circ f = r\cos\theta$ and $y\circ f = r\sin\theta$. Hence:

$$df\left(\frac{\partial}{\partial r}\right)(g) = \cos\theta\frac{\partial g}{\partial x} + \sin\theta\frac{\partial g}{\partial y}$$

This holds for all g hence we derive,

$$df\left(\frac{\partial}{\partial r}\right) = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$$

A similar calculation shows

$$df\left(\frac{\partial}{\partial \theta}\right) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

We could go through the reverse calculation for f^{-1} and derive that:

$$df^{-1}\left(\frac{\partial}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \qquad \text{and} \qquad df^{-1}\left(\frac{\partial}{\partial y}\right) = \sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial \theta}$$

These formulas go to show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ transforms to $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

10.3.2 a pragmatic formula for the push-forward

In practice I quote the following result as the definition. However, Definition 10.3.2 is prefered by many for its coordinate independence. As we've seen in the intuition example, coordinates tend to arise in actual calculations.

Proposition 10.3.3.

If $F : M \to N$ is a differentiable mapping of manifolds and (x^i) are coordinates$(i = 1,\dots,m)$ on M and (y^j) are coordinates $(j = 1,\dots,n)$ on N which contain $p \in M$ and $F(p) \in N$ respective then:

$$(dF)_p\left(\sum_{i=1}^m X^i \frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{i=1}^m\sum_{j=1}^n X^i \frac{\partial (y^j\circ F)}{\partial x^i}\frac{\partial}{\partial y^j}\Big|_{F(p)}.$$

Proof: Notice the absence of the f in this formula as compared to Defnition 10.3.2 $dF(X)(f) = X(f \circ F)$. To show equivalence, we can expand on our definition, we assume here that $X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}\Big|_p$ thus:

$$X(f \circ F) = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}\Big|_p (f \circ F)$$

note f is a function on N hence the chain rule gives:

$$X(f \circ F) = \sum_{i=1}^{m} X^i \sum_{j=1}^{n} \frac{\partial f}{\partial y^j}(F(p)) \frac{\partial (y^j \circ F)}{\partial x^i}$$

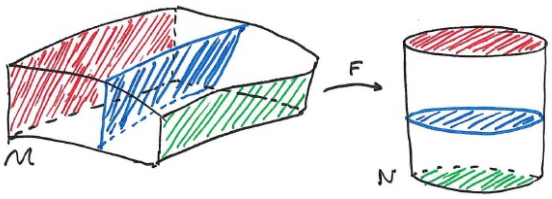
But, we can write this as:

$$X(f \circ F) = \left(\sum_{i=1}^{m} X^i \sum_{j=1}^{n} \frac{\partial (y^j \circ F)}{\partial x^i} \frac{\partial}{\partial y^j}\Big|_{F(p)} \right) (f).$$

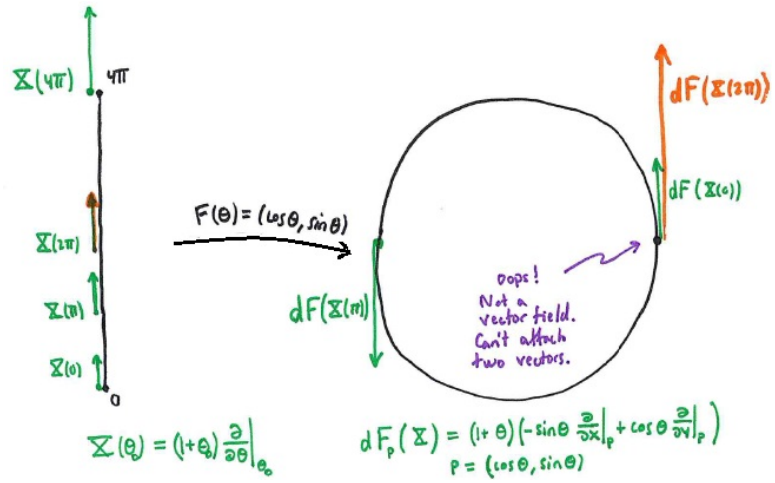
Therefore, as f is arbitrary, we have shown the claim of the proposition. □

Observe that the difference between Definition 10.3.2 and Proposition 10.3.3 is merely an application of the chain-rule.

The push-forward is more than just coordinate change. If we consider a mapping between spaces of disparate dimension then the push-forward captures something about the mapping in question and the domain and range spaces. For example, the existence of a nontrivial vector field on the whole of a manifold implies the existence of a foliation of the manifold.



If the mapping is a diffeomorphism then we expect it will carry the nontrivial vector field to the range space. However, if the mapping is not injective then there is no assurance a vector field even maps to a vector field. We could attach two vectors to a point in the range for a two-to-one map.For example, this mapping wraps around the circle and when it hits the circle the second time the vector pushed-forward does not match what was pushed forward the first time. It follows that push-forward of the vector field does not form a vector field in this case:



The pull-back (introducted in Section 10.7) is also an important tool to compare geometry of different spaces. We'll see in Section 10.10 how the pull-back even allows us to write a general formula for calculating the potential energy function of a flux-form of arbitrary degree. This captures the electric and magnetic potentials of electromagnetism and much more we have yet to discover experimentally. That said, the pull-back is formulated in terms of the push-forward we consider here thus the importance of the push-forward is hard to overstate.

Example 10.3.4. *Suppose $F : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ is defined by*

$$F(x,y) = e^x \begin{bmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{bmatrix}$$

Let $\mathbb{R}^2$ have the usual (x,y)-coordinate chart and let Z^{ij} defined by

$$Z^{ij}(A) = A_{ij}$$

for $\mathbb{R}^{2\times 2}$ form the global coordinate chart for 2×2 matrices. Let us calculate the push-forward of the coordinate vector $\partial_x|_p$:

$$dF_p(\partial_x|_p) = \sum_{i,j=1}^{2} \frac{\partial(Z^{ij}\circ F)}{\partial x} \frac{\partial}{\partial Z^{ij}}\bigg|_{F(p)} \quad \& \quad dF_p(\partial_y|_p) = \sum_{i,j=1}^{2} \frac{\partial(Z^{ij}\circ F)}{\partial y} \frac{\partial}{\partial Z^{ij}}\bigg|_{F(p)}$$

Observe that:

$$Z^{11}(F(x,y)) = e^x\cos y = Z^{22}(F(x,y)), \qquad Z^{21}(F(x,y)) = e^x \sin y = -Z^{12}(F(x,y)).$$

From which we derive,

$$dF_p(\partial_x) = e^x\cos y(\partial_{11}+\partial_{22}) + e^x \sin y(\partial_{21}-\partial_{12})$$
$$dF_p(\partial_y) = -e^x\sin y(\partial_{11}+\partial_{22}) + e^x \cos y(\partial_{21}-\partial_{12})$$

Here ∂_x, ∂_y are at $p \in \mathbb{R}^2$ whereas ∂_{ij} is at $F(p) \in \mathbb{R}^{2\times 2}$. However, these are constant vector fields so the point-dependence is not too interesting.

10.4 cotangent space

The tangent space to a smooth manifold $\mathcal{M}$ is a vector space of derivations and we denote it by $T_p\mathcal{M}$. The dual space to this vector space is called the **cotangent space** and the typical elements are called **covectors**.

Definition 10.4.1. *cotangent space $T_p\mathcal{M}^*$*

Suppose $\mathcal{M}$ is a smooth manifold and $T_p\mathcal{M}$ is the tangent space at $p \in \mathcal{M}$. We define, $T_p\mathcal{M}^* = \{\alpha_p : T_p\mathcal{M} \to \mathbb{R} \mid \alpha \text{ is linear}\}$.

If x is a local coordinate chart at p and $\frac{\partial}{\partial x^1}\big|_p, \frac{\partial}{\partial x^2}\big|_p, \ldots, \frac{\partial}{\partial x^m}\big|_p$ is a basis for $T_p\mathcal{M}$ then we denote the dual basis $d_px^1, d_px^2, \ldots, d_px^m$ where $d_px^i\big(\partial_k\big|_p\big) = \delta_{ik}$. Moreover, if α is a covector at p then[7]:

$$\alpha = \sum_{k=1}^{m} \alpha_k dx^k$$

where $\alpha_k = \alpha\left(\frac{\partial}{\partial x^k}\Big|_p\right)$ and dx^k is a short-hand for d_px^k. We should understand that covectors are defined at a point even if the point is not explicitly indicated in a particular context. This does lead to some ambiguity in the same way that the careless identification of the function f and it's value $f(x)$ does throughout calculus. That said, an abbreviated notation is often important to help us see through more difficult patterns without getting distracted by the minutia of the problem.

You might worry the notation used for the differential and our current notation for the dual basis of covectors is not consistent. After all, we have two rather different meanings for d_px^k at this time:

1. $x^k : V \to \mathbb{R}$ is a smooth function hence $d_px^k : T_p\mathcal{M} \to T_{x^k(p)}\mathbb{R}$ is defined as a *push-forward*, $d_px^k(X_p)(g) = X_p(g \circ x^k)$
2. $d_px^k : T_p\mathcal{M} \to \mathbb{R}$ where $d_px^k\big(\partial_j\big|_p\big) = \delta_{jk}$

It is customary to identify $T_{x^k(p)}\mathbb{R}$ with $\mathbb{R}$ hence there is no trouble. Let us examine how the dual-basis condition can be *derived* for the differential, suppose $g : \mathbb{R} \to \mathbb{R}$ hence $g \circ x^k : V \to \mathbb{R}$,

$$d_px^k\left(\frac{\partial}{\partial x^j}\Big|_p\right)(g) = \frac{\partial}{\partial x^j}\Big|_p (g(x^k)) = \underbrace{\frac{\partial x^k}{\partial x^j}\Big|_p \frac{dg}{dt}\Big|_{x^k(p)}}_{chain\ rule} = \delta_{jk}\frac{d}{dt}\Big|_{x^k(p)}(\ g\ ) = \delta_{jk}g$$

Where, we've made the identification $1 = \frac{d}{dt}\Big|_{x^k(p)}$ (which is the nut and bolts of $T_{x^k(p)}\mathbb{R} = \mathbb{R}$) and hence have the beautiful identity:

$$\boxed{d_px^k\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \delta_{jk}.}$$

[7] we explained this for an arbitrary vector space V and its dual V^* in a previous chapter, we simply apply those results once more here in the particular context $V = T_p\mathcal{M}$

In contrast, there is no need to derive this for case (2.) since in that context this serves as the definition for the object. Personally, I find the multiple interpretations of objects in manifold theory is one of the most difficult aspects of the theory. On the other hand, the notation is really neat once you understand how subtly it assumes many theorems. You should understand the notation we enjoy at this time is the result of generations of mathematical thought. Following a similar derivation for an arbitrary vector $X_p \in T_p\mathcal{M}$ and $f : \mathcal{M} \to \mathbb{R}$ we find

$$\boxed{d_pf(X_p) = X_p(f)}$$

This notation is completely consistent with the *total differential* as commonly discussed in multivariate calculus. Recall that if $f : \mathbb{R}^m \to \mathbb{R}$ then we defined

$$df = \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \cdots + \frac{\partial f}{\partial x^m}dx^m.$$

Notice that the j-th component of df is simply $\frac{\partial f}{\partial x^j}$. Notice that the identity $d_pf(X_p) = X_p(f)$ gives us the same component if we simply evaluate the covector d_pf on the coordinate basis $\frac{\partial}{\partial x^j}\big|_p$,

$$d_pf\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \frac{\partial f}{\partial x^j}\bigg|_p$$

10.5 differential forms

In this section we apply the results of Chapter 9 on exterior algebra to the vector space $V = T_pM$ and its dual space $V^* = T_pM^*$. In short, this involves making the identifications:

$$e_i = \frac{\partial}{\partial x^i}\bigg|_p \qquad \text{and} \qquad e^j = d_px^j$$

however, we often omit the p-dependence of d_px^j and just write dx^j. Observe that:

$$dx^j(\partial_i|_p) = \partial_i(x^j) = \delta_{ij}$$

Therefore, the coordinate basis $\{\partial_1|_p, \ldots, \partial_m|_p\}$ for T_pM is indeed dual to the differentials of the coordinates $\{dx^1, \ldots, dx^m\}$ basis for the cotangent space T_pM^*.

In contrast to Chapter 9 we have a point-dependence to our basis. We establish the following terminology: for a manifold M and $p \in M$,

(0.) $p \mapsto \mathbb{R}$ is a 0-form, or function on M

(1.) $p \mapsto dx^j$ is a 1-form on M

(2.) $p \mapsto dx^i \wedge dx^j$ is a 2-form on M

(3.) $p \mapsto dx^i \wedge dx^j \wedge dx^k$ is a 3-form on M

(4.) $p \mapsto dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ is a k-form on M

Generally a k-form is formed from taking sums of the basic differential forms given above with coefficients which are smooth functions. [8]

(1.) a one-form $\alpha = \sum_{j=1}^{m} \alpha_j dx^j$ has smooth coefficient functions α_j.

(2.) a two-form $\beta = \sum_{i,j=1}^{m} \beta_{ij} dx^i \wedge dx^j$ has smooth coefficient functions β_{ij}.

(3.) a k-form $\gamma = \sum_{i_1,\ldots,i_k}^{n} \gamma_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ has smooth coefficient functions $\gamma_{i_1,\ldots,i_k}$.

The algebra of differential forms follows the same rules as the exterior algebra we previously discussed. However, instead of having scalars as numbers we now consider scalars as functions. This comment is made explicit in the theorem to follow:

Theorem 10.5.1.

If α is a p-form, β is a k-form, and γ is a l-form on M then

1. $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$
2. $\alpha \wedge \beta = (-1)^{pk}(\beta \wedge \alpha)$
3. $\alpha \wedge (a\beta + b\gamma) = a(\alpha \wedge \beta) + b(\alpha \wedge \gamma) \quad a, b \in \mathbb{R}$

Notice that in $\mathbb{R}^3$ the set of differential forms

$$\mathcal{B} = \{1, dx, dy, dz, dy \wedge dz, dz \wedge dx, dx \wedge dy, dx \wedge dy \wedge dz\}$$

is a basis of the space of differential forms in the sense that every form on $\mathbb{R}^3$ is a linear combination of the forms in $\mathcal{B}$ with smooth real-valued functions on $\mathbb{R}^3$ as coefficients.

Example 10.5.2. *Let $\alpha = f dx + g dy$ and let $\beta = 3dx + dz$ where f, g are functions. Find $\alpha \wedge \beta$, write the answer in terms of the basis defined in the Remark above,*

$$\begin{aligned} \alpha \wedge \beta &= (f dx + g dy) \wedge (3dx + dz) \\ &= f dx \wedge (3dx + dz) + g dy \wedge (3dx + dz) \\ &= 3f dx \wedge dx + f dx \wedge dz + 3g dy \wedge dx + g dy \wedge dz \\ &= -g dy \wedge dz - f dz \wedge dx - 3g dx \wedge dy \end{aligned} \tag{10.3}$$

[8] Technically, this means the exterior algebra of differential forms is a module over the ring of smooth functions. However, the exterior algebra at a point is a vector space.

Example 10.5.3. Top form: *Let $\alpha = dx \wedge dy \wedge dz$ and let β be any other form with degree $p > 0$. We argue that $\alpha \wedge \beta = 0$. Notice that if $p > 0$ then there must be at least one differential inside β so if that differential is dx^k we can rewrite $\beta = dx^k \wedge \gamma$ for some γ. Then consider,*

$$\alpha \wedge \beta = dx \wedge dy \wedge dz \wedge dx^k \wedge \gamma \tag{10.4}$$

now k has to be either $1, 2$ or 3 therefore we will have dx^k repeated, thus the wedge product will be zero. (can you prove this?).

The proposition below and its proof are included here to remind the reader on the structure of the $\otimes$ and $\wedge$ products and components. One distinction, the components are functions now whereas they were scalars in the previous chapter.

Proposition 10.5.4.

If ω is a p-form in an n-dimensional space is written in the coordinate coframe $dx^1, \ldots, dx^n$ at p then the components of ω are given by evaluation on the coordinate frame $\partial_1, \ldots, \partial_n$. at p.

Proof: Suppose ω has component functions $\omega_{i_1 i_2 \ldots i_p}$ with respect to the tensor basis $dx^{i_1} \otimes \cdots \otimes dx^{i_p}$ for type $(0, p)$ tensors.

$$\omega = \sum_{i_1 i_2 \ldots i_p}^{n} \omega_{i_1 i_2 \ldots i_p} dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_p}$$

the functions $\omega_{i_1 i_2 \ldots i_p}$ are called the **tensor components** of ω. Consider evaluation of ω on a p-tuple of coordinate vector fields,

$$\begin{aligned}
\omega(\partial_{j_1}, \partial_{j_2} \ldots \partial_{j_p}) &= \sum_{i_1, i_2, \ldots, i_p = 1}^{n} \omega_{i_1 i_2 \ldots i_p} dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_p}(\partial_{j_1}, \partial_{j_2}, \ldots, \partial_{j_p}) \\
&= \sum_{i_1, i_2, \ldots, i_p = 1}^{n} \omega_{i_1 i_2 \ldots i_p} \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_p j_p} \\
&= \omega_{j_1 j_2 \ldots j_p}
\end{aligned}$$

If ω is a p-form that indicated ω is a completely antisymmetric tensor and by a calculation similar to those near Equation 9.10 we can express ω by a sum over all p-forms (this is not a basis expansion)

$$\omega = \sum_{i_1, i_2, \ldots, i_p = 1}^{n} \frac{1}{p!} \omega_{i_1 i_2 \ldots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}.$$

But, we just found $\omega(\partial_{j_1}, \partial_{j_2} \ldots \partial_{j_p}) = \omega_{i_1 i_2 \ldots i_p}$ hence:

$$\omega = \sum_{i_1 i_2 \ldots i_p}^{n} \frac{1}{p!} \omega(\partial_{i_1}, \partial_{i_2} \ldots \partial_{i_p}) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}.$$

which proves the assertion of the proposition. □

Note, if we work with an expansion of linearly independent p-vectors then we can write the conclusion of the proposition:

$$\omega = \sum_{i_1<i_2<\cdots<i_p}^{n} \omega(\partial_{i_1}, \partial_{i_2} \ldots \partial_{i_p}) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}.$$

10.6 the exterior derivative

The operation $\wedge$ depends only on the values of the forms point by point. We define an operator d on differential forms which depends not only on the value of the differential form at a point but on its value in an entire neighborhood of the point. Thus if β ia k-form then to define $d\beta$ at a point p we need to know not only the value of β at p but we also need to know its value at every q in a neighborhood of p.

You might note the derivative below does not directly involve the construction of differential forms from tensors. Also, the rule given below is easily taken as a starting point for formal calculations. In other words, even if you don't understand the nuts and bolts of manifold theory you can still calculate with differential forms. In the same sense that highschool students "do" calculus, you can "do" differential form calculations. I don't believe this is a futile exercise so long as you understand you have more to learn. Which is not to say we don't know some things!

Definition 10.6.1. *the exterior derivative.*

If β is a k-form and $x = (x^1, x^2, \cdots, x^n)$ is a chart and $\beta = \sum_I \frac{1}{k!}\beta_I dx^I$ and we define a $(k+1)$-form $d\beta$ to be the form

$$d\beta = \sum_I \frac{1}{k!} d\beta_I \wedge dx^I \ .$$

Where $d\beta_I$ is defined as it was in calculus III,

$$d\beta_I = \sum_{j=1}^{n} \frac{\partial \beta_I}{\partial x_j} dx^j.$$

Note that $d\beta_I$ is well-defined as

$$\beta_I = \beta_{i_1 i_2 \cdots i_k}$$

is just a real-valued function on $dom(x)$. The definition in an expanded form is given by

$$d_p\beta = \frac{1}{k!} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} (d_p\beta_{i_1 i_2 \cdots i_k}) \wedge d_p x^{i_1} \wedge \cdots \wedge d_p x^{i_k}$$

where

$$\beta_q = \frac{1}{k!}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}\beta_{i_1 i_2\cdots i_k}(q)d_q x^{i_1}\wedge\cdots\wedge d_p x^{i_k}\ .$$

Consequently we see that for each k the operator d maps $\wedge^k(M)$ into $\wedge^{k+1}(M)$. Also:

Theorem 10.6.2. *properties of the exterior derivative.*

If $\alpha\in\wedge^k(M)$, $\beta\in\wedge^l(M)$ and $a,b\in\mathbf{R}$ then

1. $d(a\alpha+b\beta)=a(d\alpha)+b(d\beta)$
2. $d(\alpha\wedge\beta)=(d\alpha\wedge\beta)+(-1)^k(\alpha\wedge d\beta)$
3. $d(d\alpha)=0$

Remark 10.6.3.

Warning: I use Einstein's repeated index notation in the proof that follows. In fact, it's a bit worse, I use I to denote a **multi-index**. This means a repeated I indicates an implicit sum over all increasing strings of indices of a particular length. This is just a brief notation to sum over the basis of coordinate p-forms. Indeed, from this point on in the notes there is occasional use of Einstein's convention.

Proof: The proof of (1) is obvious. To prove (2), let $x=(x^1,\cdots,x^n)$ be a chart on M then suppose $\alpha=\alpha_I dx^I$ and $\beta=\beta_J dx^J$

$$\begin{aligned}
d(\alpha\wedge\beta)=d(\alpha_I\beta_J)\wedge dx^I\wedge dx^J &= (\alpha_I d\beta_J+\beta_J d\alpha_I)\wedge dx^I\wedge dx^J\\
&=\alpha_I(d\beta_J\wedge dx^I\wedge dx^J)\\
&\quad+\beta_J(d\alpha_I\wedge dx^I\wedge dx^J)\\
&=\alpha_I(dx^I\wedge(-1)^k(d\beta_J\wedge dx^J))\\
&\quad+\beta_J((d\alpha_I\wedge dx^I)\wedge dx^J)\\
&=(\alpha\wedge(-1)^k d\beta)+\beta_J(d\alpha\wedge dx^J)\\
&=d\alpha\wedge\beta+(-1)^k(\alpha\wedge d\beta)\ .
\end{aligned}$$

To prove (3.) we could resort to a beautiful tensor calculation (see Equation 10.11) or:

$$d\alpha=d\alpha_I\wedge dx^I$$

hence

$$d(d\alpha)=d(d\alpha_I\wedge dx^I)=d(d\alpha_I)\wedge dx^I+\alpha_I\wedge d(dx^I).$$

Notice $d(dx^I)=d(dx^{i_1}\wedge\cdots\wedge dx^{i_k})=d(1)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}=0$. Therefore, we have reduced the problem to showing $d(d\alpha_I)=0$ for a function α_I. I leave that problem to the reader. □.

10.6.1 exterior derivatives on $\mathbb{R}^3$

We begin by noting that vector fields may correspond either to a one-form or to a two-form.

Definition 10.6.4. *dictionary of vectors verses forms on $\mathbb{R}^3$.*

Let $\vec{A} = (A^1, A^2, A^3)$ denote a vector field in $\mathbb{R}^3$. Define then,

$$\omega_A = \delta_{ij} A^i dx^j = A_i dx^i$$

which we will call the **work-form** of $\vec{A}$. Also define

$$\Phi_A = \frac{1}{2}\delta_{ik} A^k \epsilon_{ijk}(dx^i \wedge dx^j) = \frac{1}{2} A_i \epsilon_{ijk}(dx^i \wedge dx^j)$$

which we will call the **flux-form** of $\vec{A}$.

If you accept the primacy of differential forms, then you can see that vector calculus confuses two separate objects. Apparently there are two types of vector fields. In fact, if you have studied coordinate change for vector fields deeply then you will encounter the qualifiers **axial** or **polar** vector fields. Those fields which are axial correspond directly to two-forms whereas those correspondant to one-forms are called polar. Note, the magnetic field is axial whereas the electric field is polar.

Example 10.6.5. Gradient: *Consider three-dimensional Euclidean space. Let $f : \mathbb{R}^3 \to \mathbb{R}$ then*

$$df = \frac{\partial f}{\partial x^i} dx^i = \omega_{\nabla f}$$

which gives the one-form corresponding to ∇f.

Example 10.6.6. Curl: *Consider three-dimensional Euclidean space. Let $\vec{F}$ be a vector field and let $\omega_F = F_i dx^i$ be the corresponding one-form then*

$$\begin{aligned} d\omega_F &= dF_i \wedge dx^i \\ &= \partial_j F_i dx^j \wedge dx^i \\ &= \partial_x F_y dx \wedge dy + \partial_y F_x dy \wedge dx + \partial_z F_x dz \wedge dx + \partial_x F_z dx \wedge dz + \partial_y F_z dy \wedge dz + \partial_z F_y dz \wedge dy \\ &= (\partial_x F_y - \partial_y F_x) dx \wedge dy + (\partial_z F_x - \partial_x F_z) dz \wedge dx + (\partial_y F_z - \partial_z F_y) dy \wedge dz \\ &= \Phi_{\nabla \times \vec{F}}. \end{aligned}$$

Thus we recover the curl.

Example 10.6.7. Divergence: *Consider three-dimensional Euclidean space. Let $\vec{G}$ be a vector field and let $\Phi_G = \frac{1}{2}\epsilon_{ijk} G_i dx^j \wedge dx^k$ be the corresponding two-form then*

$$\begin{aligned} d\Phi_G &= d(\tfrac{1}{2}\epsilon_{ijk} G_i) \wedge dx^j \wedge dx^k \\ &= \tfrac{1}{2}\epsilon_{ijk}(\partial_m G_i) dx^m \wedge dx^j \wedge dx^k \\ &= \tfrac{1}{2}\epsilon_{ijk}(\partial_m G_i)\epsilon_{mjk} dx \wedge dy \wedge dz \\ &= \tfrac{1}{2} 2\delta_{im}(\partial_m G_i) dx \wedge dy \wedge dz \\ &= \partial_i G_i dx \wedge dy \wedge dz \\ &= (\nabla \cdot \vec{G}) dx \wedge dy \wedge dz \end{aligned}$$

and in this way we recover the divergence.

10.6.2 coordinate independence of exterior derivative

The Einstein summation convention is used in this section and throughout the remainder of this chapter, please feel free to email me if it confuses you somewhere. When an index is repeated in a single summand it is implicitly assumed there is a sum over all values of that index

It must be shown that this definition is independent of the chart used to define $d\beta$. Suppose for example, that

$$\beta_q = \overline{\beta}_J(q)(d_q\overline{x}^{j_1} \wedge \cdots \wedge d_q\overline{x}^{j_k})$$

for all q in the domain of a chart $(\overline{x}^1, \overline{x}^2, \cdots \overline{x}^n)$ where

$$dom(x) \cap dom(\overline{x}), \neq \emptyset \, .$$

We assume, of course that the coefficients $\{\overline{\beta}_J(q)\}$ are skew-symmetric in J for all q. We will have defined $d\beta$ in this chart by

$$d\beta = d\overline{\beta}_J \wedge d\overline{x}^J \, .$$

We need to show that $d_p\overline{\beta}_J \wedge d_p\overline{x}^J = d_p\beta_I \wedge d_p x^I$ for all $p \in dom(x) \cap dom(\overline{x})$ if this definition is to be meaningful. Since β is given to be a well-defined form we know

$$\beta_I(p)d_p x^I = \beta_p = \overline{\beta}_J(p)d_p\overline{x}^J \, .$$

Using the identities

$$d\overline{x}^j = \frac{\partial \overline{x}^j}{\partial x^i}dx^i$$

we have

$$\beta_I dx^I = \overline{\beta}_J \frac{\partial \overline{x}^{j_1}}{\partial x^{i_1}} \frac{\partial \overline{x}^{j_k}}{\partial x^{i_k}} \cdots \frac{\partial \overline{x}^{j_k}}{\partial x^{i_k}} \, dx^I$$

so that

$$\beta_I = \overline{\beta}_J \left(\frac{\partial \overline{x}^{j_1}}{\partial x^{i_1}} \frac{\partial \overline{x}^{j_2}}{\partial x^{i_2}} \cdots \frac{\partial \overline{x}^{j_k}}{\partial x^{i_k}} \right) \, .$$

Consequently,

$$\begin{aligned}
d\beta_J \wedge dx^J = \tfrac{\partial \beta_J}{\partial x^\lambda}(dx^\lambda \wedge dx^J) \quad &= \tfrac{\partial}{\partial x^\lambda}[\overline{\beta}_I \left(\tfrac{\partial \overline{x}^{i_1}}{\partial x^{j_1}} \cdots \tfrac{\partial \overline{x}^{i_k}}{\partial x^{j_k}} \right)](dx^\lambda \wedge dx^J) \\
&\overset{*}{=} \tfrac{\partial \overline{\beta}_I}{\partial x^\lambda} \left(\tfrac{\partial \overline{x}^{i_1}}{\partial x^{j_1}} \cdots \tfrac{\partial \overline{x}^{i_k}}{\partial x^{j_k}} \right)(dx^\lambda \wedge dx^J) \\
&\quad + \overline{\beta}_I \textstyle\sum_r \left(\tfrac{\partial \overline{x}^{i_1}}{\partial x^{j_1}} \cdots \tfrac{\partial^2 \overline{x}^{i_r}}{\partial x^\lambda \partial x^{j_r}} \cdots \tfrac{\partial \overline{x}^{i_k}}{\partial x^{j_k}} \right)(dx^\lambda \wedge dx^J) \\
&= \tfrac{\partial \overline{\beta}_I}{\partial x^\lambda} dx^\lambda \wedge \left(\tfrac{\partial \overline{x}^{i_1}}{\partial x^{j_1}} dx^{j_1} \right) \wedge \cdots \wedge \left(\tfrac{\partial \overline{x}^{i_k}}{\partial x^{j_k}} dx^{j_k} \right) \\
&= \tfrac{\partial \overline{\beta}_I}{\partial \overline{x}^p} \left[\left(\tfrac{\partial \overline{x}^p}{\partial x^\lambda} dx^\lambda \right) \wedge d\overline{x}^{i_1} \wedge \cdots \wedge d\overline{x}^{i_k} \right] \\
&= d\overline{\beta}_I \wedge d\overline{x}^I
\end{aligned}$$

where in (*) the sum $\sum_r$ is zero since:

$$\frac{\partial^2 \overline{x}^{i_r}}{\partial x^\lambda \partial x^{j_r}}(dx^\lambda \wedge dx^J) = \pm \frac{\partial^2 \overline{x}^{i_r}}{\partial x^\lambda \partial x^{j_r}}[(dx^\lambda \wedge dx^{j_r}) \wedge dx^{j_1} \wedge \cdots \wedge \widehat{dx^{j_r}} \wedge \cdots dx^{j_k}] = 0.$$

It follows that $d\beta$ is independent of the coordinates used to define it.

10.7 the pull-back

Another important operation one can perform on differential forms is the "pull-back" of a form under a map[9]. The definition is constructed in large part by a sneaky application of the push-forward (aka differential) discussed in the preceding chapter. If you are impatient for intuition, skip ahead to the end of this section and later return to the careful calculations at the outset.

Definition 10.7.1. *pull-back of a differential form.*

If $f : \mathcal{M} \to \mathcal{N}$ is a smooth map and $\omega \in \wedge^k(N)$ then $f^*\omega$ is the form on $\mathcal{M}$ defined by

$$(f^*\omega)_p(X_1, \cdots, X_k) = \omega_{f(p)}(d_pf(X_1), d_pf(X_2), \cdots, d_pf(X_k)) \; .$$

for each $p \in \mathcal{M}$ and $X_1, X_2, \ldots, X_k \in T_p\mathcal{M}$. Moreover, in the case $k = 0$ we have a smooth function $\omega : \mathcal{N} \to \mathbb{R}$ and the pull-back is accomplished by composition $(f^*\omega)(p) = (\omega \circ f)(p)$ for all $p \in \mathcal{M}$.

This operation is linear on forms and commutes with the wedge product and exterior derivative:

Theorem 10.7.2. *properties of the pull-back.*

If $f : M \to N$ is a C^1-map and $\omega \in \wedge^k(N)$, $\tau \in \wedge^l(N)$ then

1. $f^*(a\omega + b\tau) = a(f^*\omega) + b(f^*\tau) \qquad a, b \in \mathbb{R}$
2. $f^*(\omega \wedge \tau) = f^*\omega \wedge (f^*\tau)$
3. $f^*(d\omega) = d(f^*\omega)$

Proof: The proof of (1) is clear. We now prove (2).

$$\begin{aligned} f^*(\omega \wedge \tau)]_p(X_1, \cdots, X_{k+l}) &= (\omega \wedge \tau)_{f(p)}(d_pf(X_1), \cdots, d_pf(X_{k+l})) \\ &= \textstyle\sum_\sigma (\text{sgn}\sigma)(\omega \otimes \tau)_{f(p)}(d_pf(X_{\sigma_1}), \cdots, d_pf(X_{\sigma(k+l)})) \\ &= \textstyle\sum_\sigma \text{sgn}(\sigma)\omega(d_pf(X_{\sigma(1)}), \cdots d_pf(X_{\sigma(k)}))\tau(df(X_{\sigma(k+1)} \cdots df X_{\sigma(k+l)}) \\ &= \textstyle\sum_\sigma \text{sgn}(\sigma)(f^*\omega)_p(X_{\sigma(1)}, \cdots, X_{\sigma(k)})(f^*\tau_p)(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}) \\ &= [(f^*\omega) \wedge (f^*\tau)]_p(X_1, X_2, \cdots, X_{(k+l)}) \end{aligned}$$

[9] thanks to my advisor R.O. Fulp for the arguments that follow

Finally we prove (3).

$$\begin{aligned}
f^*(d\omega)]_p(X_1, X_2 \cdots, X_{k+1}) &= (d\omega)_{f(p)}(df(X_1), \cdots df(X_{k+1})) \\
&= (d\omega_I \wedge dx^I)_{f(p)}(df(X_1), \cdots, df(X_{k+1})) \\
&= \left(\left. \frac{\partial \omega_I}{\partial x^\lambda} \right|_{f(p)} \right) (dx^\lambda \wedge dx^I)_{f(p)}(df(X_1), \cdots, df(X_{k+1})) \\
&= \left(\left. \frac{\partial \omega_I}{\partial x^\lambda} \right|_{f(p)} \right) [d_p(x^\lambda \circ f) \wedge d_p(x^I \circ f)](X_1, \cdots, X_{k+1}) \\
&= [d(\omega_I \circ f) \wedge d(x^I \circ f)](X_1, \cdots, X_{k+1}) \\
&= d[(\omega_I \circ f)_p d_p(x^I \circ f)](X_1, \cdots, X_{k+1}) \\
&= d(f^*\omega)_p(X_1, \cdots, X_{k+1}) \, .
\end{aligned}$$

The theorem follows. □.

We saw that one important application of the push-forward was to change coordinates for a given vector. Similar comments apply here. If we wish to change coordinates on a given differential form then we can use the pull-back. However, given the direction of the operation we need to use the inverse coordinate transformation to pull forms forward. Let me mirror the example from the last chapter for forms on $\mathbb{R}^2$. We wish to convert from r, θ to x, y notation.

Example 10.7.3. *Suppose $F : \mathbb{R}^2_{r,\theta} \to \mathbb{R}^2_{x,y}$ is the polar coordinate transformation. In particular,*

$$F(r, \theta) = (r \cos\theta, r \sin\theta)$$

The inverse transformation, at least for appropriate angles, is given by

$$F^{-1}(x, y) = \left(\sqrt{x^2 + y^2}, \ \tan^{-1}(y/x) \right).$$

Let calculate the pull-back of dr under F^{-1}: let $p = F^{-1}(q)$

$$F^{-1*}(dr)_q = d_p r(\partial_x|p) d_p x + d_p r(\partial_y|p) d_p y$$

Again, drop the annoying point-dependence to see this clearly:

$$F^{-1*}(dr) = dr(\partial_x)dx + dr(\partial_y)dy = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy$$

Likewise,

$$F^{-1*}(d\theta) = d\theta(\partial_x)dx + d\theta(\partial_y)dy = \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy$$

Note that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ have the following partial derivatives:

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} \qquad \text{and} \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}$$

$$\frac{\partial\theta}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} \qquad \textit{and} \qquad \frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

Of course the expressions using r are pretty, but to make the point, we have changed into x, y-notation via the pull-back of the inverse transformation as advertised. We find:

$$\boxed{dr = \frac{xdx+ydy}{\sqrt{x^2+y^2}} \qquad \textit{and} \qquad d\theta = \frac{-ydx+xdy}{x^2+y^2}.}$$

Once again we have found results with the pull-back that we might previously have chalked up to substitution in multivariate calculus. That's often the idea behind an application of the pull-back. It's just a formal langauge to be precise about a substitution. It takes us past simple symbol pushing and gives us a rigorous notation for substutions. It's a bit more than that though, the *substitution* we discuss here takes us from one space to another in general.

10.7.1 intuitive computation of pull-backs

Consider the definition below: how can we understand this computationally?

$$(f^*\omega)_p(X_1, \cdots, X_k) = \omega_{f(p)}(d_pf(X_1), d_pf(X_2), \cdots, d_pf(X_k)) \; .$$

In particular, let us consider $f(u,v) = (x,y,z)$. Furthermore, let us consider a one-form to begin $\omega = adx + bdy + cdz$ the pull-back will be formed by a suitable linear combination of du and dv. We calculate,

$$df(\partial_u) = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z} \qquad \& \qquad df(\partial_v) = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z}$$

Therefore, as $\omega = adx + bdy + cdz$,

$$\omega(df(\partial_u)) = a\frac{\partial x}{\partial u} + b\frac{\partial y}{\partial u} + c\frac{\partial z}{\partial u} \qquad \& \qquad \omega(df(\partial_v)) = a\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial v} + c\frac{\partial z}{\partial v}$$

From which we deduce: by Proposition 10.5.4,

$$\begin{aligned} f^*\omega &= \left[a\frac{\partial x}{\partial u} + b\frac{\partial y}{\partial u} + c\frac{\partial z}{\partial u}\right]du + \left[a\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial v} + c\frac{\partial z}{\partial v}\right]dv \\ &= a\underbrace{\left[\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right]}_{dx \text{ for } x=x(u,v)} + b\underbrace{\left[\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right]}_{dy \text{ for } y=y(u,v)} + c\underbrace{\left[\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right]}_{dz \text{ for } z=z(u,v)}. \end{aligned} \tag{10.5}$$

This shows the pull-back of ω is accomplished by taking $\omega = adx + bdy + cdz$ and substituting the total differentials of $x = x(u,v), y = y(u,v)$ and $z = z(u,v)$ into dx, dy and dz respectively.

Continuing in our somewhat special context, consider $\Omega = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$ and use Theorem 10.7.2 to simplify our life. We already worked out the formula for the one-form case so we can use it to find the intuitive formula for the two-form:

$$\begin{aligned} f^*\Omega &= f^*\left[ady \wedge dz + bdz \wedge dx + cdx \wedge dy\right] \\ &= f^*(ady) \wedge f^*dz + f^*(bdz) \wedge f^*dx + f^*(cdx) \wedge f^*dy \ \text{ (by Theorem 10.7.2)} \\ &= a\left[\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right] \wedge \left[\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right] + b\left[\frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv\right] \wedge \left[\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right] \\ &\quad + c\left[\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right] \wedge \left[\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right] \ \text{ (by Equation 10.5)} \end{aligned}$$

The formula above simplifies considerable as certain terms vanish due to $du \wedge du = 0$ and $dv \wedge dv = 0$ and $dv \wedge du = -du \wedge dv$. Furthermore, the following standard notation[10]

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u} \quad \& \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}, \quad \& \quad \frac{\partial(z,x)}{\partial(u,v)} = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}.$$

Generally, $\frac{\partial(x_i,x_j)}{\partial(u,v)} = \det\begin{bmatrix} \partial_u x_i & \partial_v x_i \\ \partial_u x_j & \partial_v x_j \end{bmatrix}$. Ugly equations aside, this allows us to express the pull-back of the two-form in a way which is easy to remember and is readily generalized.

$$f^*\Omega = \left[a\frac{\partial(y,z)}{\partial(u,v)} + b\frac{\partial(z,x)}{\partial(u,v)} + c\frac{\partial(x,y)}{\partial(u,v)}\right] du \wedge dv$$

I should mention, throughout this calculation I have suppressed the point-dependence. Technically, a, b, c in the expression above should be understood as $a \circ f, b \circ f, c \circ f$ which are functions of u, v. The pull-back once complete trades a form in the range coordinates (x, y, z) for a new form in the domain coordinates (u, v).

The discussion thus far is somewhat limiting since the domain only supports a nontrivial two-form. To continue, let's consider pull-backs for the mapping $G : \mathbb{R}^3_{uvw} \to \mathbb{R}^4_{txyz}$. In this case, we can consider the pull-back of

$$\gamma = adt \wedge dx \wedge dz + bdx \wedge dy \wedge dz + cdy \wedge dz \wedge dt + mdz \wedge dt \wedge dy$$

and we will obtain:

$$G^*\gamma = \left[a\frac{\partial(t,x,y)}{\partial(u,v,w)} + b\frac{\partial(x,y,z)}{\partial(u,v,w)} + c\frac{\partial(y,z,t)}{\partial(u,v,w)} + m\frac{\partial(z,t,x)}{\partial(u,v,w)}\right] du \wedge dv \wedge dw$$

Where we define the coefficients by the natural generalization of the second-order case. Multiplying out the wedge product of the pull-back of the three one-forms will produce the signs of the following determinants

$$\frac{\partial(x_i,x_j,x_k)}{\partial(u,v,w)} = \det\begin{bmatrix} \partial_u x_i & \partial_v x_i & \partial_w x_i \\ \partial_u x_j & \partial_v x_j & \partial_w x_j \\ \partial_u x_k & \partial_v x_k & \partial_w x_k \end{bmatrix}$$

[10] H.M. Edwards Advanced Calculus a Differential Forms approach spends dozens of pages explaining this through intuitive geometric arguments which we do not pursue here for brevity

In words, the pull-back is formed with coefficients taken from determinants of submatrices of the Jacobian matrix of G. That probably doesn't help you much. Perhaps the following two-form pull-back with respect to G will inspire: an arbitrary two-form on $\mathbb{R}^4_{txyz}$ has the form:

$$\Xi = A_{01}dt\wedge dx + A_{02}dt\wedge dy + A_{03}dt\wedge dz + A_{23}dy\wedge dz + A_{31}dz\wedge dx + A_{12}dx\wedge dy$$

when we pull this back under G to u,v,w-space we will get a form in $dv\wedge dw$, $dw\wedge du$ and $du\wedge dv$. The coefficients are again obtained by appropriate determinants of submatrices of the Jacobian: in particular: $G^*\Xi =$

$$\begin{aligned}
&\left[A_{01}\frac{\partial(t,x)}{\partial(v,w)} + A_{02}\frac{\partial(t,y)}{\partial(v,w)} + A_{03}\frac{\partial(t,z)}{\partial(v,w)} + A_{23}\frac{\partial(y,z)}{\partial(v,w)} + A_{31}\frac{\partial(z,x)}{\partial(v,w)} + A_{12}\frac{\partial(x,y)}{\partial(v,w)}\right] dv\wedge dw \\
&+ \left[A_{01}\frac{\partial(t,x)}{\partial(w,u)} + A_{02}\frac{\partial(t,y)}{\partial(w,u)} + A_{03}\frac{\partial(t,z)}{\partial(w,u)} + A_{23}\frac{\partial(y,z)}{\partial(w,u)} + A_{31}\frac{\partial(z,x)}{\partial(w,u)} + A_{12}\frac{\partial(x,y)}{\partial(w,u)}\right] dw\wedge du \\
&+ \left[A_{01}\frac{\partial(t,x)}{\partial(u,v)} + A_{02}\frac{\partial(t,y)}{\partial(u,v)} + A_{03}\frac{\partial(t,z)}{\partial(u,v)} + A_{23}\frac{\partial(y,z)}{\partial(u,v)} + A_{31}\frac{\partial(z,x)}{\partial(u,v)} + A_{12}\frac{\partial(x,y)}{\partial(u,v)}\right] du\wedge dv.
\end{aligned}$$

The arbitrary case is perhaps a bit tiresome. Let's consider a particular example

$$\Gamma = (t^2+x^2)dy\wedge dz$$

and the mapping $G(u,v,w) = (u-1, v^2+1, w^3, uvw) = (t,x,y,z)$. Observe:

$$dt = du,\quad dx = 2vdv,\quad dy = 3w^2dw,\quad dz = vwdu + uwdv + uvdw$$

the pull-back of Γ under G is thus:

$$\begin{aligned}
G^*\Gamma &= \big((u-1)^2+(v^2+1)^2\big)3w^2dw\wedge[vwdu + uwdv + uvdw] \\
&= \big((u-1)^2+(v^2+1)^2\big)3w^2\,[vw\,dw\wedge du + uw\,dw\wedge dv] \\
&= A_{23}(u,v)(\underbrace{-3w^2uw}_{\frac{\partial(y,z)}{\partial(v,w)}})dv\wedge dw + A_{23}(u,v)(\underbrace{3w^2vw}_{\frac{\partial(y,z)}{\partial(w,u)}})dw\wedge du
\end{aligned} \tag{10.6}$$

In the above I let $A_{23}(u,v) = (u-1)^2+(v^2+1)^2$. Let's check the coefficients of Equation 10.6 in view of the general claim of $G^*\Xi$. Are the coefficients in fact the Jacobians indicated? We calculate:

$$\frac{\partial(y,z)}{\partial(v,w)} = \det\begin{bmatrix} y_v & y_w \\ z_v & z_w\end{bmatrix} = \det\begin{bmatrix} 0 & 3w^2 \\ uw & uv\end{bmatrix} = -3w^2uw$$

and

$$\frac{\partial(y,z)}{\partial(w,u)} = \det\begin{bmatrix} y_w & y_u \\ z_w & z_u\end{bmatrix} = \det\begin{bmatrix} 3w^2 & 0 \\ uv & vw\end{bmatrix} = 3w^2vw.$$

Well, that's a relief. We can either approach the calculation of a pull-back in terms of Jacobian coefficients or we can just plug in the pull-backs of each coordinate function and multiply it out. I suppose both techinques have their place. Moreover, when faced with many abstract questions I much prefer Definition 10.7.1. The thought of sorting through the proof of Theorem 10.7.2 in Jacobian notation seems hopeless.

10.7.2 implicit function theorem in view of forms

Previously we saw this theorem without the benefit of the calculus of differential forms, I hope this brings new light to the topic. This is the implicit function theorem as presented in H.M. Edwards' *Advanced Calculus: a Differential Forms Approach*.

Proposition 10.7.4.

Let us consider for $i = 1, \dots m$,

$$y_i = f_i(x_1, \dots, x_n)$$

where f_i are continuously differentiable near a point $(\bar{x}_1, \dots, \bar{x}_n)$. Furthermore, denote $\bar{y}_i = f_i(\bar{x}_1, \dots, \bar{x}_n)$ for $i = 1, \dots m$. Suppose the following conditions are satisfied:

1. $\frac{\partial(y_1,\dots,y_r)}{\partial(x_1,\dots,x_r)} \neq 0$ at $(\bar{x}_1, \dots, \bar{x}_n)$
2. the pull-back of every k-form for $k > r$ is identically zero near $(\bar{x}_1, \dots, \bar{x}_n)$

Then near $(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_m) \in \mathbb{R}^{n+m}$ there exist differentiable functions g_i, h_i which solve $y_i = f_i(x_1, \dots, x_n)$ provided we suppose:

$$x_i = g_i(y_1, \dots, y_r, x_{r+1}, \dots, x_n) \qquad (\text{for } i = 1, \dots, r)$$

$$y_i = h_i(y_1, \dots, y_r) \qquad (\text{for } i = r+1, \dots, m).$$

Condition (1.) implies the rank is at least r then condition (2.) assures us the rank is at most r hence the rank is r. If the graph $y = f(x)$ is viewed as $G(x, y) = y - f(x)$ then it is seen as a mapping from $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. We saw before that if $G(\bar{x}, \bar{y}) = 0$ and $rank(G'(\bar{x}, \bar{y})) = m$ then the solution set of $G(x, y) = 0$ near $(\bar{x}, \bar{y})$ forms an (n)-dimensional manifold. This generalizes that in a sense because it allows for redundant conditions as indicated by $r < m$. On the other hand, this is less general than the previous implicit function theorem as it assumes a linear-dependence on the m-variables $y_1, \dots, y_m$ for the m equations defining the level set in $\mathbb{R}^m \times \mathbb{R}^n$. The formulation of this Theorem in Edward's text falls inline with the general pattern of that text to emphasize equations over mappings. It is both the strength and weakness of the text in my opinion. The next two examples attempt to work within the confines of the notation put forth in the theorem above, then we transition to examples where we informally apply the theorem.

Example 10.7.5. *Just to remind us of the counting: if* $y_1 = f(x_1, x_2) = x_1 + x_2 - 1$ *then*

$$G(x_1, x_2, y_1) = y_1 - (x_1 + x_2 - 1) = -x_1 - x_2 + y_1 + 1 = 0$$

is a plane in (x_1, x_2, y_1)*-space. Consider,* $(\bar{x}_1, \bar{x}_2) = (2, 3)$ *then* $\bar{y}_1 = 2 + 3 - 1 = 4$*. A point on this plane is* $(2, 3, 4)$ *we can use* (x_1, x_2) *as coordinates hence I expect* $r = 1$*. Observe, as* $G : \mathbb{R}^3 \to \mathbb{R}$ *we consider pull-back of the one-form* dt *in the range,*

$$G^* dt = -dx_1 - dx_2 + dy_1$$

apparently, in our context here:

$$\frac{\partial(y_1,\ldots,y_r)}{\partial(x_1,\ldots,x_r)} = \frac{\partial t}{\partial x_1} = -1 \neq 0$$

and clearly there is no 2-form which pulls back under G since the only two-form in t-space is trivial. We can explicitly write $x_1 = g_1(y_1, x_2) = -x_2 + y_1 + 1$ and there is no h_i since $r + 1 = 2 > m$.

Example 10.7.6. *Let us attempt another example to unravel the meaning of the Implicit Function Theorem. Suppose $y_1 = x_1^2 - x_2^2$ and $y_2 = x_1^2 - x_2^2$ clearly these are redundant. The theorem should deal with this. Let $G(x_1, x_2, y_2, y_2) = (x_1^2 - x_2^2, x_1^2 - x_2^2)$ so we might hope the solution set is $4-2 = 2$-dimensional however, the rank of G' is usually 1 so the solution set is likely 3-dimensional. No need for guessing, let's work it out.*

$$dy_1 = 2x_1dx_1 - 2x_2dx_2 = dy_2$$

Hence, if we attempt $r = 2$ then consider:

$$\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)} = det\begin{bmatrix} 2x_1 & -2x_2 \\ 2x_1 & -2x_2 \end{bmatrix} = 0.$$

So, we are forced to look at $r = 1$ for which $\frac{\partial y_1}{\partial x_1} = 2x_1 \neq 0$ for $x_1 \neq 0$. Consider then a point such that $x_1 \neq 0$ and see if we can derive the functions such that $x_1 = g_1(y_1, x_2)$ and $y_2 = h_2(y_1)$. Assume $x_1, x_2 > 0$ for convenience (we could replicate this calculation in other quadrants with appropriate signs adjoined),

$$x_1 = \sqrt{y_1 + x_2^2} = g_1(y_1, x_2), \qquad \& \qquad y_2 = y_1 = h_1(y_1).$$

I'm mostly interested in this theorem to gain more geometric insight as to what the pull-back means. So, a better way to look at the last example is just to emphasize

$$dy_1 = 2x_1dx_1 - 2x_2dx_2 = dy_2$$

shows the pull-back of the basic one-forms can only give a one-form and where the coefficients are zero we cannot force the corresponding coordinate to be dependent. For example, $2x_1dx_1$ is trivial when $x_1 = 0$ hence x_1 cannot be solved for as a function of the remaining variables. This is the computational essence of the theorem. Ideally, I want to get us to the point of calculating without reliance on the Jacobians. Towards that end, let's consider an example for which $m = r$ hence the h function is not needed and we can focus on the pull-back geometry.

Example 10.7.7. *For which variables can we solve the following system (possibly subject some condition). Let $F(x, y, z) = (s, t)$ defined as follows:*

$$\begin{aligned} s &= x + z - 1 \\ t &= y + z - 2 \end{aligned} \tag{10.7}$$

then clearly $F^*ds = dx + dz$ *and* $F^*dt = dy + dz$.

$$F^*(ds \wedge dt) = (dx + dz) \wedge (dy + dz) = \underbrace{dx \wedge dy}_{(1.)} + \underbrace{dx \wedge dz}_{(2.)} + \underbrace{dz \wedge dy}_{(3.)}$$

Therefore, by (1.)*, we can solve for* x, y *as functions of* s, t, z*. Or, by* (2.)*, we could solve for* x, z *as functions of* s, t, y*. Or, by* (3.)*, we could solve for* z, y *as functions of* s, t, x*. The fact that the rank is maximal is implicit within the fact that* $ds \wedge dt$ *is the top-form in the range. In contrast,*

$$F^*ds = dx + dz$$

does not mean that I can solve Equation 10.7 by solving for x *as a function of* s, t, y, z*. Of course,* $x = s - z + 1$ *solves the first equation, but the second equation is not contrained by the solution for* x *what so over. Conversely, we can solve for* $z = t - y + 2$ *but we cannot also solve for* $z = s - x + 1$*. The coefficients of* 1 *in* $F^*ds = dx + dz$ *are not applicable to the Implicit Function Theorem because this form is not the highest degree which pulls-back nontrivially. That role falls to* $ds \wedge dt$ *here.*

Example 10.7.8. *For which variables can we solve the following system (possibly subject some condition). Let* $F(u, v) = (x, y, z)$ *defined as follows:*

$$\begin{aligned} x &= u^2 + v^2 \\ y &= u^2 - v^2 \\ z &= uv \end{aligned} \tag{10.8}$$

then

$$\begin{aligned} dx &= 2udu + 2vdv \\ dy &= 2udu - 2vdv \\ dz &= vdu + udv \end{aligned}$$

We can calculate,

$$dy \wedge dz = (2udu - 2vdv) \wedge (vdu + udv) = 2(u^2 + v^2)du \wedge dv$$

$$dz \wedge dx = (vdu + udv) \wedge (2udu + 2vdv) = 2(v^2 - u^2)du \wedge dv$$

$$dx \wedge dy = (2udu + 2vdv) \wedge (2udu - 2vdv) = -8uv\, du \wedge dv$$

What does this tell us geometrically? Well, notice that the top-form $dx \wedge dy \wedge dz$ *pulls-back to zero since the two-form* $du \wedge dv$ *is the top-form in the domain. Therefore,* $r = 2$*, that is* F *has rank* 2.

10.8 integration of forms

The general strategy is generally as follows:

(i) there is a natural way to calculate the integral of a k-form on a subset of $\mathbb{R}^k$

(ii) given a k-form on a manifold we can locally pull it back to a subset of $\mathbb{R}^k$ provided the manifold is an oriented[11] k-dimensional and thus by the previous idea we have an integral.

(iii) globally we should expect that we can add the results from various local charts and arrive at a total value for the manifold, assuming of course the integral in each chart is finite.

We will only investigate items (*i.*) and (*ii.*) in these notes. There are many other excellent texts which take great effort to carefully expand on point (iii.) and I do not wish to replicate that effort here. You can read Edwards and see about *pavings*, or read Munkres' where he has at least 100 pages devoted to the careful study of multivariate integration. I do not get into those topics in my notes because we simply do not have sufficient analytical power to do them justice. I would encourage the student interested in deeper ideas of integration to find time to talk to Dr. Skoumbourdis, he has thought a long time about these matters and he really understands integration in a way we dare not cover in the calculus sequence. You really should have that conversation after you've taken real analysis and have gained a better sense of what analysis' purpose is in mathematics. That said, what we do cover in this section and the next is fascinating whether or not we understand all the analytical underpinnings of the subject!

10.8.1 integration of k-form on $\mathbb{R}^k$

Note that on $U \subseteq \mathbb{R}^k$ a k-form α is the top form thus there exists some smooth function f on U such that $\alpha_x = f(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k$ for all $x \in U$. If D is a subset of U then we define the integral of α over D via the corresponding intgral of k-variables in $\mathbb{R}^k$. In particular,

$$\int_D \alpha = \int_D f(x)d^k x$$

where on the r.h.s. the symbol $d^k x$ is meant to denote the usual integral of k-variables on $\mathbb{R}^k$. It is sometimes convenient to write such an integral as:

$$\int_D f(x)d^k x = \int_D f(x)dx^1 dx^2 \cdots dx^k$$

but, to be more careful, the integration of f over D is a quantity which is independent of the particular order in which the variables on $\mathbb{R}^k$ are assigned. On the other hand, the order of the variables in the formula for α certainly can introuduce signs. Note

$$\alpha_x = -f(x)dx^2 \wedge dx^1 \wedge \cdots \wedge dx^k.$$

[11] we will discuss this as the section progresses

How can we reasonably maintain the integral proposed above? Well, the answer is to make a convention that we write the form to match the standard orientation of $\mathbb{R}^k$. The standard **orientation** of $\mathbb{R}^k$ is given by $Vol_k = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k$. If the given form is written $\alpha_x = f(x)Vol_k$ then we define $\int_D \alpha = \int_D f(x)d^kx$. Since it is always possible to write a k-form as a function multiplying Vol_k on $\mathbb{R}^k$ this definition suffices to cover all possible k-forms. For example, if $\alpha_x = f(x)dx$ on some subset $D = [a, b]$ of $\mathbb{R}$,

$$\int_D \alpha = \int_D f(x)dx = \int_a^b f(x)dx.$$

Or, if $\alpha_{(x,y)} = f(x, y)dx \wedge dy$ then for D a aubset of $\mathbb{R}^2$,

$$\int_D \alpha = \int_D f(x, y)dxdy = \int_D fdA.$$

If $\alpha_{(x,y,z)} = f(x, y, z)dx \wedge dy \wedge dz$ then for D a aubset of $\mathbb{R}^3$,

$$\int_D \alpha = \int_D f(x, y, z)dxdydz = \int_D fdV.$$

In practice we tend to break the integrals above down into an interated integral thanks to Fubini's theorems. The integrals $\int_D fdA$ and $\int_D fdV$ are not in and of themselves dependent on orientation. However the set D may be oriented the value of those integrals are the same for a fixed function f. The orientation dependence of the form integral is completely wrapped up in our rule that the form must be written as a multiple of the volume form on the given space.

10.8.2 orientations and submanifolds

Given a k-manifold $\mathcal{M}$ we say it is an **oriented** manifold iff all coordinates on $\mathcal{M}$ are **consistently oriented**. If we make a choice and say $\phi_0 : U_0 \to V_0$ is **right-handed** then any overlapping patch $\phi_1 : U_1 \to V_1$ is said to be **right-handed** iff $det(d\theta_{01}) > 0$. Otherwise, if $det(d\theta_{01}) < 0$ then the patch $\phi_1 : U_1 \to V_1$ is said to be **left-handed**. If the manifold is **orientable** then as we continue to travel across the manifold we can choose coordinates such that on each overlap the transition functions satisfy $det(d\theta_{ij}) > 0$. In this way we find an atlas for an orientable $\mathcal{M}$ which is **right-handed**.

We can also say $\mathcal{M}$ is oriented is there exists a nonzero volume-form on $\mathcal{M}$. If $\mathcal{M}$ is k-dimensional then a volume form Vol is simply a nonzero k-form. At each point $p \in \mathcal{M}$ we can judge if a given coordinate system is left or right handed. We have to make a convention to be precise and I do so at this point. We assume Vol is positive and we say a coordinate system with chart $(x^1, x^2, \ldots, x^k)$ is **positively oriented** iff $Vol(\partial_1|_p, \partial_2|_p, \ldots, \partial_k|_p) > 0$. If a coordinate system is not positively oriented then it is said to be negatively oriented and we will find $Vol(\partial_1|_p, \partial_2|_p, \ldots, \partial_k|_p) < 0$ in that case. It is important that we suppose $Vol_p \neq 0$ at each $p \in \mathcal{M}$ since that is what allows us to demarcate coordinate systems as positive or negatively oriented.

Naturally, you are probably wondering: is a positively oriented coordinate system is the same idea as a right-handed coordinate system as defined above? To answer that we should analyze how the

Vol changes coordinates on an overlap. Suppose we are given a positive volume form Vol and a point $p \in \mathcal{M}$ where two coordinate systems x and y are both defined. There must exist some function f such that

$$Vol_x = f(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k$$

To change coordinates recall $dx^j = \sum_{j=1}^k \frac{\partial x^j}{\partial y^j} dy^j$ and subsitute,

$$\begin{aligned} Vol &= \sum_{j_1,\ldots,j_k=1}^{k} (f \circ x \circ y^{-1})(y) \frac{\partial x^1}{\partial y^{j_1}} \frac{\partial x^2}{\partial y^{j_2}} \cdots \frac{\partial x^k}{\partial y^{j_k}} dy^{j_1} \wedge dy^{j_2} \wedge \cdots \wedge dy^{j_k} \\ &= (f \circ x \circ y^{-1})(y) det\left[\frac{\partial x}{\partial y}\right] dy^1 \wedge dy^2 \wedge \cdots \wedge dy^k \end{aligned} \tag{10.9}$$

If you calculate the value of Vol on $\partial_{x^I}|_p = \partial_{x_1}|_p, \partial_{x_2}|_p, \ldots, \partial_{x_k}|_p$ you'll find $Vol(\partial_{x^I}|_p) = f(x(p)$. Whereas, if you evaluate Vol on $\partial_{y^I}|_p = \partial_{y_1}|_p, \partial_{y_2}|_p, \ldots, \partial_{y_k}|_p$ then the value is $Vol(\partial_{y^I}|_p) = f(x(p))det\left[\frac{\partial x}{\partial y}(p)\right]$. But, we should recognize that $det\left[\frac{\partial x}{\partial y}\right] = det(d\theta_{ij})$ hence two coordinate systems which are positively oriented must also be consistently oriented. Why? Assume $Vol(\partial_{x^I}|_p) = f(x(p)) > 0$ then $Vol(\partial_{y^I}|_p) = f(x(p))det\left[\frac{\partial x}{\partial y}(p)\right] > 0$ iff $det\left[\frac{\partial x}{\partial y}(p)\right] > 0$ hence y is positively oriented if we are given that x is positively oriented **and** $det\left[\frac{\partial x}{\partial y}\right] > 0$.

Let $\mathcal{M}$ be an oriented k-manifold with orientation given by the volume form Vol and an associated atlas of positively oriented charts. Furthermore, let α be a p-form defined on $V \subseteq \mathcal{M}$. Suppose there exists a local parametrization $\phi : U \subseteq \mathbb{R}^k \to V \subseteq \mathcal{M}$ and $D \subset V$ then there is a smooth function h such that $\alpha_q = h(q)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k$ for each $q \in V$. We define the integral of α over D as follows:

$$\int_D \alpha = \int_{\phi^{-1}(D)} h(\phi(x))d^k x \qquad \leftarrow [\star_x]$$

Is this definition dependent on the coordinate system $\phi : U \subseteq \mathbb{R}^k \to V \subseteq \mathcal{M}$? If we instead used coordinate system $\psi : \bar{U} \subseteq \mathbb{R}^k \to \bar{V} \subseteq \mathcal{M}$ where coordinates $y^1, y^2, \ldots, y^k$ on $\bar{V}$ then the given form α has a different coefficient of $dy^1 \wedge dy^2 \wedge \cdots \wedge dy^k$

$$\alpha = h(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k = (h \circ x \circ y^{-1})(y) det\left[\frac{\partial x}{\partial y}\right] dy^1 \wedge dy^2 \wedge \cdots \wedge dy^k$$

Thus, as we change over to y coordinates the function picks up a factor which is precisely the determinant of the derivative of the transition functions.

$$\begin{aligned} \int_D \alpha &= \int_D (h \circ x \circ y^{-1})(y) det\left[\frac{\partial x}{\partial y}\right] dy^1 \wedge dy^2 \wedge \cdots \wedge dy^k \\ &= \int_{\psi^{-1}(D)} (h \circ x \circ y^{-1})(y) det\left[\frac{\partial x}{\partial y}\right] d^k y \qquad \leftarrow [\star_y] \end{aligned}$$

We need $\star_x = \star_y$ in order for the integral $\int_D \alpha$ to be well-defined. Fortunately, the needed equality is *almost* provided by the change of variables theorem for multivariate integrals on $\mathbb{R}^k$. Recall,

$$\int_R f(x)d^k x = \int_{\bar{R}} \tilde{f}(y) \left| det \frac{\partial x}{\partial y} \right| d^k y$$

where $\tilde{f}$ is more pedantically written as $\tilde{f} = f \circ y^{-1}$, notation aside its just the function f written in terms of the new y-coordinates. Likewise, $\bar{R}$ limits y-coordinates so that the corresponding x-coordinates are found in R. Applying this theorem to our pull-back expression,

$$\int_{\phi^{-1}(D)} h(\phi(x))\, d^k x = \int_{\psi^{-1}(D)} (h \circ x \circ y^{-1})(y) \left| det\left[\frac{\partial x}{\partial y}\right]\right| d^k y.$$

Equality of $\star_x$ and $\star_y$ follows from the fact that $\mathcal{M}$ is oriented and has transition functions[12] θ_{ij} which satisfy $det(d\theta_{ij}) > 0$. We see that this integral to be well-defined only for oriented manifolds. To integrate over manifolds without an orientation additional ideas are needed, but it is possible.

Perhaps the most interesting case to consider is that of an embedded k-manifold in $\mathbb{R}^n$. In this context we must deal with both the coordinates of the ambient $\mathbb{R}^n$ and the local parametrizations of the k-manifold. In multivariate calculus we often consider vector fields which are defined on an open subset of $\mathbb{R}^3$ and then we calculate the flux over a surfaces or the work along a curve. What we have defined thus-far is in essence like definition how to integrate a vector field on a surface or a vector field along a curve, no mention of the vector field off the domain of integration was made. We supposed the forms were already defined on the oriented manifold, but, what if we are instead given a formula for a differential form on $\mathbb{R}^n$? How can we restrict that differential form to a surface or line or more generally a parametrized k-dimensional submanifold of $\mathbb{R}^n$? That is the problem we concern ourselvew with for the remainder of this section.

Let's begin with a simple object. Consider a one-form $\alpha = \sum_{i=1}^n \alpha_i dx^i$ where the function $p \to \alpha_i(p)$ is smooth on some subset of $\mathbb{R}^n$. Suppose C is a curve parametrized by $X : D \subseteq \mathbb{R} \to C \subseteq \mathbb{R}^n$ then the natural chart on C is provided by the parameter t in particular we have $T_pC = span\{\frac{\partial}{\partial t}\big|_{t_o}\}$ where $X(t_o) = p$ and $T_pC^* = span\{d_{t_o}t\}$ hence a vector field along C has the form $f(t)\frac{\partial}{\partial t}$ and a differential form has the form $g(t)dt$. How can we use the one-form α on $\mathbb{R}^n$ to naturally obtain a one-form defined along C? I propose:

$$\alpha\big|_C(t) = \sum_{i=1}^n \alpha_i(X(t))\frac{\partial X^i}{\partial t} dt$$

It can be shown that $\alpha\big|_C$ is a one-form on C. If we change coordinates on the curve by reparametrizing $t \to s$ it then the component relative to s vs. the component relative to t are related:

$$\sum_{i=1}^n \alpha_i(X(t(s)))\frac{\partial X^i}{ds} = \sum_{i=1}^n \alpha_i(X(t))\frac{dt}{ds}\frac{\partial X^i}{\partial t} = \frac{dt}{ds}\left(\sum_{i=1}^n \alpha_i(X(t))\frac{\partial X^i}{\partial t}\right)$$

This is precisely the transformation rule we want for the components of a one-form.

[12] once more recall the notation $\frac{\partial x}{\partial y}$ is just the matrix of the linear transformation $d\theta_{ij}$ and the determinant of a linear transformation is the determinant of the matrix of the transformation

Example 10.8.1. *Suppose $\alpha = dx + 3x^2dy + ydz$ and C is the curve $X : \mathbb{R} \to C \subseteq \mathbb{R}^3$ defined by $X(t) = (1, t, t^2)$ we have $x = 1$, $y = t$ and $z = t^2$ hence $dx = 0, dy = dt$ and $dz = 2tdt$ on C hence $\alpha\big|_C = 0 + 3dt + t(2tdt) = (3 + 2t^2)dt$.*

Next, consider a two-form $\beta = \sum_{i,j=1}^n \frac{1}{2}\beta_{ij}dx^i \wedge dx^j$. Once more we consider a parametrized submanifold of $\mathbb{R}^n$. In particular use the notation $X : D \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^n$ where u, v serve as coordinates on the surface S. We can write an arbitrary two-form on S in the form $h(u, v)du \wedge dv$ where $h : S \to \mathbb{R}$ is a smooth function on S. How should we construct $h(u, v)$ given β? Again, I think the following formula is quite natural, honestly, what else would you do[13]?

$$\beta\big|_S(u, v) = \sum_{i,j=1}^n \beta_{ij}(X(u, v))\frac{\partial X^i}{\partial u}\frac{\partial X^j}{\partial v}du \wedge dv$$

The coefficient function of $du \wedge dv$ is smooth because we assume β_{ij} is smooth on $\mathbb{R}^n$ and the local parametrization is also assumed smooth so the functions $\frac{\partial X^i}{\partial u}$ and $\frac{\partial X^i}{\partial v}$ are smooth. Moreover, the component function has the desired coordinate change property with respect to a reparametrization of S. Suppose we reparametrize by s, t, then suppressing the point-dependence of β_{ij},

$$\beta\big|_S = \sum_{i,j=1}^n \beta_{ij}\frac{\partial Y^i}{\partial s}\frac{\partial Y^j}{\partial t}ds \wedge dt = \frac{du}{ds}\frac{dv}{dt}\sum_{i,j=1}^n \beta_{ij}\frac{\partial X^i}{\partial u}\frac{\partial X^j}{\partial v}ds \wedge dt = \sum_{i,j=1}^n \beta_{ij}\frac{\partial X^i}{\partial u}\frac{\partial X^j}{\partial v}du \wedge dv.$$

Therefore, the restriction of β to S is coordinate independent and we have thus constructed a two-form on a surface from the two-form in the ambient space.

Example 10.8.2. *Consider $\beta = y^2dt \wedge dx + zdx \wedge dy + (x + y + z + t)dt \wedge dz$. Suppose $S \subseteq \mathbb{R}^4$ is parametrized by*

$$X(u, v) = (1,\ u^2v^2,\ 3u,\ v)$$

In other words, we are given that

$$t = 1,\ x = u^2v^2,\ y = 3u,\ z = v$$

Hence, $dt = 0$, $dx = 2uv^2du + 2u^2vdv$, $dy = 3du$ and $dz = dv$. Computing $\beta\big|_S$ is just a matter of substuting in all the formulas above, fortunately $dt = 0$ so only the $zdx \wedge dy$ term is nontrivial:

$$\beta\big|_S = v(2uv^2du + 2u^2vdv) \wedge (3du) = 6u^2v^2dv \wedge du = -6u^2v^2du \wedge dv.$$

It is fairly clear that we can restrict any p-form on $\mathbb{R}^n$ to a p-dimensional parametrized submanifold by the procedure we explained above for $p = 1, 2$. That is the underlying idea in the definitions which follow. Beyond that, once we have restricted the p-form β on $\mathbb{R}^n$ to $\beta|_{\mathcal{M}}$ then we pull-back the restricted form to an open subset of $\mathbb{R}^p$ and reduce the problem to an ordinary multivariate integral.

[13] include the $\frac{1}{2}$ you say?, we'll see why not soon enough

Remark 10.8.3. .

Just a warning, Einstein summation convention is used in what follows, by my count there are over a dozen places where we implicitly indicate a sum.

Our goal in the remainder of the section is to make contact with the[14] integrals we study in calculus. Note that an embedded manifold with a single patch is almost trivially oriented since there is no overlap to consider. In particular, if $\phi : U \subseteq \mathbb{R}^k \to \mathcal{M} \subseteq \mathbb{R}^n$ is a local parametrization with $\phi^{-1} = (u^1, u^2, \ldots, u^k)$ then $du^1 \wedge du^2 \wedge \cdots \wedge du^k$ is a volume form for $\mathcal{M}$. This is the natural generalization of the normal-vector field construction for surfaces in $\mathbb{R}^3$.

Definition 10.8.4. integral of one-form along oriented curve:

Let $\alpha = \alpha_i dx^i$ be a one form and let C be an oriented curve with parametrization $X(t) : [a, b] \to C$ then we define the integral of the one-form α along the curve C as follows,

$$\int_C \alpha \equiv \int_a^b \alpha_i(X(t)) \frac{dX^i}{dt}(t) dt$$

where $X(t) = (X^1(t), X^2(t), \ldots, X^n(t))$ so we mean X^i to be the i^{th} component of $X(t)$. Moreover, the indices are understood to range over the dimension of the ambient space, if we consider forms in $\mathbb{R}^2$ then $i = 1, 2$ if in $\mathbb{R}^3$ then $i = 1, 2, 3$ if in Minkowski $\mathbb{R}^4$ then i should be replaced with $\mu = 0, 1, 2, 3$ and so on.

Example 10.8.5. One form integrals vs. line integrals of vector fields: *We begin with a vector field $\vec{F}$ and construct the corresponding one-form $\omega_{\vec{F}} = F_i dx^i$. Next let C be an oriented curve with parametrization $X : [a, b] \subset \mathbb{R} \to C \subset \mathbb{R}$, observe*

$$\int_C \omega_{\vec{F}} = \int_a^b F_i(X(t)) \frac{dX^i}{dt}(t) dt = \int_C \vec{F} \cdot d\vec{l}$$

You may note that the definition of a line integral of a vector field is not special to three dimensions, we can clearly construct the line integral in n-dimensions, likewise the correspondance ω can be written between one-forms and vector fields in any dimension, provided we have a metric to lower the index of the vector field components. The same cannot be said of the flux-form correspondance, it is special to three dimensions for reasons we have explored previously.

[14] hopefully known to you already from multivariate calculus

Definition 10.8.6. integral of two-form over an oriented surface:

Let $\beta = \frac{1}{2}\beta_{ij}dx^i \wedge dx^j$ be a two-form and let S be an oriented piecewise smooth surface with parametrization $X(u,v) : D_2 \subset \mathbb{R}^2 \to S \subset \mathbb{R}^n$ then we define the integral of the two-form β over the surface S as follows,

$$\int_S \beta \equiv \int_{D_2} \beta_{ij}(X(u,v)) \frac{\partial X^i}{\partial u}(u,v) \frac{\partial X^j}{\partial v}(u,v) du dv$$

where $X(u,v) = (X^1(u,v), X^2(u,v), \ldots, X^n(u,v))$ so we mean X^i to be the i^{th} component of $X(u,v)$. Moreover, the indices are understood to range over the dimension of the ambient space, if we consider forms in $\mathbb{R}^2$ then $i,j = 1,2$ if in $\mathbb{R}^3$ then $i,j = 1,2,3$ if in Minkowski $\mathbb{R}^4$ then i,j should be replaced with $\mu,\nu = 0,1,2,3$ and so on.

Example 10.8.7. Two-form integrals vs. surface integrals of vector fields in $\mathbb{R}^3$: *We begin with a vector field $\vec{F}$ and construct the corresponding two-form $\Phi_{\vec{F}} = \frac{1}{2}\epsilon_{ijk}F_k dx^i \wedge dx^j$ which is to say $\Phi_{\vec{F}} = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$. Next let S be an oriented piecewise smooth surface with parametrization $X : D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$, then*

$$\int_S \Phi_{\vec{F}} = \int_S \vec{F} \cdot d\vec{A}$$

Proof: *Recall that the normal to the surface S has the form,*

$$N(u,v) = \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} = \epsilon_{ijk} \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v} e_k$$

at the point $X(u,v)$. This gives us a vector which points along the outward normal to the surface and it is nonvanishing throughout the whole surface by our assumption that S is oriented. Moreover the vector surface integral of $\vec{F}$ over S was defined by the formula,

$$\int_S \vec{F} \cdot d\vec{A} \equiv \int\int_D \vec{F}(X(u,v)) \cdot \vec{N}(u,v)\ du dv.$$

now that the reader is reminded what's what, lets prove the proposition, dropping the (u,v) depence to reduce clutter we find,

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int\int_D \vec{F} \cdot \vec{N}\ du dv \\
&= \int\int_D F_k N_k\ du dv \\
&= \int\int_D F_k \epsilon_{ijk} \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v}\ du dv \\
&= \int\int_D (\Phi_{\vec{F}})_{ij} \frac{\partial X^i}{\partial u} \frac{\partial X^j}{\partial v}\ du dv \\
&= \int_S \Phi_{\vec{F}}
\end{aligned}$$

notice that we have again used our convention that $(\Phi_{\vec{F}})_{ij}$ refers to the tensor components of the 2-form $\Phi_{\vec{F}}$ meaning we have $\Phi_{\vec{F}} = (\Phi_{\vec{F}})_{ij}dx^i \otimes dx^j$ whereas with the wedge product $\Phi_{\vec{F}} = \frac{1}{2}(\Phi_{\vec{F}})_{ij}dx^i \wedge dx^j$, I mention this in case you are concerned there is a half in $\Phi_{\vec{F}}$ yet we never found a half in the integral. Well, we don't expect to because we defined the integral of the form with respect to the tensor components of the form, again they don't contain the half.

Example 10.8.8. *Consider the vector field $\vec{F} = (0, 0, 3)$ then the corresponding two-form is simply $\Phi_F = 3dx \wedge dy$. Lets calculate the surface integral and two-form integrals over the square $D = [0, 1]\times[0, 1]$ in the xy-plane, in this case the parameters can be taken to be x and y so $X(x, y) = (x, y)$ and,*

$$N(x, y) = \frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y} = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$$

which is nice. Now calculate,

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int\int_D \vec{F} \cdot \vec{N}\ dxdy \\
&= \int\int_D (0, 0, 3) \cdot (0, 0, 1)dxdy \\
&= \int_0^1\int_0^1 3dxdy \\
&= 3.
\end{aligned}$$

Consider that $\Phi_F = 3dx \wedge dy = 3dx \otimes dy - 3dy \otimes dx$ therefore we may read directly that $(\Phi_F)_{12} = -(\Phi_F)_{21} = 3$ and all other components are zero,

$$\begin{aligned}
\int_S \Phi_F &= \int\int_D (\Phi_F)_{ij}\frac{\partial X^i}{\partial x}\frac{\partial X^j}{\partial y}\ dxdy \\
&= \int\int_D \left(3\frac{\partial X^1}{\partial x}\frac{\partial X^2}{\partial y} - 3\frac{\partial X^2}{\partial x}\frac{\partial X^1}{\partial y}\right)\ dxdy \\
&= \int_0^1\int_0^1 \left(3\frac{\partial x}{\partial x}\frac{\partial y}{\partial y} - 3\frac{\partial y}{\partial x}\frac{\partial x}{\partial y}\right)\ dxdy \\
&= 3.
\end{aligned}$$

Definition 10.8.9. integral of a three-form over an oriented volume:

Let $\gamma = \frac{1}{6}\beta_{ijk}dx^i \wedge dx^j \wedge dx^k$ be a three-form and let V be an oriented piecewise smooth volume with parametrization $X(u,v,w) : D_3 \subset \mathbb{R}^3 \to V \subset \mathbb{R}^n$ then we define the integral of the three-form γ in the volume V as follows,

$$\int_V \gamma \equiv \int_{D_3} \gamma_{ijk}(X(u,v,w))\frac{\partial X^i}{\partial u}\frac{\partial X^j}{\partial v}\frac{\partial X^k}{\partial w}\, dudvdw$$

where $X(u,v,w) = (X^1(u,v,w), X^2(u,v,w), \ldots, X^n(u,v,w))$ so we mean X^i to be the i^{th} component of $X(u,v,w)$. Moreover, the indices are understood to range over the dimension of the ambient space, if we consider forms in $\mathbb{R}^3$ then $i,j,k = 1,2,3$ if in Minkowski $\mathbb{R}^4$ then i,j,k should be replaced with $\mu,\nu,\sigma = 0,1,2,3$ and so on.

Finally we define the integral of a p-form over an p-dimensional subspace of $\mathbb{R}$, we assume that $p \leq n$ so that it is possible to embed such a subspace in $\mathbb{R}$,

Definition 10.8.10. integral of a p-form over an oriented volume:

Let $\gamma = \frac{1}{p!}\beta_{i_1\ldots i_p}dx^{i_1} \wedge \cdots dx^{i_p}$ be a p-form and let S be an oriented piecewise smooth subspace with parametrization $X(u_1,\ldots,u_p) : D_p \subset \mathbb{R}^p \to S \subset \mathbb{R}^n$ (for $n \geq p$) then we define the integral of the p-form γ in the subspace S as follows,

$$\int_S \gamma \equiv \int_{D_p} \beta_{i_1\ldots i_p}(X(u_1,\ldots,u_p))\frac{\partial X^{i_1}}{\partial u_1}\cdots\frac{\partial X^{i_p}}{\partial u_p}\, du_1\cdots du_p$$

where $X(u_1,\ldots,u_p) = (X^1(u_1,\ldots,u_p), X^2(u_1,\ldots,u_p), \ldots, X^n(u_1,\ldots,u_p))$ so we mean X^i to be the i^{th} component of $X(u_1,\ldots,u_p)$. Moreover, the indices are understood to range over the dimension of the ambient space.

Integrals of forms play an important role in modern physics. I hope you can begin to appreciate that forms recover all the formulas we learned in multivariate calculus and give us a way to extend calculation into higher dimensions with ease. I include a toy example at the conclusion of this chapter just to show you how electromagnetism is easily translated into higher dimensions.

10.9 Generalized Stokes Theorem

The generalized Stokes theorem contains within it most of the main theorems of integral calculus, namely the fundamental theorem of calculus, the fundamental theorem of line integrals (a.k.a the FTC in three dimensions), Greene's Theorem in the plane, Gauss' Theorem and also Stokes Theorem, not to mention a myriad of higher dimensional not so commonly named theorems. The breadth of its application is hard to overstate, yet the statement of the theorem is simple,

Theorem 10.9.1. *Generalized Stokes Theorem:*

Let S be an oriented, piecewise smooth (p+1)-dimensional subspace of $\mathbb{R}^n$ where $n \geq p+1$ and let ∂S be it boundary which is consistently oriented then for a p-form α which behaves reasonably on S we have that

$$\boxed{\int_S d\alpha = \int_{\partial S} \alpha}$$

The proof of this theorem (and a more careful statement of it) can be found in a number of places, Susan Colley's *Vector Calculus* or Steven H. Weintraub's *Differential Forms: A Complement to Vector Calculus* or Spivak's *Calculus on Manifolds* just to name a few. I believe the argument in Edward's text is quite complete. In any event, you should already be familar with the idea from the usual Stokes Theorem where we must insist the boundary curve to the surface is related to the surface's normal field according to the right-hand-rule. Explaining how to orient the boundary $\partial\mathcal{M}$ given an oriented $\mathcal{M}$ is the problem of generalizing the right-hand-rule to many dimensions. I leave it to your homework for the time being.

Lets work out how this theorem reproduces the main integral theorems of calculus.

Example 10.9.2. Fundamental Theorem of Calculus in $\mathbb{R}$: *Let $f : \mathbb{R} \to \mathbb{R}$ be a zero-form then consider the interval $[a,b]$ in $\mathbb{R}$. If we let $S = [a,b]$ then $\partial S = \{a,b\}$. Further observe that $df = f'(x)dx$. Notice by the definition of one-form integration*

$$\int_S df = \int_a^b f'(x)dx$$

However on the other hand we find (the integral over a zero-form is taken to be the evaluation map, perhaps we should have defined this earlier, oops., but its only going to come up here so I'm leaving it.)

$$\int_{\partial S} f = f(b) - f(a)$$

Hence in view of the definition above we find that

$$\int_a^b f'(x)dx = f(b) - f(a) \quad \Longleftrightarrow \quad \int_S df = \int_{\partial S} f$$

Example 10.9.3. Fundamental Theorem of Calculus in $\mathbb{R}^3$: *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a zero-form then consider a curve C from $p \in \mathbb{R}^3$ to $q \in \mathbb{R}^3$ parametrized by $\phi : [a,b] \to \mathbb{R}^3$. Note that $\partial C = \{\phi(a) = p, \phi(b) = q\}$. Next note that*

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Then consider that the exterior derivative of a function corresponds to the gradient of the function thus we are not to surprised to find that

$$\int_C df = \int_a^b \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} dt = \int_C (\nabla f) \cdot d\vec{l}$$

On the other hand, we use the definition of the integral over a a two point set again to find

$$\int_{\partial C} f = f(q) - f(p)$$

Hence if the Generalized Stokes Theorem is true then so is the FTC in three dimensions,

$$\int_C (\nabla f) \cdot d\vec{l} = f(q) - f(p) \quad \iff \quad \int_C df = \int_{\partial C} f$$

another popular title for this theorem is the "fundamental theorem for line integrals". As a final thought here we notice that this calculation easily generalizes to 2,4,5,6,... dimensions.

Example 10.9.4. Greene's Theorem: *Let us recall the statement of Greene's Theorem as I have not replicated it yet in the notes, let D be a region in the xy-plane and let ∂D be its consistently oriented boundary then if $\vec{F} = (M(x,y), N(x,y), 0)$ is well behaved on D*

$$\int_{\partial D} M dx + N dy = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

We begin by finding the one-form corresponding to $\vec{F}$ namely $\omega_F = M dx + N dy$ consider then that

$$d\omega_F = d(M dx + N dy) = dM \wedge dx + dN \wedge dy = \frac{\partial M}{\partial y} dy \wedge dx + \frac{\partial N}{\partial x} dx \wedge dy$$

which simplifies to,

$$d\omega_F = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \wedge dy = \Phi_{(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\hat{k}}$$

Thus, using our discussion in the last section we recall

$$\int_{\partial D} \omega_F = \int_{\partial D} \vec{F} \cdot d\vec{l} = \int_{\partial D} M dx + N dy$$

where we have reminded the reader that the notation in the rightmost expression is just another way of denoting the line integral in question. Next observe,

$$\int_D d\omega_F = \int_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}) \hat{k} \cdot d\vec{A}$$

And clearly, since $d\vec{A} = \hat{k}dxdy$ *we have*

$$\int_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\hat{k} \cdot d\vec{A} = \int_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy$$

Therefore,

$$\int_{\partial D} Mdx + Ndy = \int\int_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \quad \iff \quad \int_D d\omega_F = \int_{\partial D} \omega_F$$

Example 10.9.5. Gauss Theorem: *Let us recall Gauss Theorem to begin, for suitably defined* $\vec{F}$ *and* V*,*

$$\int_{\partial V} \vec{F} \cdot d\vec{A} = \int_V \nabla \cdot \vec{F} \; d\tau$$

First we recall our earlier result that

$$d(\Phi_F) = (\nabla \cdot \vec{F})dx \wedge dy \wedge dz$$

Now note that we may integrate the three form over a volume,

$$\int_V d(\Phi_F) = \int_V (\nabla \cdot \vec{F})dxdydz$$

whereas,

$$\int_{\partial V} \Phi_F = \int_{\partial V} \vec{F} \cdot d\vec{A}$$

so there it is,

$$\int_V (\nabla \cdot \vec{F})d\tau = \int_{\partial V} \vec{F} \cdot d\vec{A} \quad \iff \quad \int_V d(\Phi_F) = \int_{\partial V} \Phi_F$$

I have left a little detail out here, I may assign it for homework.

Example 10.9.6. Stokes Theorem: *Let us recall Stokes Theorem to begin, for suitably defined* $\vec{F}$ *and* S*,*

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{A} = \int_{\partial S} \vec{F} \cdot d\vec{l}$$

Next recall we have shown in the last chapter that,

$$d(\omega_F) = \Phi_{\nabla \times \vec{F}}$$

Hence,

$$\int_S d(\omega_F) = \int_S (\nabla \times \vec{F}) \cdot d\vec{A}$$

whereas,

$$\int_{\partial S} \omega_F = \int_{\partial S} \vec{F} \cdot d\vec{l}$$

which tells us that,

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{A} = \int_{\partial S} \vec{F} \cdot d\vec{l} \quad \iff \quad \int_S d(\omega_F) = \int_{\partial S} \omega_F$$

The Generalized Stokes Theorem is perhaps the most persausive argument for mathematicians to be aware of differential forms, it is clear they allow for more deep and sweeping statements of the calculus. The generality of differential forms is what drives modern physicists to work with them, string theorists for example examine higher dimensional theories so they are forced to use a language more general than that of the conventional vector calculus.

10.10 Poincare lemma

This section is in large part inspired by M. Gockeler and T. Schucker's *Differential geometry, gauge theories, and gravity* page 20-22. The converse calculation is modelled on the argument found in H. Flanders *Differential Forms with Applications to the Physical Sciences*. The original work was done around the dawn of the twentieth century and can be found in many texts besides the two I mentioned here.

10.10.1 exact forms are closed

Proposition 10.10.1.

The exterior derivative of the exterior derivative is zero. $d^2 = 0$

Proof: Let α be an arbitrary p-form then

$$d\alpha = \frac{1}{p!}(\partial_m \alpha_{i_1 i_2 \ldots i_p}) dx^m \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \tag{10.10}$$

then differentiate again,

$$\begin{aligned} d(d\alpha) &= d\left[\tfrac{1}{p!}(\partial_m \alpha_{i_1 i_2 \ldots i_p}) dx^m \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}\right] \\ &= \tfrac{1}{p!}(\partial_k \partial_m \alpha_{i_1 i_2 \ldots i_p}) dx^k \wedge dx^m \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p} \\ &= 0 \end{aligned} \tag{10.11}$$

since the partial derivatives commute whereas the wedge product anticommutes so we note that the pair of indices (k,m) is symmetric for the derivatives but antisymmetric for the wedge, as we know the sum of symmetric against antisymmetric vanishes (see equation **??** part *iv* if you forgot.)

Definition 10.10.2.

A differential form α is **closed** iff $d\alpha = 0$. A differential form β is **exact** iff there exists γ such that $\beta = d\gamma$.

Proposition 10.10.3.

All exact forms are closed. However, there exist closed forms which are not exact.

Proof: Exact implies closed is easy, let β be exact such that$\beta = d\gamma$ then

$$d\beta = d(d\gamma) = 0$$

using the theorem $d^2 = 0$. To prove that there exists a closed form which is not exact it suffices to give an example. A popular example (due to its physical significance to magnetic monopoles, Dirac Strings and the like..) is the following differential form in $\mathbb{R}^2$

$$\phi = \frac{1}{x^2+y^2}(xdy - ydx) \tag{10.12}$$

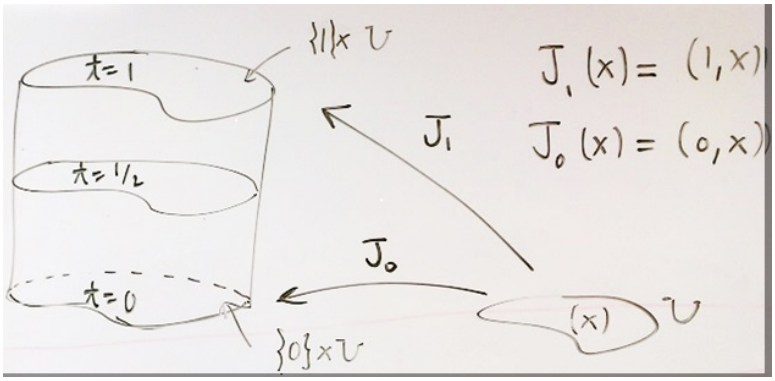
You may verify that $d\phi = 0$ in homework. Observe that if ϕ were exact then there would exist f such that $\phi = df$ meaning that

$$\frac{\partial f}{\partial x} = -\frac{y}{x^2+y^2}, \qquad \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$$

which are solved by $f = arctan(y/x) + c$ where c is arbitrary. Observe that f is ill-defined along the y-axis $x = 0$ (this is the Dirac String if we put things in context), however the natural domain of ϕ is $\mathbb{R}^{n\times n} - \{(0,0)\}$. $\square$

10.10.2 potentials for closed forms

Poincare' suggested the following partial converse, he said closed implies exact provided we place a topological restriction on the domain of the form. In particular, if the domain of a closed form is smoothly deformable to a point then each closed form is exact. We'll work out a proof of that result for a subset of $\mathbb{R}^n$. Be patient, we have to build some toys before we play.



Suppose $U \subseteq \mathbb{R}^n$ and $I = [0,1]$ we denote a typical point in $I \times U$ as (t,x) where $t \in I$ and $x \in \mathbb{R}^n$. Define maps,

$$J_1 : U \to I \times U, \qquad J_0 : U \to I \times U$$

by $J_1(x) = (1,x)$ and $J_0(x) = (0,x)$ for each $x \in U$. Flanders encourages us to view $I \times U$ as a cylinder and where the map J_1 maps U to the top and J_0 maps U to the base. We can pull-back

forms on the cylinder to the U on the top ($t = 1$) or to the base ($t = 0$). For instance, if we consider $\omega = (x + t)dx + dt$ for the case $n = 1$ then

$$J_0^*\omega = xdx \qquad J_1^*\omega = (x + 1)dx.$$

Define a smooth mapping K of $(p + 1)$ forms on $I \times U$ to p-forms on U as follows:

$$(1.)\ K(a(t, x)dx^I) = 0, \qquad (2.)\ K(a(t, x)dt \wedge dx^J) = \left(\int_0^1 a(t, x)dt\right)dx^J$$

for multi-indices I of length $(p + 1)$ and J of length p. The cases (1.) and (2.) simply divide the possible monomial[15] inputs from $\Lambda^{p+1}(I \times U)$ into forms which have dt and those which don't. Then K is defined for a general $(p + 1)$-form on $I \times U$ by linearly extending the formulas above to multinomials of the basic monomials.

It turns out that the following identity holds for K:

Lemma 10.10.4. *the K-lemma.*

If ω is a differential form on $I \times U$ then

$$K(d\omega) + d(K(\omega)) = J_1^*\omega - J_0^*\omega.$$

Proof: Since the equation is given for linear operations it suffices to check the formula for monomials since we can extend the result linearly once those are affirmed. As in the definition of K there are two basic categories of forms on $I \times U$:

Case 1: If $\omega = a(t, x)dx^I$ then clearly $K(\omega) = 0$ hence $d(K(\omega)) = 0$. Observe,

$$d\omega = da \wedge dx^I = \sum_j \frac{\partial a}{\partial x^j}dx^j \wedge dx^I + \frac{\partial a}{\partial t}dt \wedge dx^I$$

Hence $K(d\omega)$ is calculated as follows:

$$\begin{aligned} K(d\omega) &= K\left(\sum_j \frac{\partial a}{\partial x^j}dx^j \wedge dx^I\right) + K\left(\frac{\partial a}{\partial t}dt \wedge dx^I\right) \\ &= \left(\int_0^1 \frac{\partial a}{\partial t}dt\right)dx^I \\ &= \big[a(x, 1) - a(x, 0)\big]dx^I \\ &= J_1^*\omega - J_0^*\omega \end{aligned} \tag{10.13}$$

where we used the FTC in the next to last step. The pull-backs in this case just amount to evaluation at $t = 0$ or $t = 1$ as there is no dt-type term to squash in ω. The identity follows.

[15] $dx \wedge dy$ is a monomial whereas $dx + dy$ is a binomial in this context

Case 2: Suppose $\omega = a(t,x)dt \wedge dx^J$. Calculate,

$$d\omega = \sum_j \frac{\partial a}{\partial x^k} dx^k \wedge dt \wedge dx^J + \frac{\partial a}{\partial t} \underbrace{dt \wedge dt}_{zero\ !} \wedge dx^J$$

Thus, using $dx^k \wedge dt = -dt \wedge dx^k$, we calculate:

$$\begin{aligned} K(d\omega) &= K\left(-\sum_k \frac{\partial a}{\partial x^k} dt \wedge dx^k \wedge dx^I \right) \\ &= -\sum_k \left(\int_0^1 \frac{\partial a}{\partial x^k} dt \right) dx^k \wedge dx^I \end{aligned}$$

at which point we cannot procede further since a is an arbitrary function which can include a nontrivial time-dependence. We turn to the calculation of $d(K(\omega))$. Recall we defined

$$K(\omega) = \left(\int_0^1 a(t,x)dt \right) dx^J.$$

We calculate the exterior derivative of $K(\omega)$:

$$\begin{aligned} d(K(\omega)) &= d\left(\int_0^1 a(t,x)dt \right) \wedge dx^J \\ &= \left(\frac{\partial}{\partial t} \Big[\underbrace{\int_0^1 a(\tau,x)d\tau}_{constant\ in\ t} \Big] dt + \sum_k \frac{\partial}{\partial x^k} \left[\int_0^1 a(t,x)dt \right] dx^k \right) \wedge dx^J \\ &= \sum_k \left(\int_0^1 \frac{\partial a}{\partial x^k} dt \right) dx^k \wedge dx^J. \end{aligned} \tag{10.14}$$

Therefore, $K(d\omega) + d(K(\omega)) = 0$ and clearly $J_0^*\omega = J_1^*\omega = 0$ in this case since the pull-backs squash the dt to zero. The lemma follows. $\square$.

Definition 10.10.5.

A subset $U \subseteq \mathbb{R}^n$ is deformable to a point P if there exists a smooth mapping $G : I \times U \to U$ such that $G(1,x) = x$ and $G(0,x) = P$ for all $x \in U$.

The map G deforms U smoothly into the point P. Recall that $J_1(x) = (1,x)$ and $J_0(x) = (0,x)$ hence the conditions on the deformation can be expressed as:

$$G(J_1(x)) = x \qquad G(J_0(x)) = P$$

Denoting Id for the identity on U and P as the constant mapping with value P on U we have

$$G \circ J_1 = Id \qquad G \circ J_0 = P$$

Therefore, if γ is a $(p+1)$-form on U we calculate,

$$(G \circ J_1)^* \gamma = Id^* \gamma \quad \Rightarrow \quad J_1^*[G^* \gamma] = \gamma$$

whereas,

$$(G \circ J_0)^* \gamma = P^* \gamma = 0 \quad \Rightarrow \quad J_0^*[G^* \gamma] = 0$$

Apply the K-lemma to the form $\omega = G^* \gamma$ on $I \times U$ and we find:

$$K(d(G^* \gamma)) + d(K(G^* \gamma)) = \gamma.$$

However, recall that we proved that pull-backs and exterior derivatives commute thus

$$d(G^* \gamma) = G^*(d\gamma)$$

and we find an extremely interesting identity,

$$\boxed{K(G^*(d\gamma)) + d(K(G^* \gamma)) = \gamma.}$$

Proposition 10.10.6.

If $U \subseteq \mathbb{R}$ is deformable to a point P then a p-form γ on U is closed iff ϕ is exact.

Proof: Suppose γ is exact then $\gamma = d\beta$ for some $(p-1)$-form β on U hence $d\gamma = d(d\beta) = 0$ by Proposition 10.10.1 hence γ is closed. Conversely, suppose γ is closed. Apply the boxed consequence of the K-lemma, note that $K(G^*(0)) = 0$ since we assume $d\gamma = 0$. We find,

$$d(K(G^* \gamma)) = \gamma$$

identify that $G^* \gamma$ is a p-form on $I \times U$ whereas $K(G^* \gamma)$ is a $(p-1)$-form on U by the very construction of K. It follows that γ is exact since we have shown how it is obtained as the exterior derivative of another differential form of one degree less. □

Where was deformability to a point P used in the proof above? The key is the existence of the mapping G. In other words, if you have a space which is not deformable to a point then no deformation map G is available and the construction via K breaks down. Basically, if the space has a hole which you get stuck on as you deform loops to a point then it is not deformable to a point. Often we call such spaces *simply connected.* Careful definition of these terms is too difficult for calculus, deformation of loops and higher dimensional objects is properly covered in algebraic topology. In any event, the connection of the deformation and exactness of closed forms allows topologists to use differential forms detect holes in spaces. In particular:

Definition 10.10.7. de Rham cohomology:

We define several real vector spaces of differential forms over some subset U of $\mathbb{R}$,

$$Z^p(U) \equiv \{\phi \in \Lambda^p U \mid \phi \text{ closed}\}$$

the space of closed p-forms on U. Then,

$$B^p(U) \equiv \{\phi \in \Lambda^p U \mid \phi \text{ exact}\}$$

the space of exact p-forms where by convention $B^0(U) = \{0\}$ The de Rham cohomology groups are defined by the quotient of closed/exact,

$$H^p(U) \equiv Z^p(U)/B^p(U).$$

the $dim(H^pU) = p^{th}$ Betti number of U.

We observe that simply connected regions have all the Betti numbers zero since $Z^p(U) = B^p(U)$ implies that $H^p(U) = \{0\}$. Of course there is much more to say about de Rahm Cohomology, I just wanted to give you a taste and alert you to the fact that differential forms can be used to reveal aspects of topology. Not all algebraic topology uses differential forms though, there are several other calculational schemes based on triangulation of the space, or studying singular simplexes. One important event in 20-th century mathematics was the discovery that all these schemes described the same homology groups. The Steenrod reduced the problem to a few central axioms and it was shown that all the calculational schemes adhere to that same set of axioms.

One interesting aspect of the proof we (copied from Flanders [16]) is that it is not a mere existence proof. It actually lays out how to calculate the form which provides exactness. Let's call β the potential form of γ if $\gamma = d\beta$. Notice this is totally reasonable langauge since in the case of classical mechanics we consider conservative forces $\vec{F}$ which as derivable from a scalar potential V by $\vec{F} = -\nabla V$. Translated into differential forms we have $\omega_{\vec{F}} = -dV$. Let's explore how the K-mapping and proof indicate the potential of a vector field ought be calculated.

Suppose U is deformable to a point and F is a smooth conservative vector field on U. Perhaps you recall that for conservative F are irrotational hence $\nabla \times F = 0$. Recall that $d\omega_F = \Phi_{\nabla \times F} = \Phi_0 = 0$ thus the one-form corresponding to a conservative vector field is a closed form. Apply the identity: let $G : I \times U \to U \subseteq \mathbb{R}^3$ be the deformation of U to a point P,

$$d(K(G^*\omega_F)) = \omega_F$$

Hence, including the minus to make energy conservation natural,

$$V = -K(G^*\omega_F)$$

[16] I don't know the complete history of this calculation at the present. It would be nice to find it since I doubt Flanders is the originator.

For convenience, lets suppose the space considered is the unit-ball B and lets use a deformation to the origin. Explicitly, $G(t,r) = tr$ for all $r \in \mathbb{R}^3$ such that $||r|| \leq 1$. Note that clearly $G(0,r) = 0$ whereas $G(1,r) = r$ and G has a nice formula so it's smooth[17]. We wish to calculate the pull-back of $\omega_F = Pdx + Qdy + Rdz$ under G, from the definition of pull-back we have

$$(G^*\omega_F)(X) = \omega_F(dG(X))$$

for each smooth vector field X on $I \times B$. Differential forms on $I \times B$ are written as linear combinations of dt, dx, dy, dz with smooth functions as coefficients. We can calculate the coefficents by evalutaion on the corresponding vector fields $\partial_t, \partial_x, \partial_y, \partial_z$. Observe, since $G(t,x,y,z) = (tx, ty, tz)$ we have

$$dG(\partial_t) = \frac{\partial G^1}{\partial t}\frac{\partial}{\partial x} + \frac{\partial G^2}{\partial t}\frac{\partial}{\partial y} + \frac{\partial G^3}{\partial t}\frac{\partial}{\partial z} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

wheras,

$$dG(\partial_x) = \frac{\partial G^1}{\partial x}\frac{\partial}{\partial x} + \frac{\partial G^2}{\partial x}\frac{\partial}{\partial y} + \frac{\partial G^3}{\partial x}\frac{\partial}{\partial z} = t\frac{\partial}{\partial x}$$

and similarly,

$$dG(\partial_y) = t\frac{\partial}{\partial y} \qquad dG(\partial_x) = t\frac{\partial}{\partial z}$$

Furthermore,

$$\omega_F(dG(\partial_t)) = \omega_F(x\partial_x + y\partial_y + z\partial_z) = xP + yQ + zR$$

$$\omega_F(dG(\partial_x)) = \omega_F(t\partial_x) = tP, \qquad \omega_F(dG(\partial_y)) = \omega_F(t\partial_y) = tQ, \qquad \omega_F(dG(\partial_z)) = \omega_F(t\partial_z) = tR$$

Therefore,

$$G^*\omega_F = (xP + yQ + zR)dt + tPdx + tQdy + tRdz = (xP + yQ + zR)dt + t\omega_F$$

Now we can calculate $K(G^*\omega_F)$ and hence V [18]

$$K(G^*\omega_F)(t,x,y,z) = K\bigg(\big(xP(tx,ty,tz) + yQ(tx,ty,tz) + zR(tx,ty,tz)\big)dt\bigg)$$

Therefore,

$$V(x,y,z) = -K(G^*\omega_F) = -\int_0^1 \big(xP(tx,ty,tz) + yQ(tx,ty,tz) + zR(tx,ty,tz)\big)dt$$

Notice this is precisely the line-integral of $F =< P, Q, R >$ along the line C with direction $< x, y, z >$ from the origin to (x,y,z). In particular, if $\vec{r}(t) =< tx, ty, tz >$ then $\frac{d\vec{r}}{dt} =< x, y, z >$ hence we identify

$$V(x,y,z) = -\int_0^1 \vec{F}\big(\vec{r}(t)\big) \cdot \frac{d\vec{r}}{dt}\, dt = -\int_C \vec{F} \cdot d\vec{r}$$

[17] there is of course a deeper meaning to the word, but, for brevity I gloss over this.

[18] Note that only the coefficient of dt gives a nontrivial contribution so in retrospect we did a bit more calculation than necessary. That said, I'll just keep it as a celebration of extreme youth for calculation. Also, I've been a bit careless in omiting the point up to this point, let's include the point dependence since it will be critical to properly understand the formula.

Perhaps you recall this is precisely how we calculate the potential function for a conservative vector field provided we take the origin as the zero for the potential.

Actually, this calculation is quite interesting. Suppose we used a different deformation $\tilde{G} : I \times U \to U$. For fixed point Q we travel to from the origin to the point by the path $t \mapsto \tilde{G}(t, Q)$. Of course this path need not be a line. The space considered might look like a snake where a line cannot reach from the base point P to the point Q. But, the same potential is derived. Why? Path independence of the vector field is one answer. The criteria $\nabla \times F = 0$ suffices for a simply connected region. However, we see something deeper. The criteria of a closed form paired with a simply connected (deformable) domain suffices to construct a potential for the given form. This result reproduces the familar case of conservative vector fields derived from scalar potentials and *much more*. In Flanders he calculates the potential for a closed two-form. This ought to be the mathematics underlying the construction of the so-called **vector potential** of magnetism. In junior-level electromagnetism[19] the magnetic field B satisfies $\nabla \cdot B = 0$ and thus the two-form Φ_B has exterior derivative $d\Phi_B = \nabla \cdot B dx \wedge dy \wedge dz = 0$. The magnetic field corresponds to a closed form. Poincare's lemma shows that there exists a one-form ω_A such that $d\omega_A = \Phi_B$. But this means $\Phi_{\nabla \times A} = \Phi_B$ hence in the langauge of vector fields we expect the vector potential A generated the magnetic field B throught the curl $B = \nabla \times A$. Indeed, this is precisely what a typical junior level physics student learns in the magnetostatic case. Appreciate that it goes deeper still, the Poincare lemma holds for p-forms which correspond to objects which don't match up with the quaint vector fields of 19-th century physics. We can be confident to find potential for 3-form fluxes in a 10-dimensional space, or wherever our imagination takes us. I explain at the end of this chapter how to translate electromagnetics into the langauge of differential forms, it may well be that in the future we think about forms the way we currently think about vectors. This is one of the reasons I like Flanders text, he really sticks with the langauge of differential forms throughout. In contrast to these notes, he just does what is most interesting. I think undergraduates need to see more detail and not just the most clever calculations, but, I can hardly blame Flanders! He makes no claim to be an undergraduate work.

Finally, I should at least mention that though we can derive a potential β for a given closed form α on a simply connected domain it need not be unique. In fact, it will not be unique unless we add further criteria for the potential. This ambuity is called **gauge freedom** in physics. Mathematically it's really simple give form language. If $\alpha = d\beta$ where β is a $(p-1)$-form then we can take any smooth $(p-2)$ form and calculate that

$$d(\alpha + d\lambda) = d\beta + d^2\lambda = d\beta = \alpha$$

Therefore, if β is a potential-form for α then $\beta + d\lambda$ is also a potential-form for α.

[19] just discussing magnetostatic case here to keep it simple

10.11 introduction to geometric differential equations

Differential forms, pull-backs and submanifolds provide a language in which the general theory of partial differential equations is naturally expressed. That said, let us begin with an application to the usual differential equations course.

10.11.1 exact differential equations

Throughout what follows assume that M, N, I, F are continuously differentiable functions of x, y, perhaps just defined for some open subset of $\mathbb{R}^2$. Recall (or learn) that a differential equation $Mdx + Ndy = 0$ is said to be **exact** iff there is a function F such that $dF = Mdx + Ndy$. This is a very nice kind of differential equation since the solution is simply $F(x, y) = c$. A convenient test for exactness was provided from the fact that partial derivatives commute. This exchange of partial derivatives implies that $\partial_y M = \partial_x N$. Observe, we can recover this condition via exterior differentiation:

Proposition 10.11.1.

If a differential equation $Mdx + Ndy = 0$ is exact then $d(Mdx + Ndy) = 0$ (this zero is derived from M, N alone, independent of the given differential equation).

Proof: if $Mdx + Ndy = 0$ is exact then by definition $dF = Mdx + Ndy$ hence $d(Mdx + Ndy) = d(dF) = 0$. □

Pfaff did pioneering work in the theory of differential forms. One Theorem due to Pfaff states that any first order differential equation can be made exact by the method of integrating factors. In particular, if $Mdx + Ndy = 0$ is not exact then there exists I such that $IMdx + INdy = 0$ is an exact differential equation. The catch, it is as hard or harder to find I as it is to solve the given differential equation. That said, the integrating factor method is an example of this method. Although, we don't usually think of linear ordinary differential equations as an exact equation, it can be viewed as such[20].

A differential equation in $x_1, x_2, \ldots, x_n$ of the form:

$$M_1 dx_1 + M_2 dx_2 + \cdots + M_n dx_n = 0$$

can be written as $dF = 0$ locally iff

$$d(M_1 dx_1 + M_2 dx_2 + \cdots + M_n dx_n) = 0.$$

The fact that exact implies closed is just $d^2 = 0$. The converse direction, assuming closed near a point, only gives the existence of a potential form F close to the point. Globally, there could be a topological obstruction as we saw in the Poincare Lemma section.

[20] see my differential equations notes, it's in there

Example 10.11.2. Problem: *Suppose $xdy + ydx - xdz = 0$. Is there a point(s) in $\mathbb{R}^3$ near which there exists F such that $dF = xdy + ydx - xdz$?*

Solution: *If there was the the differential form of the DEqn would vanish identically. However:*

$$d(xdy + ydx - xdz) = dx \wedge dy + dy \wedge dx - dx \wedge dz = dz \wedge dx. \qquad \textit{Therefore, no.}$$

We can try to find an integrating factor. Let's give it a shot, this problem is simple enough it may be possible to work it out. We want I such that $I(xdy + ydx - xdz)$ is a closed one-form. Use the Leibniz rule and our previous calculation:

$$\begin{aligned} d[I(xdy + ydx - xdz)] &= dI \wedge (xdy + ydx - xdz) + Idz \wedge dx \\ &= (I_x dx + I_y dy + I_z dz) \wedge (xdy + ydx - xdz) + Idz \wedge dx \\ &= (xI_x - yI_y)dx \wedge dy + (xI_x + yI_z + I)dz \wedge dx + (-xI_y - xI_z)dy \wedge dz \end{aligned}$$

Therefore, our integrating factor must satisfy the following partial differential equations,

$$(I.)\ xI_x - yI_y = 0, \qquad (II.)\ xI_x + yI_z + I = 0, \qquad (III.)\ -xI_y - xI_z = 0.$$

I'll leave this to the reader. I'm not sure if it has a solution. It seems possible that the differential consquences of this system are nonsensical. I just wanted to show how differential forms allow us to extend to higher dimensional problems. Notice, we could just as well have not solved a problem with 4 or 5 variables.

10.11.2 differential equations via forms

I follow Example 1.2.3. of *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems* by Thomas A. Ivey and J.M Landsberg. I suspect understanding this text makes you quite a bit more than a *beginner*. Consider,

$$u_x = A(x, y, u) \tag{10.15}$$

$$u_y = B(x, y, u) \tag{10.16}$$

This section is not finished.

Chapter 11

Electromagnetism in differential form

Warning: I will use Einstein's implicit summation convention throughout this section. I have made a point of abstaining from Einstein's convention in these notes up to this point. However, I just can't bear the summations in this section. They're just too ugly.

11.1 differential forms in Minkowski space

The logic here follows fairly close to the last section, however the wrinkle is that the metric here demands more attention. We must take care to raise the indices on the forms when we Hodge dual them. First we list the basis differential forms, we have to add time to the mix (again $c = 1$ so $x^0 = ct = t$ if you worried about it)

Name	Degree	Typical Element	"Basis" for $\Lambda^p(\mathbb{R}^4)$
function	$p = 0$	f	1
one-form	$p = 1$	$\alpha = \alpha_\mu dx^\mu$	dt, dx, dy, dz
two-form	$p = 2$	$\beta = \frac{1}{2}\beta_{\mu\nu} dx^\mu \wedge dx^\nu$	$dy \wedge dz,\ dz \wedge dx,\ dx \wedge dy$ $dt \wedge dx,\ dt \wedge dy,\ dt \wedge dz$
three-form	$p = 3$	$\gamma = \frac{1}{3!}\gamma_{\mu\nu\alpha} dx^\mu \wedge dx^\nu \wedge dx^\alpha$	$dx \wedge dy \wedge dz, dt \wedge dy \wedge dz$ $dt \wedge dx \wedge dz, dt \wedge dx \wedge dy$
four-form	$p = 4$	$g dt \wedge dx \wedge dy \wedge dz$	$dt \wedge dx \wedge dy \wedge dz$

Greek indices are defined to range over $0, 1, 2, 3$. Here the top form is degree four since in four dimensions we can have four differentials without a repeat. Wedge products work the same as they have before, just now we have dt to play with. Hodge duality may offer some surprises though.

Definition 11.1.1. *The antisymmetric symbol in* **flat** $\mathbb{R}^4$ *is denoted* $\epsilon_{\mu\nu\alpha\beta}$ *and it is defined by the value*

$$\epsilon_{0123} = 1$$

plus the demand that it be completely antisymmetric.

We must not assume that this symbol is invariant under a cyclic exhange of indices. Consider,

$$\begin{aligned} \epsilon_{0123} &= -\epsilon_{1023} \qquad \text{flipped (01)} \\ &= +\epsilon_{1203} \qquad \text{flipped (02)} \\ &= -\epsilon_{1230} \qquad \text{flipped (03).} \end{aligned} \tag{11.1}$$

Example 11.1.2. *We now compute the Hodge dual of $\gamma = dx$ with respect to the Minkowski metric $\eta_{\mu\nu}$. First notice that dx has components $\gamma_\mu = \delta^1_\mu$ as is readily verified by the equation $dx = \delta^1_\mu dx^\mu$. We raise the index using η, as follows*

$$\gamma^\mu = \eta^{\mu\nu}\gamma_\nu = \eta^{\mu\nu}\delta^1_\nu = \eta^{1\mu} = \delta^{1\mu}.$$

Beginning with the definition of the Hodge dual we calculate

$$\begin{aligned} {}^*(dx) &= \tfrac{1}{(4-1)!}\delta^{1\mu}\epsilon_{\mu\nu\alpha\beta}dx^\nu \wedge dx^\alpha \wedge dx^\beta \\ &= (1/6)\epsilon_{1\nu\alpha\beta}dx^\nu \wedge dx^\alpha \wedge dx^\beta \\ \\ &= (1/6)[\epsilon_{1023}dt \wedge dy \wedge dz + \epsilon_{1230}dy \wedge dz \wedge dt + \epsilon_{1302}dz \wedge dt \wedge dy \\ &\qquad +\epsilon_{1320}dz \wedge dy \wedge dt + \epsilon_{1203}dy \wedge dt \wedge dz + \epsilon_{1032}dt \wedge dz \wedge dy] \\ \\ &= (1/6)[-dt \wedge dy \wedge dz - dy \wedge dz \wedge dt - dz \wedge dt \wedge dy \\ &\qquad +dz \wedge dy \wedge dt + dy \wedge dt \wedge dz + dt \wedge dz \wedge dy] \\ \\ &= -dy \wedge dz \wedge dt. \end{aligned} \tag{11.2}$$

The difference between the three and four dimensional Hodge dual arises from two sources, for one we are using the Minkowski metric so indices up or down makes a difference, and second the antisymmetric symbol has more possibilities than before because the Greek indices take four values.

Example 11.1.3. *We find the Hodge dual of $\gamma = dt$ with respect to the Minkowski metric $\eta_{\mu\nu}$. Notice that dt has components $\gamma_\mu = \delta^0_\mu$ as is easily seen using the equation $dt = \delta^0_\mu dx^\mu$. Raising the index using η as usual, we have*

$$\gamma^\mu = \eta^{\mu\nu}\gamma_\nu = \eta^{\mu\nu}\delta^0_\nu = -\eta^{0\mu} = -\delta^{0\mu}$$

where the minus sign is due to the Minkowski metric. Starting with the definition of Hodge duality we calculate

$$\begin{aligned} {}^*(dt) &= -(1/6)\delta^{0\mu}\epsilon_{\mu\nu\alpha\beta}dx^\nu \wedge dx^\alpha \wedge dx^\beta \\ &= -(1/6)\epsilon_{0\nu\alpha\beta}dx^\nu \wedge dx^\alpha \wedge dx^\beta \\ \\ &= -(1/6)\epsilon_{0ijk}dx^i \wedge dx^j \wedge dx^k \\ &= -(1/6)\epsilon_{ijk}dx^i \wedge dx^j \wedge dx^k \\ &= -dx \wedge dy \wedge dz. \end{aligned} \tag{11.3}$$

for the case here we are able to use some of our old three dimensional ideas. The Hodge dual of dt cannot have a dt in it which means our answer will only have dx, dy, dz in it and that is why we were able to shortcut some of the work, (compared to the previous example).

Example 11.1.4. *Finally, we find the Hodge dual of $\gamma = dt \wedge dx$ with respect to the Minkowski metric $\eta_{\mu\nu}$. Recall that ${}^*(dt \wedge dx) = \frac{1}{(4-2)!}\epsilon_{01\mu\nu}\gamma^{01}(dx^\mu \wedge dx^\nu)$ and that $\gamma^{01} = \eta^{0\lambda}\eta^{1\rho}\gamma_{\lambda\rho} = (-1)(1)\gamma_{01} = -1$. Thus*

$$\begin{aligned} {}^*(dt \wedge dx) &= -(1/2)\epsilon_{01\mu\nu}dx^\mu \wedge dx^\nu \\ &= -(1/2)[\epsilon_{0123}dy \wedge dz + \epsilon_{0132}dz \wedge dy] \\ &= -dy \wedge dz. \end{aligned} \tag{11.4}$$

Notice also that since $dt \wedge dx = -dx \wedge dt$ we find $*(dx \wedge dt) = dy \wedge dz$

The other Hodge duals of the basic two-forms follow from similar calculations. Here is a table of all the basic Hodge dualities in Minkowski space, In the table the terms are grouped as they are to

${}^*1 = dt \wedge dx \wedge dy \wedge dz$	${}^*(dt \wedge dx \wedge dy \wedge dz) = -1$
${}^*(dx \wedge dy \wedge dz) = -dt$	${}^*dt = -dx \wedge dy \wedge dz$
${}^*(dt \wedge dy \wedge dz) = -dx$	${}^*dx = -dy \wedge dz \wedge dt$
${}^*(dt \wedge dz \wedge dx) = -dy$	${}^*dy = -dz \wedge dx \wedge dt$
${}^*(dt \wedge dx \wedge dy) = -dz$	${}^*dz = -dx \wedge dy \wedge dt$
${}^*(dz \wedge dt) = dx \wedge dy$	${}^*(dx \wedge dy) = -dz \wedge dt$
${}^*(dx \wedge dt) = dy \wedge dz$	${}^*(dy \wedge dz) = -dx \wedge dt$
${}^*(dy \wedge dt) = dz \wedge dx$	${}^*(dz \wedge dx) = -dy \wedge dt$

emphasize the isomorphisms between the one-dimensional $\Lambda^0(M)$ and $\Lambda^4(M)$, the four-dimensional $\Lambda^1(M)$ and $\Lambda^3(M)$, the six-dimensional $\Lambda^2(M)$ and itself. Notice that the dimension of $\Lambda(M)$ is 16 which just happens to be 2^4.

Now that we've established how the Hodge dual works on the differentials we can easily take the Hodge dual of arbitrary differential forms on Minkowski space. We begin with the example of the 4-current $\mathcal{J}$

Example 11.1.5. Four Current: *often in relativistic physics we would even just call the four current simply the current, however it actually includes the charge density ρ and current density $\vec{J}$. Consequently, we define,*

$$(\mathcal{J}^\mu) \equiv (\rho, \vec{J}),$$

moreover if we lower the index we obtain,

$$(\mathcal{J}_\mu) = (-\rho, \vec{J})$$

which are the components of the current one-form,

$$\mathcal{J} = \mathcal{J}_\mu dx^\mu = -\rho dt + J_x dx + J_y dy + J_z dz$$

This equation could be taken as the definition of the current as it is equivalent to the vector definition. Now we can rewrite the last equation using the vectors $\mapsto$ forms mapping as,

$$\mathcal{J} = -\rho dt + \omega_{\vec{J}}.$$

Consider the Hodge dual of $\mathcal{J}$,

$$\begin{aligned} {}^*\mathcal{J} &= {}^*(-\rho dt + J_x dx + J_y dy + J_z dz) \\ &= -\rho {}^*dt + J_x {}^*dx + J_y {}^*dy + J_z {}^*dz \\ &= \rho dx \wedge dy \wedge dz - J_x dy \wedge dz \wedge dt - J_y dz \wedge dx \wedge dt - J_z dx \wedge dy \wedge dt \\ &= \rho dx \wedge dy \wedge dz - \Phi_{\vec{J}} \wedge dt. \end{aligned} \tag{11.5}$$

we will find it useful to appeal to this calculation in a later section.

Example 11.1.6. Four Potential: *often in relativistic physics we would call the four potential simply the potential, however it actually includes the scalar potential V and the vector potential $\vec{A}$ (discussed at the end of chapter 3). To be precise we define,*

$$(A^\mu) \equiv (V, \vec{A})$$

we can lower the index to obtain,

$$(A_\mu) = (-V, \vec{A})$$

which are the components of the current one-form,

$$A = A_\mu dx^\mu = -V dt + A_x dx + A_y dy + A_z dz$$

Sometimes this equation is taken as the definition of the four potential. We can rewrite the four potential vector field using the vectors $\mapsto$ forms mapping as,

$$A = -V dt + \omega_{\vec{A}}.$$

The Hodge dual of A is

$${}^*A = V dx \wedge dy \wedge dz - \Phi_{\vec{A}} \wedge dt. \tag{11.6}$$

Several steps were omitted because they are identical to the calculation of the dual of the 4-current above.

Definition 11.1.7. *Faraday tensor.*

Given an electric field $\vec{E} = (E_1, E_2, E_3)$ and a magnetic field $\vec{B} = (B_1, B_2, B_3)$ we define a 2-form F by

$$F = \omega_E \wedge dt + \Phi_B.$$

This 2-form is often called the **electromagnetic field tensor or the Faraday tensor.** If we write it in tensor components as $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ and then consider its matrix $(F_{\mu\nu})$ of components then it is easy to see that

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \tag{11.7}$$

Convention: Notice that when we write the matrix version of the tensor components we take the first index to be the row index and the second index to be the column index, that means $F_{01} = -E_1$ whereas $F_{10} = E_1$.

Example 11.1.8. *In this example we demonstrate various conventions which show how one can transform the field tensor to other type tensors. Define a type* $(1,1)$ *tensor by raising the first index by the inverse metric* $\eta^{\alpha\mu}$ *as follows,*

$$F^{\alpha}{}_{\nu} = \eta^{\alpha\mu} F_{\mu\nu}$$

The zeroth row,

$$(F^{0}{}_{\nu}) = (\eta^{0\mu} F_{\mu\nu}) = (0, E_1, E_2, E_3)$$

Then row one is unchanged since $\eta^{1\mu} = \delta^{1\mu}$*,*

$$(F^{1}{}_{\nu}) = (\eta^{1\mu} F_{\mu\nu}) = (E_1, 0, B_3, -B_2)$$

and likewise for rows two and three. In summary the (1,1) tensor $F' = F^{\alpha}_{\nu}(\frac{\partial}{\partial x^{\alpha}} \otimes dx^{\nu})$ *has the components below*

$$(F^{\alpha}{}_{\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{11.8}$$

At this point we raise the other index to create a $(2,0)$ *tensor,*

$$\boxed{F^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu} F_{\mu\nu}} \tag{11.9}$$

and we see that it takes one copy of the inverse metric to raise each index and $F^{\alpha\beta} = \eta^{\beta\nu} F^{\alpha}{}_{\nu}$ *so we can pick up where we left off in the* $(1,1)$ *case. We could proceed case by case like we did with the* $(1,1)$ *case but it is better to use matrix multiplication. Notice that* $\eta^{\beta\nu} F^{\alpha}{}_{\nu} = F^{\alpha}{}_{\nu}\eta^{\nu\beta}$ *is just the* (α, β) *component of the following matrix product,*

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{11.10}$$

So we find a $(2,0)$ *tensor* $F'' = F^{\alpha\beta}(\frac{\partial}{\partial x^{\alpha}} \otimes \frac{\partial}{\partial x^{\beta}})$*. Other books might even use the same symbol* F *for* F' *and* F''*, it is in fact typically clear from the context which version of* F *one is thinking about. Pragmatically physicists just write the components so its not even an issue for them.*

Example 11.1.9. Field tensor's dual: *We now calculate the Hodge dual of the field tensor,*

$$\begin{aligned} {}^*F &= {}^*(\omega_E \wedge dt + \Phi_B) \\ &= E_x{}^*(dx \wedge dt) + E_y{}^*(dy \wedge dt) + E_z{}^*(dz \wedge dt) \\ &\quad + B_x{}^*(dy \wedge dz) + B_y{}^*(dz \wedge dx) + B_z{}^*(dx \wedge dy) \\ &= E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy \\ &\quad - B_x dx \wedge dt - B_y dy \wedge dt - B_z dz \wedge dt \\ &= \Phi_E - \omega_B \wedge dt. \end{aligned}$$

we can also write the components of ${}^*F$ in matrix form:

$$({}^*F_{\mu\nu}) = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0. \end{pmatrix} \tag{11.11}$$

Notice that the net-effect of Hodge duality on the field tensor was to make the exchanges $\vec{E} \mapsto -\vec{B}$ and $\vec{B} \mapsto \vec{E}$.

11.2 exterior derivatives of charge forms, field tensors, and their duals

In the last chapter we found that the single operation of the exterior differentiation reproduces the gradiant, curl and divergence of vector calculus provided we make the appropriate identifications under the "work" and "flux" form mappings. We now move on to some four dimensional examples.

Example 11.2.1. Charge conservation: *Consider the 4-current we introduced in example* 11.1.5*. Take the exterior derivative of the dual of the current to get,*

$$\begin{aligned} d({}^*\mathcal{J}) &= d(\rho dx \wedge dy \wedge dz - \Phi_{\vec{J}} \wedge dt) \\ &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - d[(J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy) \wedge dt] \\ &= d\rho \wedge dx \wedge dy \wedge dz \\ &\quad -\partial_x J_x dx \wedge dy \wedge dz \wedge dt - \partial_y J_y dy \wedge dz \wedge dx \wedge dt - \partial_z J_z dz \wedge dx \wedge dy \wedge dt \\ &= (\partial_t \rho + \nabla \cdot \vec{J}) dt \wedge dx \wedge dy \wedge dz. \end{aligned}$$

We work through the same calculation using index techniques,

$$\begin{aligned} d({}^*\mathcal{J}) &= d(\rho dx \wedge dy \wedge dz - \Phi_{\vec{J}} \wedge dt) \\ &= d(\rho) \wedge dx \wedge dy \wedge dz - d[\tfrac{1}{2}\epsilon_{ijk} J_i dx^j \wedge dx^k \wedge dt) \\ &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - \tfrac{1}{2}\epsilon_{ijk} \partial_\mu J_i dx^\mu \wedge dx^j \wedge dx^k \wedge dt \\ &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - \tfrac{1}{2}\epsilon_{ijk} \partial_m J_i dx^m \wedge dx^j \wedge dx^k \wedge dt \\ &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - \tfrac{1}{2}\epsilon_{ijk}\epsilon_{mjk} \partial_m J_i dx \wedge dy \wedge dz \wedge dt \\ &= (\partial_t \rho) dt \wedge dx \wedge dy \wedge dz - \tfrac{1}{2} 2\delta_{im} \partial_m J_i dx \wedge dy \wedge dz \wedge dt \\ &= (\partial_t \rho + \nabla \cdot \vec{J}) dt \wedge dx \wedge dy \wedge dz. \end{aligned}$$

Observe that we can now phrase charge conservation by the following equation

$$d({}^*\mathcal{J}) = 0 \qquad \Longleftrightarrow \qquad \partial_t \rho + \nabla \cdot \vec{J} = 0.$$

In the classical scheme of things this was a derived consequence of the equations of electromagnetism, however it is possible to build the theory regarding this equation as fundamental. Rindler describes that formal approach in a late chapter of "Introduction to Special Relativity".

Proposition 11.2.2.

If $(A_\mu) = (-V, \vec{A})$ is the vector potential (which gives the magnetic field) and $A = -V dt + \omega_{\vec{A}}$, then $dA = \omega_{\vec{E}} + \Phi_{\vec{B}} = F$ where F is the electromagnetic field tensor. Moreover, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Proof: The proof uses the definitions $\vec{E} = -\nabla V - \partial_t A$ and $\vec{B} = \nabla \times \vec{A}$ and some vector identities:

$$\begin{aligned}
dA &= d(-V dt + \omega_{\vec{A}}) \\
&= -dV \wedge dt + d(\omega_{\vec{A}}) \\
&= -dV \wedge dt + (\partial_t A_i) dt \wedge dx^i + (\partial_j A_i) dx^j \wedge dx^i \\
&= \omega_{(-\nabla V)} \wedge dt - \omega_{\partial_t \vec{A}} \wedge dt + \Phi_{\nabla \times \vec{A}} \\
&= (\omega_{(-\nabla V)} - \omega_{\partial_t \vec{A}}) \wedge dt + \Phi_{\nabla \times \vec{A}} \\
&= \omega_{(-\nabla V - \partial_t \vec{A})} \wedge dt + \Phi_{\nabla \times \vec{A}} \\
&= \omega_{\vec{E}} \wedge dt + \Phi_{\vec{B}} \\
&= F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu.
\end{aligned}$$

Moreover we also have:

$$\begin{aligned}
dA &= d(A_\nu) \wedge dx^\nu \\
&= \partial_\mu A_\nu dx^\mu \wedge dx^\nu \\
&= \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu + \tfrac{1}{2}(\partial_\mu A_\nu + \partial_\nu A_\mu) dx^\mu \wedge dx^\nu \\
&= \tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu.
\end{aligned}$$

Comparing the two identities we see that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the proposition follows.

Example 11.2.3. Exterior derivative of the field tensor: *We have just seen that the field tensor is the exterior derivative of the potential one-form. We now compute the exterior derivative of the field tensor expecting to find Maxwell's equations since the derivative of the fields are governed by Maxwell's equations,*

$$\begin{aligned}
dF &= d(E_i dx^i \wedge dt) + d(\Phi_{\vec{B}}) \\
&= \partial_m E_i (dx^m \wedge dx^i \wedge dt) + (\nabla \cdot \vec{B}) dx \wedge dy \wedge dz + \tfrac{1}{2} \epsilon_{ijk} (\partial_t B_i)(dt \wedge dx^j \wedge dx^k).
\end{aligned} \tag{11.12}$$

W pause here to explain our logic. In the above we dropped the $\partial_t E_i dt \wedge \cdots$ term because it was wedged with another dt in the term so it vanished. Also we broke up the exterior derivative on the flux form of $\vec{B}$ into the space and then time derivative terms and used our work in example 10.6.7.

Continuing the calculation,

$$\begin{aligned} dF &= [\partial_j E_k + \tfrac{1}{2}\epsilon_{ijk}(\partial_t B_i)]dx^j \wedge dx^k \wedge dt + (\nabla \cdot \vec{B})dx \wedge dy \wedge dz \\ &= [\partial_x E_y - \partial_y E_x + \epsilon_{i12}(\partial_t B_i)]dx \wedge dy \wedge dt \\ &\quad +[\partial_z E_x - \partial_x E_z + \epsilon_{i31}(\partial_t B_i)]dz \wedge dx \wedge dt \\ &\quad +[\partial_y E_z - \partial_z E_y + \epsilon_{i23}(\partial_t B_i)]dy \wedge dz \wedge dt \\ &\quad +(\nabla \cdot \vec{B})dx \wedge dy \wedge dz \\ &= (\nabla \times \vec{E} + \partial_t \vec{B})_i \Phi_{e_i} \wedge dt + (\nabla \cdot \vec{B})dx \wedge dy \wedge dz \\ &= \Phi_{\nabla \times \vec{E} + \partial_t \vec{B}} \wedge dt + (\nabla \cdot \vec{B})dx \wedge dy \wedge dz \end{aligned} \tag{11.13}$$

where we used the fact that Φ is an isomorphism of vector spaces (at a point) and $\Phi_{e_1} = dy \wedge dz$, $\Phi_{e_2} = dz \wedge dx$, and $\Phi_{e_3} = dx \wedge dy$. Behold, we can state two of Maxwell's equations as

$$\boxed{dF = 0 \quad \Longleftrightarrow \quad \nabla \times \vec{E} + \partial_t \vec{B} = 0, \quad \nabla \cdot \vec{B} = 0} \tag{11.14}$$

Example 11.2.4. We now compute the exterior derivative of the dual to the field tensor:

$$\begin{aligned} d^*F &= d(-B_i dx^i \wedge dt) + d(\Phi_{\vec{E}}) \\ &= \Phi_{-\nabla \times \vec{B} + \partial_t \vec{E}} \wedge dt + (\nabla \cdot \vec{E})dx \wedge dy \wedge dz \end{aligned} \tag{11.15}$$

This follows directly from the last example by replacing $\vec{E} \mapsto -\vec{B}$ and $\vec{B} \mapsto \vec{E}$. We obtain the two inhomogeneous Maxwell's equations by setting d^*F equal to the Hodge dual of the 4-current,

$$\boxed{d^*F = \mu_o {}^*\mathcal{J} \quad \Longleftrightarrow \quad -\nabla \times \vec{B} + \partial_t \vec{E} = -\mu_o \vec{J}, \quad \nabla \cdot \vec{E} = \rho} \tag{11.16}$$

Here we have used example 11.1.5 *to find the RHS of the Maxwell equations.*

We now know how to write Maxwell's equations via differential forms. The stage is set to prove that Maxwell's equations are Lorentz covariant, that is they have the same form in all inertial frames.

11.3 coderivatives and comparing to Griffith's relativitic E & M

Optional section, for those who wish to compare our tensorial E & M with that of Griffith's, you may skip ahead to the next section if not interested

I should mention that this is not the only way to phrase Maxwell's equations in terms of differential forms. If you try to see how what we have done here compares with the equations presented in Griffith's text it is not immediately obvious. He works with $F^{\mu\nu}$ and $G^{\mu\nu}$ and J^μ none of which are the components of differential forms. Nevertheless he recovers Maxwell's equations as $\partial_\mu F^{\mu\nu} = J^\nu$ and $\partial_\mu G^{\mu\nu} = 0$. If we compare the components of ${}^*F$ with equation 12.119 (the matrix form of $G^{\mu\nu}$) in Griffith's text,

$$(G^{\mu\nu}(c=1)) = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & -E_3 & 0 & -E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix} = -({}^*F^{\mu\nu}). \tag{11.17}$$

we find that we obtain the negative of Griffith's "dual tensor" (recall that raising the indices has the net-effect of multiplying the zeroth row and column by -1). The equation $\partial_\mu F^{\mu\nu} = J^\nu$ does not follow directly from an exterior derivative, rather it is the component form of a "coderivative". The coderivative is defined $\delta = {}^*d^*$, it takes a p-form to an $(n-p)$-form then d makes it a $(n-p+1)$-form then finally the second Hodge dual takes it to an $(n-(n-p+1))$-form. That is δ takes a p-form to a $p-1$-form. We stated Maxwell's equations as

$$dF = 0 \qquad\qquad d^*F = {}^*\mathcal{J}$$

Now we can take the Hodge dual of the inhomogeneous equation to obtain,

$${}^*d^*F = \delta F = {}^{**}\mathcal{J} = \pm\mathcal{J}$$

where I leave the sign for you to figure out. Then the other equation

$$\partial_\mu G^{\mu\nu} = 0$$

can be understood as the component form of $\delta^* F = 0$ but this is really $dF = 0$ in disguise,

$$0 = \delta^* F = {}^*d^{**}F = \pm {}^*dF \iff dF = 0$$

so even though it looks like Griffith's is using the dual field tensor for the homogeneous Maxwell's equations and the field tensor for the inhomogeneous Maxwell's equations it is in fact not the case. The key point is that there are **coderivatives** implicit within Griffith's equations, so you have to read between the lines a little to see how it matched up with what we've done here. I have not entirely proved it here, to be complete we should look at the component form of $\delta F = \mathcal{J}$ and explicitly show that this gives us $\partial_\mu F^{\mu\nu} = J^\nu$, I don't think it is terribly difficult but I'll leave it to the reader.

Comparing with Griffith's is fairly straightforward because he uses the same metric as we have. Other texts use the mostly negative metric, its just a convention. If you try to compare to such a book you'll find that our equations are almost the same up to a sign. One good careful book is Reinhold A. Bertlmann's *Anomalies in Quantum Field Theory* you will find much of what we have done here done there with respect to the other metric. Another good book which shares our conventions is Sean M. Carroll's *An Introduction to General Relativity: Spacetime and Geometry*, that text has a no-nonsense introduction to tensors forms and much more over a curved space (in contrast to our approach which has been over a vector space which is flat). By now there are probably thousands of texts on tensors; these are a few we have found useful here.

11.4 Maxwell's equations are relativistically covariant

Let us begin with the definition of the field tensor once more. We define the components of the field tensor in terms of the 4-potentials as we take the view-point those are the basic objects (not the fields). If

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu,$$

then the field tensor $F = F_{\mu\nu} dx^\mu \otimes dx^\nu$ is a tensor, or is it ? We should check that the components transform as they ought according to the discussion in section **??**. Let $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ then we observe,

$$\begin{aligned} &(1.)\ \bar{A}_\mu = (\Lambda^{-1})^\alpha_\mu A_\alpha \\ &(2.)\ \frac{\partial}{\partial \bar{x}^\nu} = \frac{\partial x^\beta}{\partial \bar{x}^\nu}\frac{\partial}{\partial x^\beta} = (\Lambda^{-1})^\beta_\nu \frac{\partial}{\partial x^\beta} \end{aligned} \tag{11.18}$$

where (2.) is simply the chain rule of multivariate calculus and (1.) is not at all obvious. We will assume that (1.) holds, that is we assume that the 4-potential transforms in the appropriate way for a one-form. In principle one could prove that from more base assumptions. After all electromagnetism is the study of the interaction of charged objects, we should hope that the potentials are derivable from the source charge distribution. Indeed, there exist formulas to calculate the potentials for moving distributions of charge. We could take those as definitions for the potentials, then it would be possible to actually calculate if (1.) is true. We'd just change coordinates via a Lorentz transformation and verify (1.). For the sake of brevity we will just assume that (1.) holds. We should mention that alternatively one can show the electric and magnetic fields transform as to make $F_{\mu\nu}$ a tensor. Those derivations assume that charge is an invariant quantity and just apply Lorentz transformations to special physical situations to deduce the field transformation rules. See Griffith's chapter on special relativity or look in Resnick for example.

Let us find how the field tensor transforms assuming that (1.) and (2.) hold, again we consider $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$,

$$\begin{aligned} \bar{F}_{\mu\nu} &= \bar{\partial}_\mu \bar{A}_\nu - \bar{\partial}_\nu \bar{A}_\mu \\ &= (\Lambda^{-1})^\alpha_\mu \partial_\alpha((\Lambda^{-1})^\beta_\nu A_\beta) - (\Lambda^{-1})^\beta_\nu \partial_\beta((\Lambda^{-1})^\alpha_\mu A_\alpha) \\ &= (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &= (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu F_{\alpha\beta}. \end{aligned} \tag{11.19}$$

therefore the field tensor really is a tensor over Minkowski space.

Proposition 11.4.1.

The dual to the field tensor is a tensor over Minkowski space. For a given Lorentz transformation $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ it follows that

$$^*\bar{F}_{\mu\nu} = (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu {}^*F_{\alpha\beta}$$

Proof: homework (just kidding in 2010), it follows quickly from the definition and the fact we already know that the field tensor is a tensor.

Proposition 11.4.2.

The four-current is a four-vector. That is under the Lorentz transformation $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ we can show,

$$\bar{\mathcal{J}}_\mu = (\Lambda^{-1})^\alpha_\mu \mathcal{J}_\alpha$$

Proof: follows from arguments involving the invariance of charge, time dilation and length contraction. See Griffith's for details, sorry we have no time.

Corollary 11.4.3.

The dual to the four current transforms as a 3-form. That is under the Lorentz transformation $\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ we can show,

$$*\bar{\mathcal{J}}_{\mu\nu\sigma} = (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu (\Lambda^{-1})^\gamma_\sigma \mathcal{J}_{\alpha\beta\gamma}$$

Up to now the content of this section is simply an admission that we have been a little careless in defining things upto this point. The main point is that if we say that something is a tensor then we need to make sure that is in fact the case. With the knowledge that our tensors are indeed tensors the proof of the covariance of Maxwell's equations is trivial.

$$dF = 0 \qquad\qquad d^*F = {}^*\mathcal{J}$$

are coordinate invariant expressions which we have already proved give Maxwell's equations in one frame of reference, thus they must give Maxwell's equations in all frames of reference.
The essential point is simply that

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}\bar{F}_{\mu\nu}d\bar{x}^\mu \wedge d\bar{x}^\nu$$

Again, we have no hope for the equation above to be true unless we know that
$\bar{F}_{\mu\nu} = (\Lambda^{-1})^\alpha_\mu (\Lambda^{-1})^\beta_\nu F_{\alpha\beta}$. That transformation follows from the fact that the four-potential is a four-vector. It should be mentioned that others prefer to "prove" the field tensor is a tensor by studying how the electric and magnetic fields transform under a Lorentz transformation. We in contrast have derived the field transforms based ultimately on the seemingly innocuous assumption that the four-potential transforms according to $\bar{A}_\mu = (\Lambda^{-1})^\alpha_\mu A_\alpha$. OK enough about that.

So the fact that Maxwell's equations have the same form in all **relativistically inertial** frames of reference simply stems from the fact that we found Maxwell's equation were given by an arbitrary frame, and the field tensor looks the same in the new barred frame so we can again go through all the same arguments with barred coordinates. Thus we find that Maxwell's equations are the same in all relativistic frames of reference, that is if they hold in one inertial frame then they will hold in any other frame which is related by a Lorentz transformation.

11.5 Electrostatics in Five dimensions

We will endeavor to determine the electric field of a point charge in 5 dimensions where we are thinking of adding an extra spatial dimension. Lets call the fourth spatial dimension the w-direction so that a typical point in space time will be (t, x, y, z, w). First we note that the electromagnetic field tensor can still be derived from a one-form potential,

$$A = -\rho dt + A_1 dx + A_2 dy + A_3 dz + A_4 dw$$

we will find it convenient to make our convention for this section that $\mu, \nu, ... = 0, 1, 2, 3, 4$ whereas $m, n, ... = 1, 2, 3, 4$ so we can rewrite the potential one-form as,

$$A = -\rho dt + A_m dx^m$$

This is derived from the vector potential $A^\mu = (\rho, A^m)$ under the assumption we use the natural generalization of the Minkowski metric, namely the 5 by 5 matrix,

$$(\eta_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = (\eta^{\mu\nu}) \tag{11.20}$$

we could study the linear isometries of this metric, they would form the group $O(1, 4)$. Now we form the field tensor by taking the exterior derivative of the one-form potential,

$$F = dA = \frac{1}{2}(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) dx^\mu \wedge dx^\nu$$

now we would like to find the electric and magnetic "fields" in 4 dimensions. Perhaps we should say 4+1 dimensions, just understand that I take there to be 4 spatial directions throughout this discussion if in doubt. Note that we are faced with a dilemma of interpretation. There are 10 independent components of a 5 by 5 antisymmetric tensor, naively we wold expect that the electric and magnetic fields each would have 4 components, but that is not possible, we'd be missing two components. The solution is this, the time components of the field tensor are understood to correspond to the electric part of the fields whereas the remaining 6 components are said to be magnetic. This aligns with what we found in 3 dimensions, its just in 3 dimensions we had the fortunate quirk that the number of linearly independent one and two forms were equal at any point. This definition means that the magnetic field will in general not be a vector field but rather a "flux" encoded by a 2-form.

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z & -E_w \\ E_x & 0 & B_z & -B_y & H_1 \\ E_y & -B_z & 0 & B_x & H_2 \\ E_z & B_y & -B_x & 0 & H_3 \\ E_w & -H_1 & -H_2 & -H_3 & 0 \end{pmatrix} \tag{11.21}$$

Now we can write this compactly via the following equation,

$$F = E \wedge dt + B$$

I admit there are subtle points about how exactly we should interpret the magnetic field, however I'm going to leave that to your imagination and instead focus on the electric sector. What is the generalized Maxwell's equation that E must satisfy?

$$d^*F = \mu_o{}^* \mathcal{J} \implies d^*(E \wedge dt + B) = \mu_o{}^* \mathcal{J}$$

where $\mathcal{J} = -\rho dt + J_m dx^m$ so the 5 dimensional Hodge dual will give us a $5-1=4$ form, in particular we will be interested in just the term stemming from the dual of dt,

$$^*(-\rho dt) = \rho dx \wedge dy \wedge dz \wedge dw$$

the corresponding term in d^*F is $d^*(E \wedge dt)$ thus, using $\mu_o = \frac{1}{\epsilon_o}$,

$$d^*(E \wedge dt) = \frac{1}{\epsilon_o} \rho dx \wedge dy \wedge dz \wedge dw \tag{11.22}$$

is the 4-dimensional Gauss's equation. Now consider the case we have an isolated point charge which has somehow always existed at the origin. Moreover consider a 3-sphere that surrounds the charge. We wish to determine the generalized Coulomb field due to the point charge. First we note that the solid 3-sphere is a 4-dimensional object, it the set of all $(x, y, z, w) \in \mathbb{R}^4$ such that

$$x^2 + y^2 + z^2 + w^2 \leq r^2$$

We may parametrize a three-sphere of radius r via generalized spherical coordinates,

$$\begin{array}{ll} x & = rsin(\theta)cos(\phi)sin(\psi) \\ y & = rsin(\theta)sin(\phi)sin(\psi) \\ z & = rcos(\theta)sin(\psi) \\ w & = rcos(\psi) \end{array} \tag{11.23}$$

Now it can be shown that the volume and surface area of the radius r three-sphere are as follows,

$$vol(S^3) = \frac{\pi^2}{2} r^4 \qquad\qquad area(S^3) = 2\pi^2 r^3$$

We may write the charge density of a smeared out point charge q as,

$$\rho = \begin{cases} 2q/\pi^2 a^4, & 0 \leq r \leq a \\ 0, & r > a \end{cases}. \tag{11.24}$$

Notice that if we integrate ρ over any four-dimensional region which contains the solid three sphere of radius a will give the enclosed charge to be q. Then integrate over the Gaussian 3-sphere S^3 with radius r call it M,

$$\int_M d^*(E \wedge dt) = \frac{1}{\epsilon_o} \int_M \rho dx \wedge dy \wedge dz \wedge dw$$

now use the Generalized Stokes Theorem to deduce,

$$\int_{\partial M} {}^*(E \wedge dt) = \frac{q}{\epsilon_o}$$

but by the "spherical" symmetry of the problem we find that E must be independent of the direction it points, this means that it can only have a radial component. Thus we may calculate the integral

with respect to generalized spherical coordinates and we will find that it is the product of $E_r \equiv E$ and the surface volume of the four dimensional solid three sphere. That is,

$$\int_{\partial M} {}^*(E \wedge dt) = 2\pi^2 r^3 E = \frac{q}{\epsilon_o}$$

Thus,

$$\boxed{E = \frac{q}{2\pi^2 \epsilon_o r^3}}$$

the Coulomb field is weaker if it were to propogate in 4 spatial dimensions. Qualitatively what has happened is that the have taken the same net flux and spread it out over an additional dimension, this means it thins out quicker. A very similar idea is used in some *brane world* scenarios. String theorists posit that the gravitational field spreads out in more than four dimensions while in contrast the standard model fields of electromagnetism, and the strong and weak forces are confined to a four-dimensional brane. That sort of model attempts an explaination as to why gravity is so weak in comparison to the other forces. Also it gives large scale corrections to gravity that some hope will match observations which at present don't seem to fit the standard gravitational models.

This example is but a taste of the theoretical discussion that differential forms allow. As a final comment I remind the reader that we have done things for flat space for the most part in this course, when considering a curved space there are a few extra considerations that must enter. Coordinate vector fields e_i must be thought of as derivations $\partial/\partial x^\mu$ for one. Also the metric is not a constant tensor like δ_{ij} or $\eta_{\mu\nu}$ rather is depends on position, this means Hodge duality aquires a coordinate dependence as well. Doubtless I have forgotten something else in this brief warning. One more advanced treatment of many of our discussions is Dr. Fulp's Fiber Bundles 2001 notes which I have posted on my webpage. He uses the other metric but it is rather elegantly argued, all his arguments are coordinate independent. He also deals with the issue of the magnetic induction and the dielectric, issues which we have entirely ignored since we always have worked in free space.

References and Acknowledgements:

I have drawn from many sources to assemble the content of the last couple chapters, the references are listed approximately in the order of their use to the course, additionally we are indebted to Dr. Fulp for his course notes from many courses (ma 430, ma 518, ma 555, ma 756, ...). Also Manuela Kulaxizi helped me towards the correct (I hope) interpretation of 5-dimensional E&M in the last example.

Vector Calculus, Susan Jane Colley

Introduction to Special Relativity, Robert Resnick

Differential Forms and Connections, R.W.R. Darling

Differential geometry, gauge theories, and gravity, M. Göckerler & T. Schücker

Anomalies in Quantum Field Theory, Reinhold A. Bertlmann

"The Differential Geometry and Physical Basis for the Applications of Feynman Diagrams", S.L. Marateck, Notices of the AMS, Vol. 53, Number 7, pp. 744-752

Abstract Linear Algebra, Morton L. Curtis

Gravitation, Misner, Thorne and Wheeler

Introduction to Special Relativity, Wolfgang Rindler

Differential Forms A Complement to Vector Calculus, Steven H. Weintraub

Differential Forms with Applications to the Physical Sciences, Harley Flanders

Introduction to Electrodynamics, (3rd ed.) David J. Griffiths

The Geometry of Physics: An Introduction, Theodore Frankel

An Introduction to General Relativity: Spacetime and Geometry, Sean M. Carroll

Gauge Theory and Variational Principles, David Bleeker

Group Theory in Physics, Wu-Ki Tung

Chapter 12

Geometry by frames and forms

In this chapter we restrain ourselves to study three dimensional space and surfaces embedded within that context. That said, once this chapter is mastered the next natural step to consider is the study of geometric surfaces in which the ambient space is not used to frame the geometry of the surface.

The idea of this chapter is to take a guided tour through the second edition of Barret Oneil's *Elementary Differential Geometry*. Obviously we do not cover the entire text. Instead, I have carefully chosen a thread which allows us to see the central argument of the text for embedded surfaces in three dimensional space.

To begin we review once more the concept of a vector field and we adopt the notation used by Oneil[1]. Frames in $\mathbb{R}^3$ are studied and the cartesian frame U_1, U_2, U_3 is employed to define the covariant derivative. This will be almost familar to anyone who has studied vector calculus in non-cartesian coordinates. Next we develop the connection formulas for $\mathbb{R}^3$ which are based on matrices of differential forms. Once the structure equations are settled we turn to the theory of surfaces. We quickly introduce the shape operator as it is derived from the normal vector field of a regular surface. Gauss and mean curvatures are defined. Principle curvature and umbilic points are described. Isometry of surfaces is introduced and we see how isometric surfaces can be surprisingly different. Finally, we turn to the problem of formulating the theory of surfaces in terms of the connection form formalism of Cartan. After some fairly simple calculation we arrive at the result that the shape operator and much of the ambient theory can be replaced with a few simple formulas. From that point the proof of Gauss' celebrated theorem on intrinsic curvature is simple.

It is certainly the case that some of the definitions in this chapter have been previously given in greater generality. However, I intend this chapter to be self-contained as much as is possible. Furthermore, whenever a definition seems unjust, you can read more background in Oneil when time permits.

[1] in this way I hope reading his text is a natural extension of this study for those interested

12.1 basic terminology and vector fields

We begin by describing the directional deirvative in $\mathbb{R}^3$ as a derivation: if v is a vector at p then

$$v_p[f] = \lim_{h\to 0}\left[\frac{f(p+tv)-f(p)}{h}\right] \tag{12.1}$$

and for functions f smooth near p we have

$$v_p[f] = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p) = \left[v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z}\right]\Big|_p [f]$$

Clearly v_p acts on a function to reveal the rate of change of f at p as we allow p to move in the $\langle v_1, v_2, v_3\rangle$-direction. A vector field on $\mathbb{R}^3$ is rule which attaches a vector to each point in $\mathbb{R}^3$. For example, $p \mapsto \frac{\partial}{\partial x}\big|_p$, $p \mapsto \frac{\partial}{\partial y}\big|_p$ and $p \mapsto \frac{\partial}{\partial z}\big|_p$. Each of these define a global vector field on $\mathbb{R}^3$ and together the triple of vector fields is known as the **cartesian frame**. A frame is set of three vector fields which provides an **orthonormal basis**[2] for $T_p\mathbb{R}^3$ at each $p \in \mathbb{R}^3$. Given the importance of the cartesian frame, we introduce a simple notation for future use:

Definition 12.1.1. notation for the cartesian frame

$$\mathbf{U_1} = \frac{\partial}{\partial x}, \qquad \mathbf{U_2} = \frac{\partial}{\partial y}, \qquad \mathbf{U_3} = \frac{\partial}{\partial z}.$$

The cartesian frame provides a convenient background for what follows.

Example 12.1.2. *Let $v = \langle a, b, c\rangle$ then $V = a\mathbf{U_1} + b\mathbf{U_2} + c\mathbf{U_3}$ is a constant vector field on $\mathbb{R}^3$ where $V(p) = v$ for all $p \in \mathbb{R}^3$.*

Example 12.1.3. *If $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ then $\nabla f = \langle x/f, y/f, z/f\rangle$ hence $||(\nabla f)(p)|| = 1$ for all $p \in \mathbb{R}^3$. We can write this global gradient field as $\nabla f = \frac{1}{\sqrt{x^2+y^2+z^2}}[x\mathbf{U_1} + y\mathbf{U_2} + z\mathbf{U_3}]$. This vector field is part of the* **spherical frame field** *which we discuss in Example 12.1.10.*

Vectors as derivations measure rates of change of functions as we move a point. The next logical study is that of vector fields. How can we capture the change of a vector field W at a point p as we move in the v-direction? This cannot be a number since a vector field has three components this will again be a vector[3]

Definition 12.1.4. covariant differentiation of vector field

Let W be a vector field and v_p a vector at p then the **covariant derivative of W in the direction of v at p** is defined as follows: $(\nabla_v W)(p) = W'(p + tv)(0)$.

[2] there is a generalization of this to arbitrary frames in the study of the frame-bundle, we leave that for another course, all our frames are nice and orthonormal.

[3] just like the change in a path in $\mathbb{R}^3$ is described by a velocity vector not just a single slope

Let the derivative in the definition above be understood as differentiation of each component. You could see it as the directional derivative of each component in the v-direction. If $W = w_1\mathbf{U_1} + w_2\mathbf{U_2} + w_3\mathbf{U_3}$ then (by definition)

$$W'(p+tv)(0) = w_1'(p+tv)(0)\mathbf{U_1} + w_2'(p+tv)(0)\mathbf{U_2} + w_3'(p+tv)(0)\mathbf{U_3}$$

however, as we discussed in Equation 12.1 these directional derivatives are best understood in terms of derivations. We tend to use the following as a method to calculate covariant derivatives:

Proposition 12.1.5.

If W is a vector field defined near p and $W = \sum_{i=1}^{3} w_i\mathbf{U_i}$ the the covariant derivative of W in the direction of v at p is given by: $(\nabla_v W)(p) = \sum_{i=1}^{3} v[w_i]\mathbf{U_i}$.

Given two vector fields W, V we can study the change in W as it moves in the direction of V at each point. In particular, at $p \in \mathbb{R}^3$ study how the vector field W evolves in the $V(p)$ direction at p. This construction suggests the covariant derivative converts a pair of vector fields into an new vector field defined by the following:

$$(\nabla_V W)(p) = (\nabla_{V(p)} W)(p)$$

Let us pause to describe some useful properties of the covariant derivative. The proofs of these follow from properties of partial differentiation and Proposition 12.1.5

Proposition 12.1.6. *Properties of the covariant derivative*

Suppose V, W, Y and Z are vector fields on $\mathbb{R}^3$, f, g are functions and $a, b \in \mathbb{R}$,

(a.) $\nabla_{fV+gW}Y = f\nabla_V Y + g\nabla_W Y$

(b.) $\nabla_V(aY + bZ) = a\nabla_V Y + b\nabla_V Z$

(c.) $\nabla_V(fY) = V[f]Y + f\nabla_V Y$

(d.) $V[Y \bullet Z] = (\nabla_V Y) \bullet Z + Y \bullet (\nabla_V Z)$.

Proof: We examine part $(d.)$. Let Y, Z be vector fields with component functions Y_i, Z_i. We have $Y = \sum_{i=1}^{3} Y_i\mathbf{U_i}$ and $Z = \sum_{i=1}^{3} Z_i\mathbf{U_i}$. We calculate

$$Y \bullet Z = \left(\sum_{i=1}^{3} Y_i\mathbf{U_i}\right) \bullet \left(\sum_{j=1}^{3} Z_j\mathbf{U_j}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} Y_i Z_j \underbrace{\mathbf{U_i} \bullet \mathbf{U_j}}_{\delta_{ij}} = \sum_{i=1}^{3} Y_i Z_i.$$

Suppose V be a vector field. By Proposition 12.1.5 we calculate:

$$V[Y \bullet Z] = V\left[\sum_{i=1}^{3} Y_i Z_i\right] = \sum_{i=1}^{3} V\left[Y_i Z_i\right] = \sum_{i=1}^{3} \big(V\left[Y_i\right] Z_i + Y_i V\left[Z_i\right]\big).$$

Then, note $(\nabla_V Y)_i = V[Y_i]$ and $(\nabla_V Z)_i = V[Z_i]$ hence

$$V[Y \bullet Z] = \sum_{i=1}^{3} (\nabla_V Y)_i Z_i + \sum_{i=1}^{3} Y_i (\nabla_V Z)_i = (\nabla_V Y) \bullet Z + Y \bullet (\nabla_V Z).$$

If you understand this then you can write proofs for the other parts without much trouble □.

Definition 12.1.7.

If a triple of vector fields $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ on $\mathbb{R}^3$ satisfy $\mathbf{E_i} \bullet \mathbf{E_j} = \delta_{ij}$ for all i, j then they define a **frame field** on $\mathbb{R}^3$.

If $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ is a frame field on $\mathbb{R}^3$ then at each point $p \in \mathbb{R}^3$ they provide an orthonormal basis $\{\mathbf{E_1}(p), \mathbf{E_2}(p), \mathbf{E_3}(p)\}$ for $T_p\mathbb{R}^3$. Therefore, a given frame field allows us to have cartesian-like coordinates based at each point in $\mathbb{R}^3$. In particular, we can calculate dot-products and vector lengths with respect to frame field coordinates just as we do with cartesian coordinates. Moreover, we can select frame field coordinate functions for vector fields via dot-products just as we could select cartesian coordinates of vectors by dot-products. The choice to work with orthonormal frames pays off big here. We'll use this proposition multiple times in our future work.

Proposition 12.1.8.

Suppose $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ is a frame. If V is a vector field on $\mathbb{R}^3$ then the component functions of V w.r.t. the $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ frame are given by $V \bullet \mathbf{E_j}$ for $1 \leq j \leq 3$; that is $V = \sum_{j=1}^{3} (V \bullet \mathbf{E_j}) E_j$.

Proof: left to the reader. Or, see my linear algebra notes □.

The method I used to find these frames was to calculate the coordinates derivatives by the push-forward formulas[4] we discussed in Subsection 10.3.1.

Example 12.1.9. *For cylindrical coordinates $x = r\cos\theta$, $x = r\sin\theta$ and $z = z$. The cylindrical frame field is defined as follows:*

$$\begin{aligned} \mathbf{E_1} &= \cos\theta \mathbf{U_1} + \sin\theta \mathbf{U_2} \\ \mathbf{E_2} &= -\sin\theta \mathbf{U_1} + \cos\theta \mathbf{U_2} \\ \mathbf{E_3} &= \mathbf{U_3} \end{aligned}$$

Example 12.1.10. *If spherical coordinates ρ, ϕ, θ are given by $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ then the spherical frame field is given by:*

$$\begin{aligned} \mathbf{F_1} &= \sin\phi\big(\cos\theta \mathbf{U_1} + \cos\theta \mathbf{U_2}\big) + \cos\phi \mathbf{U_3} \\ \mathbf{F_2} &= -\sin\theta \mathbf{U_1} + \cos\theta \mathbf{U_2} \\ \mathbf{F_3} &= \cos\phi\big(\cos\theta \mathbf{U_1} + \cos\theta \mathbf{U_2}\big) - \sin\phi \mathbf{U_3} \end{aligned}$$

[4] although, to be honest, I used a better method to calculate the needed partial derivatives. The best way is to compute the total differentials of each coordinate function and read off the derivatives there. Again, probably a homework problem

Observe that $\mathbf{E_1}$ *and* $\mathbf{E_2}$ *of Example 12.1.9 appear here and* $\mathbf{F_2} = \mathbf{E_2}$ *as can be expected since geometrically they play identical roles.*

12.2 connection forms on $\mathbb{R}^3$

Let $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ be a frame field and consider a point $p \in \mathbb{R}^3$ and a vector v at p. Let us calculate the covariant derivatives of the frame field at p in the v-direction. Furthermore, we shall express the covariant derivatives in terms of the $\mathbf{E}$-frame in view of the technique of Proposition 12.1.8

$$\begin{aligned}
\nabla_v \mathbf{E_1} &= [(\nabla_v \mathbf{E_1}) \bullet \mathbf{E_1}]\, \mathbf{E_1} + [(\nabla_v \mathbf{E_1}) \bullet \mathbf{E_2}]\, \mathbf{E_2} + [(\nabla_v \mathbf{E_1}) \bullet \mathbf{E_3}]\, \mathbf{E_3} \\
\nabla_v \mathbf{E_2} &= [(\nabla_v \mathbf{E_2}) \bullet \mathbf{E_1}]\, \mathbf{E_1} + [(\nabla_v \mathbf{E_2}) \bullet \mathbf{E_2}]\, \mathbf{E_2} + [(\nabla_v \mathbf{E_2}) \bullet \mathbf{E_3}]\, \mathbf{E_3} \\
\nabla_v \mathbf{E_3} &= [(\nabla_v \mathbf{E_3}) \bullet \mathbf{E_1}]\, \mathbf{E_1} + [(\nabla_v \mathbf{E_3}) \bullet \mathbf{E_2}]\, \mathbf{E_2} + [(\nabla_v \mathbf{E_3}) \bullet \mathbf{E_3}]\, \mathbf{E_3}
\end{aligned} \tag{12.2}$$

Think about this: for each i, j the coefficient $(\nabla_v \mathbf{E_i}) \bullet \mathbf{E_j}$ represents a linear function in $v \in T_p\mathbb{R}^3$ which assigns a particular real number. In other words, $v \mapsto (\nabla_v \mathbf{E_i}) \bullet \mathbf{E_j}$ is a one-form at p. If we allow p to vary then we obtain a one form on $\mathbb{R}^3$ for each i, j.

Definition 12.2.1. matrix of connection forms of $\mathbb{R}^3$

Let $1 \leq i, j \leq 3$ and define a one-form ω_{ij} by $\big(\omega_{ij}(p)\big)(v) = [(\nabla_v \mathbf{E_i}) \bullet \mathbf{E_j}]\,(p)$ for each vector v at $p \in \mathbb{R}^3$. Futhermore, the matrix of one-forms ω_{ij} will be denoted by ω.

This matrix of one-forms will be at the center stage of our thinking for most of the remainder of this chapter. Perhaps the most important property is given next:

Proposition 12.2.2.

Let ω_{ij} be the connection forms w.r.t. frame $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ then $\omega_{ij} = -\omega_{ji}$ for all i, j.

Proof: follows from orthonormality of the frame paired with properties of covariant differentiation. In particular, fix i, j and let $v \in T_p\mathbb{R}^3$. Calculate

$$v[\mathbf{E_i} \bullet \mathbf{E_j}] = [\nabla_v \mathbf{E_i}] \bullet \mathbf{E_j} + \mathbf{E_i} \bullet [\nabla_v \mathbf{E_j}]\,.$$

However, observe that $\mathbf{E_i} \bullet \mathbf{E_j} = \delta_{ij}$ is a constant function thus $v[\delta_{ij}] = 0$. Hence $[\nabla_v \mathbf{E_i}] \bullet \mathbf{E_j} = -\mathbf{E_i} \bullet [\nabla_v \mathbf{E_j}]$. Therefore, $\omega_{ij}(v) = -\omega_{ji}(v)$ for arbitrary v and p. The proposition follows. □.

The proposition above can be formulated[5] as $\omega^T = -\omega$. In other words, the connection matrix is **antisymmetric**. This immediately implies that $\omega_{11} = \omega_{22} = \omega_{33} = 0$ for any connection form ω.

The frame $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ is related to the cartesian frame $\mathbf{U_1}, \mathbf{U_2}, \mathbf{U_3}$ by the **attitude matrix**. It turns out this matrix provides a convenient computational tool in what follows.

[5] here Oneil uses the notation ${}^t\omega$ for ω^T

Definition 12.2.3. attitude matrix of E-frame

Let $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ be a frame field and let a_{ij} be the functions for which

$$\mathbf{E_i} = \sum_{j=1}^{3} a_{ij} \mathbf{U_j}.$$

We say $A = (a_{ij})$ is called the **attitude matrix** of the $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ frame.

The definition above makes the i-th row of the attitude matrix the components of $\mathbf{E_i}$.

Example 12.2.4. *(continuing Ex. 12.1.9) We may calculate the attitude matrix for the cylindrical frame field by inspection: (zeros and one added for clarity of A-construction)*

$$\begin{aligned} \mathbf{E_1} &= \cos\theta \mathbf{U_1} + \sin\theta \mathbf{U_2} + 0\mathbf{U_3} \\ \mathbf{E_2} &= -\sin\theta \mathbf{U_1} + \cos\theta \mathbf{U_2} + 0\mathbf{U_3} \\ \mathbf{E_3} &= 0\mathbf{U_1} + 0\mathbf{U_2} + 1\mathbf{U_3} \end{aligned} \qquad \Rightarrow \qquad A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I invite the reader to verify $AA^T = I$.

Example 12.2.5. *(continuing Ex. 12.1.10) The spherical frame* $\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}$ *has attitude matrix:*

$$\begin{aligned} \mathbf{F_1} &= \sin\phi\big(\cos\theta \mathbf{U_1} + \sin\theta \mathbf{U_2}\big) + \cos\phi \mathbf{U_3} \\ \mathbf{F_2} &= -\sin\theta \mathbf{U_1} + \cos\theta \mathbf{U_2} \\ \mathbf{F_3} &= \cos\phi\big(\cos\theta \mathbf{U_1} + \sin\theta \mathbf{U_2}\big) - \sin\phi \mathbf{U_3} \end{aligned} \quad \Rightarrow \quad A = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ -\sin\theta & \cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{bmatrix}$$

Once more, I invite the reader to verify that $AA^T = I$.

The attitude matrix is a matrix of functions. Notice we can construct a matrix of forms from A by taking the differential of each component. The matrix dA is then a matrix of one-forms, much like the connection matrix ω. That said, we should pause to define the operations we need to connect dA and ω explicitly.

When working with a matrix of one-forms the common notation in such a context is that matrix multiplication proceeds normally except that the component-wise multiplication is understood to be wedge products. We also calculate exterior derivatives of matrix-valued forms. The rule is simply to calculate the exterior derivative of each entry. In summary, if A is an $l \times m$ matrix of p-forms and B is an $m \times n$ matrix of q-forms then AB is defined to be a $l \times n$ matrix of $(p+q)$-forms:

$$(AB)_{ij} = \sum_{k=1}^{m} A_{ik} \wedge B_{kj}$$

moreover, dA is a $l \times m$ matrix of p-forms where we define $(dA)_{ij} = dA_{ij}$. Now we have all the terminology needed to explicitly connect the exterior derivative of the attitude matrix and the connection form:

Proposition 12.2.6.

Let $A = (a_{ij})$ be the attitude matrix for the frame field $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ then the connection matrix ω is given by: $\omega = dAA^T$.

Proof: observe $\omega = dAA^T$ indicates that $\omega(V) = dA(V)A^T$ for all vector fields V. In particular, for each i, j we should show $\omega_{ij}(V) = (dA(V)A^T)_{ij}$. Let us begin:

$$\begin{aligned}
\omega_{ij}(V) &= (\nabla_V \mathbf{E_i}) \bullet \mathbf{E_j} \\
&= \nabla_V \left(\sum_{k=1}^{3} a_{ik} \mathbf{U_k} \right) \bullet \mathbf{E_j} \\
&= \left[\sum_{k=1}^{3} V[a_{ik}] \mathbf{U_k} \right] \bullet \left[\sum_{l=1}^{3} a_{jl} \mathbf{U_l} \right] \\
&= \sum_{k=1}^{3} \sum_{l=1}^{3} V[a_{ik}] a_{jl} \mathbf{U_k} \bullet \mathbf{U_l} \\
&= \sum_{k=1}^{3} da_{ik}[V] a_{jk} \\
&= \sum_{k=1}^{3} (dA[V])_{ik} (A^T)_{kj} \\
&= \left(dA[V] A^T \right)_{ij} .
\end{aligned}$$

Therefore, as V and i, j were arbitrary the proposition follows $\square$.

Example 12.2.7. *(continuing Ex. 12.2.4 and Ex. 12.1.9) we calculate the connection form for the cylindrical frame via Proposition 12.2.6.*

$$\omega = dAA^T = \begin{bmatrix} -\sin\theta d\theta & \cos\theta d\theta & 0 \\ -\cos\theta d\theta & -\sin\theta d\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 12.2.8. *(continuing Ex. 12.2.5 and Ex. 12.1.10) we calculate the connection form for the spherical frame via Proposition 12.2.6. To begin, recall* $A = \begin{bmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ -\sin\theta & \cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{bmatrix}$ *and take the exterior derivative: (the lines are just for convenience)*

$$dA = \left[\begin{array}{c|c|c} \cos\phi\cos\theta d\phi - \sin\phi\sin\theta d\theta & \cos\phi\sin\theta d\phi + \sin\phi\cos\theta d\theta & -\sin\phi d\phi \\ \hline -\cos\theta d\theta & -\sin\theta d\theta & 0 \\ \hline -\sin\phi\cos\theta d\phi - \cos\phi\sin\theta d\theta & -\sin\phi\sin\theta d\phi + \cos\phi\cos\theta d\theta & -\cos\phi d\phi \end{array}\right]$$

The transpose is simply $A^T = \begin{bmatrix} \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta \\ \cos\phi & 0 & -\sin\phi \end{bmatrix}$. *Now, multiply. I'll just consider the interesting terms* $\omega_{12}, \omega_{13}, \omega_{23}$ *from* $\omega = dAA^T$. *I leave the details to the reader*[6]

$$\omega = \begin{bmatrix} 0 & \cos\phi d\theta & d\phi \\ -\cos\phi d\theta & 0 & \sin\phi d\phi \\ -d\phi & -\sin\phi d\phi & 0 \end{bmatrix}.$$

We now return to the task of covariant differentiation. This result is important because it relates the change in the frame field $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ in terms of the frame field $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$. In some sense, this is like the Frenet Serret equations from calculus III.

Proposition 12.2.9. *Connection equations of frame field* $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ *on* $\mathbb{R}^3$

Let ω_{ij} be the connection forms with respect to the frame field $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ on $\mathbb{R}^3$ for any vector field V,

$$\nabla_V \mathbf{E_i} = \sum_{j=1}^{3} \omega_{ij}(V)\mathbf{E_j}$$

Proof: Suppose $\mathbf{E_i} = \sum_{j=1}^{3} a_{ij}\mathbf{U_j}$ where a_{ij} is the attitude matrix of an orthonormal frame $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$. Note this implies $A = (a_{ij})$ is orthonormal ($A^TA = I$) and so the inverse relation is given by the transpose and we may express: $\mathbf{U_i} = \sum_{j=1}^{3} a_{ji}\mathbf{E_j}$.

$$\begin{aligned} \nabla_V \mathbf{E_i} &= \nabla_V \left[\sum_{j=1}^{3} a_{ij}\mathbf{U_j} \right] \\ &= \sum_{j=1}^{3} \nabla_V \left[a_{ij}\mathbf{U_j} \right] \\ &= \sum_{j=1}^{3} \left[V[a_{ij}]\mathbf{U_j} + a_{ij}\nabla_V \mathbf{U_j} \right] \\ &= \sum_{j=1}^{3} V[a_{ij}] \sum_{k=1}^{3} a_{kj}\mathbf{E_k} \\ &= \sum_{k=1}^{3}\sum_{j=1}^{3} da_{ij}[V]a_{kj}\mathbf{E_k} = \sum_{k=1}^{3} (dA[V]A^T)_{ik}\mathbf{E_k} = \sum_{k=1}^{3} \omega_{ik}[V]\mathbf{E_k} \quad \square. \end{aligned}$$

[6] do not attempt this unless you plan to work carefully, it's not hard, but it does require attention to detail

It is worth expanding these for future reference:

$$\begin{aligned}\nabla_V \mathbf{E_1} &= \omega_{12}[V]\mathbf{E_2} + \omega_{13}[V]\mathbf{E_3}\\ \nabla_V \mathbf{E_2} &= -\omega_{12}[V]\mathbf{E_1} + \omega_{23}[V]\mathbf{E_3}\\ \nabla_V \mathbf{E_3} &= -\omega_{13}[V]\mathbf{E_1} - \omega_{23}[V]\mathbf{E_2}\end{aligned}$$

Compare these against the Frenet Serret Equations: (assume curve is arclength parametrized)

$$\begin{aligned}\frac{dT}{dt} &= \kappa N\\ \frac{dN}{dt} &= -\kappa T + \tau B\\ \frac{dB}{dt} &= -\tau N.\end{aligned}$$

12.2.1 Frenet Serret equations

Let's pause to see how the Frenet Serret equations are derived from the connection equations for a frame field. Note that the Frenet Frame on a curve can be extended beyond the curve relative to some open tubular neighborhood for a regular curve. This is geometrically evident, but I'd rather not prove it for our purposes here. The derivative with respect to arclength is naturally identified with ∇_T. Let the frame adapted to the curve be given by the Frenet frame in the usual order: $\mathbf{E_1} = T$, $\mathbf{E_2} = N$ and $\mathbf{E_3} = B$ In particular, again using t for arclength, let $V = \sum_{i=1}^3 v_i \mathbf{U_i}$ be a vector field along the curve,

$$\nabla_T V = T(v_1)\mathbf{U_1} + T(v_2)\mathbf{U_2} + T(v_3)\mathbf{U_3} = \frac{dV}{dt}$$

Here we understand, on the curve, the $\frac{d}{dt} = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}$ and $\mathbf{U_1}, \mathbf{U_2}, \mathbf{U_3}$ we recall $\frac{\partial}{\partial x} = \mathbf{U_1}$, $\frac{\partial}{\partial y} = \mathbf{U_2}$ and $\frac{\partial}{\partial z} = \mathbf{U_3}$. Also, observe: $\nabla_T \mathbf{E_i} = \sum_{j=1}^3 \omega_{ij}[T]\mathbf{E_j}$ hence $\omega_{ij}[T] = \frac{d\mathbf{E_i}}{dt} \bullet \mathbf{E_j}$. But, the defintion of curvature κ and can be formulated as $T' = \kappa N$ hence $\frac{dT}{dt} \bullet N = \kappa$ and $\frac{dT}{dt} \bullet B = 0$. Moreover, torsion τ is defined by $\frac{dB}{dt} \bullet N = -\tau$. Therefore, $\omega_{12}[T] = \kappa$, $\omega_{13}[T] = 0$ and $\omega_{23}[T] = \tau$ and we see how the connection equations imply the Frenet Serret equations along a curve.

$$\begin{aligned}\nabla_T \mathbf{E_1} &= \ \omega_{12}[T]\mathbf{E_2} + \omega_{13}[T]\mathbf{E_3} \quad &\Rightarrow \quad \frac{dT}{dt} &= \kappa N\\ \nabla_T \mathbf{E_2} &= -\omega_{12}[T]\mathbf{E_1} + \omega_{23}[T]\mathbf{E_3} \quad &\Rightarrow \quad \frac{dN}{dt} &= -\kappa T + \tau B\\ \nabla_T \mathbf{E_3} &= -\omega_{13}[T]\mathbf{E_1} - \omega_{23}[T]\mathbf{E_2} \quad &\Rightarrow \quad \frac{dB}{dt} &= -\tau N.\end{aligned}$$

12.3 structure equations for frame field on $\mathbb{R}^3$

The cartesian frame $\mathbf{U_1}, \mathbf{U_2}, \mathbf{U_3}$ was defined by $\frac{\partial}{\partial x} = \mathbf{U_1}$, $\frac{\partial}{\partial y} = \mathbf{U_2}$ and $\frac{\partial}{\partial z} = \mathbf{U_3}$ hence the natural dual to the cartesian frame is dx, dy, dz. Remember we proved $dx_i(\partial_j) = \delta_{ij}$ in much greater generality in a previous chapter.

Definition 12.3.1. coframe dual to a given frame

Let $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ be a frame field then **coframe field dual to** $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ is $\theta_\mathbf{1}, \theta_\mathbf{2}, \theta_\mathbf{3}$ which is a triple of one-forms such that

$$\theta_\mathbf{i}(V) = V \bullet \mathbf{E_i}$$

for all vector fields V. Naturally, $\theta_\mathbf{i}(\mathbf{E_j}) = \delta_{ij}$.

The term *one-form* in in some sense unfortunate since what might be a better term is *one-form field*. In particular, notice the coframe $\theta_\mathbf{1}, \theta_\mathbf{2}, \theta_\mathbf{3}$ provides a basis $\{\theta_\mathbf{1}(p), \theta_\mathbf{2}(p), \theta_\mathbf{3}(p)\}$ for $(T_p\mathbb{R}^3)^*$ at each $p \in \mathbb{R}^3$. We often suppress the point dependence in what follows.

Suppose γ is a one-form on $\mathbb{R}^3$ then $\gamma = \sum_{j=1}^3 \gamma_j \theta_\mathbf{j}$ for some functions γ_j. Furthermore,

$$\gamma(\mathbf{E_i}) = \sum_{j=1}^3 \gamma_j \theta_\mathbf{j}(\mathbf{E_i}) = \sum_{j=1}^3 \gamma_j \delta_{ij} = \gamma_i.$$

This discussion proves the following analog to Proposition 12.1.8.

Proposition 12.3.2.

Suppose $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ is a frame with coframe $\theta_\mathbf{1}, \theta_\mathbf{2}, \theta_\mathbf{3}$ on $\mathbb{R}^3$. If γ is a one-form on $\mathbb{R}^3$ then

$$\gamma = \sum_{i=1}^3 \gamma(\mathbf{E_i})\theta_\mathbf{i}.$$

Naturally we are curious how an arbitrary coframe relates to the cartesian coframe dx, dy, dz. It turns out that the frame and coframe share the same attitude.

Proposition 12.3.3.

Suppose $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ is a frame with coframe $\theta_\mathbf{1}, \theta_\mathbf{2}, \theta_\mathbf{3}$ on $\mathbb{R}^3$. If A is the attitude matrix, meaning $\mathbf{E_i} = \sum_{j=1}^3 a_{ij}\mathbf{U_j}$, then $\theta_\mathbf{i} = \sum_{j=1}^3 a_{ij}\mathbf{dx_j}$.

Proof: Here we use the natural convention $x_1 = x, x_2 = y, x_3 = z$ in the proposition above. The proof of this proposition filters through the duality condition $\theta_\mathbf{i}(\mathbf{E_j}) = \delta_{ij}$ since $dx_i(\mathbf{U_j}) = \delta_{ij}$ and δ_{ij} is common ground. In particular, by orthonormality we have inverse relations $\mathbf{U_j} = \sum_{k=1}^3 a_{kj}\mathbf{E_k}$

$$\theta_\mathbf{i}(\mathbf{U_j}) = \theta_\mathbf{i}\left(\sum_{k=1}^3 a_{kj}\mathbf{E_k}\right) = \sum_{k=1}^3 a_{kj}\theta_\mathbf{i}\left(\mathbf{E_k}\right) = \sum_{k=1}^3 a_{kj}\delta_{ik} = a_{ij}.$$

Thus, by Proposition 12.3.3, $\theta_{\mathbf{i}} = \sum_{j=1}^{3} a_{ij} dx_j$ $\square$.

A brief history: classification of curves by curvature and torsion was elegantly treated by Frenet and Serret independently around 1860. The work on frames attached to curves then prompted Darboux to try to generalize the method of frames to surfaces. Then the general method of frames was promoted and championed by E. Cartan around 1900. Research continues to this day.

Theorem 12.3.4. *Cartan's Structural Equations*

Suppose $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$ is a frame with coframe $\theta_{\mathbf{1}}, \theta_{\mathbf{2}}, \theta_{\mathbf{3}}$ on $\mathbb{R}^3$. If ω is the connection form of the given frame then the following **structure equations** hold for all i, j

$$(1.)\quad d\theta_{\mathbf{i}} = \sum_{j=1}^{3} \omega_{ij} \wedge \theta_{\mathbf{i}} \qquad (2.)\quad d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}.$$

Proof: It is convenient to reformulate these equations in a matrix/column notation. In particular,

$$\theta = \begin{bmatrix} \theta_{\mathbf{1}} \\ \theta_{\mathbf{2}} \\ \theta_{\mathbf{3}} \end{bmatrix} \qquad d\xi = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

The equation $\theta_i = \sum_{j=1}^{3} a_{ij} dx_j$ gives $\theta = A d\xi$. Observe that $d^2 = 0$ is true for matrices of forms since the exterior differentiation is done component-wise. Thus, $d(d\xi) = 0$. Hence,

$$d\theta = d(A d\xi) = dA \wedge d\xi + A d(d\xi) = dA\, d\xi = dA A^T\, A d\xi.$$

where we use $A^T A = I$ for the last step. Next, apply Proposition 12.2.6 and $\theta = A d\xi$ to see:

$$d\theta = dA A^T A d\xi = \omega\theta.$$

The equation above is just (1.) in matrix notation. Next consider the exterior derivatve of[7] $\omega = dA A^T$, again use $d^2 = 0$ and the Leibniz rule,

$$d\omega = d(dA A^T) = d^2(A) A^T - dA d(A^T) = -\underbrace{dA A^T}_{\omega}\underbrace{A dA^T}_{-\omega}$$

To see why $A dA^T = -\omega$, use the socks-shoes property of the matrix transpose to see that $A dA^T = (dA A^T)^T = \omega^T$. But, by antisymmetry of the connection form says $\omega^T = -\omega$. We've shown

$$d\omega = -\omega(-\omega) = \omega\omega$$

this is precisely the second struture equation written in matrix notation $\square$.

This is one of my top ten favorite favorite theorems. However, to appreciate why I would enjoy such a strange set of formulas we probably need to spend an hour or two discussing the structure of a regular surface and a bit on diffeomorphisms. Once those issues are addressed we'll return and develop an intrinsic calculus via these equations.

[7] note that $d(A^T) = (dA)^T$ hence dA^T is non an ambiguous notation.

12.4 surfaces in $\mathbb{R}^3$

A surface M in $\mathbb{R}^3$ is a subset which locally looks like the plane. Moreover, we suppose the surface is **oriented**. The orientation is given by a unit-normal vector field which is defined to point in the upward direction. Let us review a few technical details to make this paragraph a bit more precise.

A surface $M \subset \mathbb{R}^3$ is called regular if it can be covered by compatible regular patches. A patch $X : D \subseteq \mathbb{R}^2 \to M$ is regular iff $\partial_u X \times \partial_v X \neq 0$ for all $(u, v) \in D$. If the vector field $\partial_u X \times \partial_v X \neq 0$ then we may normalize it to form a unit-normal field $U = \frac{1}{||\partial_u X \times \partial_v X||} \partial_u X \times \partial_v X$. Furthermore, if the surface is oriented then the normal vector field on each patch may be smoothly joined to adjacent patches as to construct a global unit normal field. Throughout this section we use the notation that U is the unit-normal to the surface considered. It is geometrically clear that $U(p)$ is orthogonal to the tangent space at $p \in M$. It is analytically clear since

$$T_pM = span\{\partial_u|_p, \partial_v|_p\}$$

and $U(p)$ is formed from the cross product of these tangent vectors. Of course, we can expand the vectors in the ambient frame $\mathbf{U_1}, \mathbf{U_2}, \mathbf{U_3}$ if necessary,

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\mathbf{U_1} + \frac{\partial y}{\partial u}\mathbf{U_2} + \frac{\partial z}{\partial u}\mathbf{U_3}, \qquad \frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\mathbf{U_1} + \frac{\partial y}{\partial v}\mathbf{U_2} + \frac{\partial z}{\partial v}\mathbf{U_3}$$

The shape of the surface is revealed from the covariant derivatives of the normal vector field. It can be shown that if v is tangent to M then $\nabla_v U$ is also tangent to M. It follows the definition below is reasonable:

Definition 12.4.1. shape operator

Let U be the unit-normal vector field to a sufrace $M \subset \mathbb{R}^3$. We define a mapping $S_p : T_pM \to T_pM$ by

$$S_p(v) = -\nabla_v U$$

for all $v \in T_pM$. This is the **shape operator at** p.

Naturally, the mapping $p \mapsto S_p$ is also called the **shape operator**. Geometrically, the value of the shape operator reflects how the normal vector field changes as p moves in the direction of v near p.

Example 12.4.2. Plane: *The shape operator on a plane is zero since U is constant.*

Example 12.4.3. Sphere: *of radius R centered at the origin has $U = \frac{1}{R}(x\mathbf{U_1} + y\mathbf{U_2} + z\mathbf{U_3})$ therefore $\nabla_v U = \frac{1}{R}\left(v[x]\mathbf{U_1} + v[y]\mathbf{U_2} + v[z]\mathbf{U_3}\right) = v/R$. This equation expresses the geometrically obvious fact that the normal vector field on a sphere bends in the direction of v as we move from p in the v-direction. Thus $S_p(v) = -v/R$.*

The sphere example is sufficiently simple in cartesian coordinates that we did not need to do much calculation. I didn't even provide a patch for the sphere, we reasoned geometrically. To understand what follows, please review the cylindrical and spherical frames. We introduced them in Examples 12.1.9 and 12.1.10.

Example 12.4.4. Cylinder: *of radius R with equation $x^2+y^2=R^2$ has natural parametrization by θ, z for which $x = R\cos\theta$, $y = R\sin\theta$ and $z = z$. The cylindrical frame field then provides vector fields tangent to the cylinder. In particular,*

$$\mathbf{E_2} = -\sin\theta\mathbf{U_1} + \cos\theta\mathbf{U_2}, \qquad \mathbf{E_3} = \mathbf{U_3}$$

are clearly tangent to the cylinder. On the other hand, it's geometrically clear that the normal vector field $U = \mathbf{E_1} = \cos\theta\mathbf{U_1} + \sin\theta\mathbf{U_2}$. Calculate the covariant derivatives in view of the fact that $U = \frac{1}{R}(x\mathbf{U_1} + y\mathbf{U_2})$,

$$\nabla_v U = \frac{1}{R}\left[v[x]\mathbf{U_1} + v[y]\mathbf{U_2}\right] = \frac{1}{R}(v_x\mathbf{U_1} + v_y\mathbf{U_2}) \tag{12.3}$$

where I denote $v = v_x\partial_x + v_y\partial_y + v_z\partial_z$.

I now work on converting the formula into the cylindrical frame. We can invert the equations for $\mathbf{E_1} = \cos\theta\mathbf{U_1} + \sin\theta\mathbf{U_2}$ and $\mathbf{E_2} = -\sin\theta\mathbf{U_1} + \cos\theta\mathbf{U_2}$ for $\mathbf{U_1}$ and $\mathbf{U_2}$. We obtain[8]:

$$\mathbf{U_1} = \cos\theta\mathbf{E_1} - \sin\theta\mathbf{E_2}, \qquad \mathbf{U_2} = \sin\theta\mathbf{E_1} + \cos\theta\mathbf{E_2}. \tag{12.4}$$

Consider $v = \mathbf{E_2} = -\sin\theta\mathbf{U_1} + \cos\theta\mathbf{U_2}$ has $v_x = -\sin\theta$ and $v_y = \cos\theta$ thus:

$$\begin{aligned}
\nabla_{\mathbf{E_2}} U &= \frac{1}{R}(-\sin\theta\mathbf{U_1} + \cos\theta\mathbf{U_2}) \qquad \textit{(by Eqn. 12.3)} \\
&= \frac{1}{R}\big[-\sin\theta(\cos\theta\mathbf{E_1} - \sin\theta\mathbf{E_2}) + \cos\theta(\sin\theta\mathbf{E_1} + \cos\theta\mathbf{E_2})\big] \qquad \textit{(by Eqn. 12.4)} \\
&= \frac{1}{R}\mathbf{E_1}.
\end{aligned}$$

In summary, if $v = a\mathbf{E_2} + b\mathbf{E_3}$ where $\mathbf{E_2} = \frac{1}{R}\partial_\theta$ and $\mathbf{E_3} = \partial_z$ then we find:

$$S_p(v) = -\nabla_v U = -\frac{a}{R}\mathbf{E_1}.$$

The matrix of this shape operator has the form: $[S_p] = \begin{bmatrix} -1/R & 0 \\ 0 & 0 \end{bmatrix}$. The shape operator reflects our intuition that the cylinder is curved in the θ-direction whereas in the z-direction it is flat.

I will use some calculus which is developed in my calculus III lecture notes in what follows. In those notes I show how $\widehat{\rho}$, $\widehat{\theta}$ and $\widehat{\phi}$ form the spherical frame. The difference between that context and our current one is that the vector fields were viewed as passive objects. In calculus III we did not think of vector fields as derivations. That said, all the algebra/geometric formulas derived for the passive frame hold for the derivation-based frame we consider here. In particular, when I consider the cone below, it is geometrically obvious that $\widehat{\phi}|_p$ is the normal vector field at each $p \neq 0$

[8] to see $\mathbf{U_1}$, multiply $\mathbf{E_1}$ by $\cos\theta$ and $\mathbf{E_2}$ by $-\sin\theta$ the terms attached to $\mathbf{U_2}$ cancel. To see $\mathbf{U_2}$, multiply $\mathbf{E_1}$ by $\sin\theta$ and $\mathbf{E_2}$ by $\cos\theta$ the terms attached to $\mathbf{U_1}$ cancel

for the cone M defined by $\phi = \phi_o$. Orthonormality immediately informs be that $\widehat{\phi}|_p, \widehat{\rho}|_p$ form an orthonormal basis for T_pM at each $p \neq 0$ on the cone M.

Recall for what follows that: (here $\mapsto$ denotes the transition from calculus III notation to our current derivation-based formalism for vectors)

$$\widehat{\rho} \mapsto \mathbf{F_1} = \sin\phi\big(\cos\theta\mathbf{U_1} + \sin\theta\mathbf{U_2}\big) + \cos\phi\mathbf{U_3}$$
$$\widehat{\theta} \mapsto \mathbf{F_2} = -\sin\theta\mathbf{U_1} + \cos\theta\mathbf{U_2}$$
$$\widehat{\phi} \mapsto \mathbf{F_3} = \cos\phi\big(\cos\theta\mathbf{U_1} + \sin\theta\mathbf{U_2}\big) - \sin\phi\mathbf{U_3}$$

Example 12.4.5. Cone *at angle ϕ_o. We consider points other than the origin, the cone is not smooth at that singular point. To be more accurate, the results we derive in this example apply to some open subset of the cone. In cylindrical coordinates $r = \rho\sin(\phi_o)$ thus the cartesian equation of this cone is easily derived from $r^2 = \rho^2\sin^2(\phi_o)$ gives $x^2 + y^2 = \sin^2(\phi_o)(x^2 + y^2 + z^2)$ hence, for $\phi_o \neq \pi/2$, we find $x^2 + y^2 = \tan^2(\phi_o)z^2$. In cylindrical coordinates this cone has equation $r = \tan(\phi_o)z$. From spherical coordinates we find a natural parametrization,*

$$x = \rho\cos(\theta)\sin(\phi_o), \qquad y = \rho\sin(\theta)\sin(\phi_o), \qquad z = \rho\cos(\phi_o)$$

Furthermore, this gives $U = \mathbf{F_3} = \cos\phi\big(\cos\theta\mathbf{U_1} + \sin\theta\mathbf{U_2}\big) - \sin\phi\mathbf{U_3}$. Consider that:

$$\nabla_v\mathbf{F_3} = v[\cos\phi\cos\theta]\mathbf{U_1} + v[\cos\phi\sin\theta]\mathbf{U_2} + v[\sin\phi]\mathbf{U_3}$$

On the cone $\phi = \phi_o$ we have $\sin\phi_o$ is constant thus the $\mathbf{U_3}$-term vanishes. In what follows we consider tangent vectors on the cone, hence $\phi = \phi_o$ throughout. If $v = \partial_\rho$:

$$\nabla_{\partial_\rho}U = \partial_\rho[\cos\phi_o\cos\theta]\mathbf{U_1} + \partial_\rho[\cos\phi_o\sin\theta]\mathbf{U_2} = 0$$

Note $\mathbf{F_1} = \partial_\rho$ hence $\nabla_{\mathbf{F_1}}U = 0$. Continuing, let $v = \partial_\theta$ and consider:

$$\nabla_{\partial_\theta}U = \partial_\theta[\cos\phi_o\cos\theta]\mathbf{U_1} + \partial_\theta[\cos\phi_o\sin\theta]\mathbf{U_2} = \cos\phi_o(-\sin\theta\mathbf{U_1} + \cos\theta\mathbf{U_2}) = \cos\phi_o\mathbf{F_2}$$

However, as $\frac{\partial}{\partial\theta} = -\rho\sin\phi\sin\theta\mathbf{U_1} + \rho\sin\phi\cos\theta\mathbf{U_2}$ we see $\mathbf{F_2} = \frac{1}{\rho\sin\phi_o}\frac{\partial}{\partial\theta}$. Thus, by part (a.) of Proposition 12.1.6,

$$\nabla_{\mathbf{F_2}}U = \frac{\cos\phi_o}{\rho\sin\phi_o}\mathbf{F_2}$$

In summary, if $v = a\mathbf{F_1} + b\mathbf{F_2}$ then

$$S_p(v) = \frac{b\cos\phi_o}{\rho\sin\phi_o}\mathbf{F_2}$$

The matrix of this shape operator has the form: $[S_p] = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{\cos\phi_o}{\rho\sin\phi_o} \end{bmatrix}$. Observe that if we hold θ fixed on the cone we trace out a line as ρ varies. The line is not curved and thus the normal

is transported parallel to such lines as is reflected in the triviality of the covariant derivative in the $\mathbf{F_1}$*-direction on the cone. On the other hand, if we fix* ρ *and vary* θ *on the cone then we trace out a circle of radius* $\rho \sin \phi_o$*. The component of the normal which is radial to this circle is* $\cos \phi_o U$*. Notice how that component (*$\cos \phi_o U$*) of the normal bends in the direction of* $\mathbf{F_2}$ *at a rate proportional to the reciprocal of the radius (*$\rho \sin \phi_o$*) of the circle. This is the same as we saw in the example of the cylinder. In contrast to the cylinder, we have a family of surfaces to consider here, as* ϕ *varies we have distinct cones. Consider that as* $\phi \to \pi/2$ *we find* $S_p = 0$*. This verifies, modulo the deleted point at the origin, that the plane does have a trivial shape operator.*

Definition 12.4.6.

If S is the shape operator on a regular surface M then the Gaussian curvature K and the mean curvature H are defined by $K = \det(S)$ and $H = \text{trace}(S)$.

The eigenvalues of a 2×2 matrix correspond uniquely to the trace and determinant of the matrix. They also have geometric meaning here. Principle curvatures are defined in the theorem:

Theorem 12.4.7.

The shape operator is symmetric thus there exist real eigenvalues and an orthonormal eigenbasis for T_pM with respect to S at each $p \in M$. The orthonormal eigenvectors define the **principle directions** and the corresponding eigenvalues are called the **principle curvatures**. If the principle curvatures are k_1, k_2 then $K = k_1 k_2$ and $H = k_1 + k_2$.

Recall, eigenvalues of a diagonal matrix are simply the diagonal entries.

Example 12.4.8. Plane: *the principle curvatures are* $k_1 = k_2 = 0$*. It follows that the mean curvature* $H = 0$ *and the Gaussian curvature* $K = 0$ *on the entire plane.*

Example 12.4.9. Sphere: *the principle curvatures are* $k_1 = k_2 = -1/r$*. It follows that the mean curvature* $H = -2/r$ *and the Gaussian curvature* $K = 1/r^2$ *on the entire sphere.*

Example 12.4.10. Cylinder: *the principle curvatures are* $k_1 = -1/r$ *and* $k_2 = 0$*. It follows that the mean curvature* $H = -1/r$ *and the Gaussian curvature* $K = 0$ *on the entire clylinder.*

Example 12.4.11. Cone: *(without the point) the principle curvatures are* $k_1 = 0$ *and* $k_2 = -\frac{\cos \phi_o}{\rho \sin \phi_o}$*. It follows that the mean curvature* $H = -\frac{\cos \phi_o}{\rho \sin \phi_o}$ *and the Gaussian curvature* $K = 0$ *on the entire cone. Here we have an example of a surface which has non-constant mean curvature whilst*[9] *having constant Gaussian curvature.*

Surfaces with constant Gaussian curvature play special roles in many applications. All the examples above have constant gaussian curvature. As a point of language, we usually intend the unqualified term "curvature" to mean Gaussian curvature. It has a deeper meaning than the mean curvature. For example, any surface with constant Gaussian curvature $K = 0$ is called **flat** because it is essentially a plane from an intrinsic point of view. In contrast, the principle and mean curvatures are

[9] yes there is doubtless a better way to write this sentence

not intrinsic; the principle and mean curvatures have to do with how the surface is embedded in some ambient space. Here I've touched on an idea you may not have encountered before[10]. What do we mean by *intrinsic*? One of my goals in the remainder of this chapter is to unwrap that idea. We'll not do it here just yet.

Another important concept to the study of surface geometry is the **umbilic point**. A point is said to be an **umbilic point** if the principle curvatures are equal at the point in question. If every point on the surface is umbilic then the surface is called, shockingly, an **umbilic surface**. Planes and spheres are umbilic surfaces. Are there others? We'll discuss this in the final section of this chapter where I present a number of theorems without proof[11].

12.5 isometry

this chapter is unfinished, on the other hand, I have several hundred pages written which expand on this chapter at my elementary differential geometry course website. I leave those materials for those who wish to self-study differential geometry. Having this course helps, but, it is not a necessary prerequisite

[10] if you've seen Flatland then you may have at least a fictional concept of it

[11] most of the proofs are found in Oneil