EXAMPLES OF ALTERNATING SERIES

If $b_n > 0$ for $n = 1, 2, \ldots$ then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 + \cdots$ is an <u>alternating series</u>.

$\boxed{\begin{array}{l}\text{Th}^{m}\text{/ Given alternating series } b_1 - b_2 + b_3 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} b_n \\ \text{if } b_{n+1} \le b_n \text{ for all } n \ \underline{\text{and}} \ \lim_{n\to\infty} (b_n) = 0 \text{ then} \\ \text{the series } \sum_{n=1}^{\infty} (-1)^{n-1} b_n \ \underline{\text{converges}}.\end{array}}$

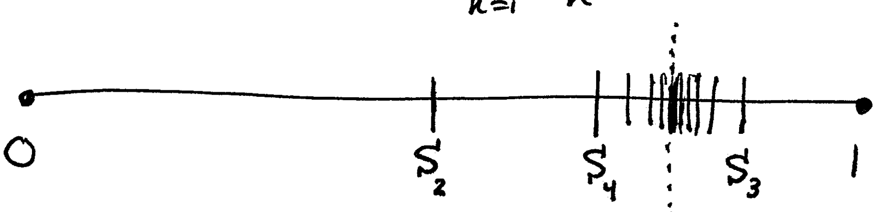
Note $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ with $b_n \not\to 0$ is divergent by the $n^{\underline{th}}$ term test as $a_n = (-1)^{n-1} b_n \not\to 0$ as $n \to \infty$.

1.) $S = \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Identify this is alternating series with $b_n = \frac{1}{n}$ and $b_{n+1} = \dfrac{1}{n+1} \le \dfrac{1}{n} = b_n$ where $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

thus S converges by the Alt. Series Test.

<u>Remark</u>: $1 - \frac{1}{2} + \frac{1}{3} + \cdots$ is the <u>alternating harmonic series</u> in contrast to $p=1$ series, it converges. Consider the partial sums $S_n = \sum_{k=1}^{n} \dfrac{(-1)^{k-1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \dfrac{(-1)^{n-1}}{n}$,



not obvious why this is true (yet)

$$\lim_{n\to\infty} (S_n) = \ln(2)$$

Th^m/ Alternating Series Estimation Theorem (ASET)

If $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1} b_n$ is a convergent alternating series and $S_n = \sum_{k=1}^{n} (-1)^{k-1} b_k$ then

$$\left| \sum_{m=1}^{\infty} (-1)^{m-1} b_m - \sum_{k=1}^{n} (-1)^{k-1} b_k \right| \leq b_{n+1}$$

- In words, the error in the n^{th} partial sum is at most b_{n+1}.

2.) Find $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ to within 0.01 of its exact value.

We want $b_n = \frac{1}{n} \sim 0.01 \Rightarrow n = \frac{1}{0.01} = 100$

hence the 100^{th} partial sum has at most an error of $\frac{1}{101}$

I'll let technology help here!

$$S_{100} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99} - \frac{1}{100} \cong \underline{0.6882}$$

(btw, $\ln(2) \cong 0.6931$ and $|\ln(2) - S_{100}| \cong 0.005$.)

3.) $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} + \cdots$

identify $b_n = \frac{1}{n^2}$ and note $\frac{1}{(n+1)^2} \leq \frac{1}{n^2}$ and

$\lim_{n \to \infty} \left(\frac{1}{n^2}\right) = 0$ hence S converges by the alt. series test.

Furthermore, $S_3 = 1 - \frac{1}{4} + \frac{1}{9} = 0.861$ and the ASET gives $|S - S_3| \leq b_4 = \frac{1}{16} = 0.0625$. This means

$$\underline{1 - \frac{1}{4} + \frac{1}{9} + \cdots = 0.861 \pm 0.0625.}$$

3.) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ is alternating series with

$b_n = \frac{1}{\sqrt{n+1}}$. Notice $b_{n+1} = \frac{1}{\sqrt{n+2}} \leq \frac{1}{\sqrt{n+1}} = b_n$

and $\lim_{n \to \infty} \left(\frac{1}{\sqrt{n+1}} \right) = 0 \therefore \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ converges by A.S.T.

4.) approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$ to 4 certain decimal places.

We want an error of at most 0.00001

Since $b_n = \frac{1}{(2n)!}$ we need to find n for which $b_n < 0.00001$ then the A.S.E.T. indicates S_{n-1}' suffices to give the desired accuracy,

$$b_1 = \frac{1}{2!} = 0.5$$

$$b_2 = \frac{1}{4!} = \frac{1}{24} = 0.042$$

$$b_3 = \frac{1}{6!} = \frac{1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \frac{1}{720} = 1.39 \times 10^{-3}$$

$$b_4 = \frac{1}{8!} = \frac{1}{40,320} = 2.48 \times 10^{-5}$$

(this is probably small enough, but I go one more to be absolutely sure.)

$$b_5 = \frac{1}{10!} = 2.75 \times 10^{-7} < 1 \times 10^{-5}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \cong -0.5 + \frac{1}{24} - \frac{1}{720} + \frac{1}{8!} \pm 2.75 \times 10^{-7}$$

$$\cong \boxed{-0.4597}$$