# Mission 2: Topics in Analysis: Solution

## P21 Ex. 1.3.1a

Claim: $1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1) \leftarrow P_n$

Observe $P_1$ true since $1^2 = \frac{1}{6}(1)(2)(3) = \frac{6}{6} = 1$.

Suppose $P_n$ true for some $n \in \mathbb{N}$. Consider,

$$\begin{aligned}
1^2+2^2+\cdots+n^2+(n+1)^2 &= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \quad : \text{by induction hypothesis.}\\
&= (n+1)\left[\frac{1}{6}n(2n+1) + n+1\right]\\
&= (n+1)\left[\frac{1}{6}(2n^2+n+6n+6)\right]\\
&= \frac{1}{6}(n+1)(n+2)(2n+3)\\
&= \frac{1}{6}(n+1)(n+1+1)(2(n+1)+1).
\end{aligned}$$

Thus $P_n \Rightarrow P_{n+1}$ and we find $P_n$ true $\forall n \in \mathbb{N}$ by PMI.//

## P22 Ex. 1.3.2 / Prove $9^n - 5^n$ is divisible by 4 $\forall n \in \mathbb{N}$

Let $P_n$ be the claim $9^n - 5^n$ is divisible by 4 for some $n \in \mathbb{N}$.
Observe $P_1$ true since $9^1 - 5^1 = 9-5 = 4 = 4\cdot 1$. Suppose $P_n$ true for some $n \in \mathbb{N}$ with $n > 1$. Consider,

$$\begin{aligned}
9^{n+1} - 5^{n+1} &= 9\cdot 9^n - 5^n\cdot 5 \quad : \text{def}^{\text{n}} \text{ of exponents.}\\
&= 9(5^n+4j) - 5^n\cdot 5 \quad : \left\{\begin{array}{l}\text{by induction hypothesis}\\ \exists j \in \mathbb{Z} \text{ such that}\\ 9^n - 5^n = 4j\end{array}\right\}\\
&= 5^n(9-5) + 9\cdot 4j\\
&= (5^n + 9j)4
\end{aligned}$$

Thus $9^{n+1} - 5^{n+1}$ is divisible by 4 as $5^n + 9j \in \mathbb{Z}$.
Hence $P_n \Rightarrow P_{n+1}$ and we conclude $P_n$ true $\forall n \in \mathbb{N}$ by PMI.//

$\boxed{P23}$ Exercise 1.3.4c | Prove $n^3 \leq 3^n$ for all $n \in \mathbb{N}$ |

Let $P_n$ be the claim $n^3 \leq 3^n$. Observe $1 < 3$ and $8 < 27$ and $27 \leq 27$ thus $P_1$, $P_2$ and $P_3$ are true. Suppose $n^3 \leq 3^n$ for some $n \geq 4$. Observe that,

$$
\begin{aligned}
(n+1)^3 &< \left(n + \tfrac{n}{4}\right)^3 && : \underbrace{\text{since } n \geq 4 \text{ we have } 1 \leq \tfrac{n}{4}}_{\text{this was key idea.}} \\
&= \left(\tfrac{5n}{4}\right)^3 \\
&= \frac{125 n^3}{64} \\
&< \frac{128 n^3}{64} && \text{nice step.} \\
&= 2n^3 \\
&< 3n^3 \\
&\leq 3 \cdot 3^n && : \text{by induction hypothesis.}
\end{aligned}
$$

Thus $(n+1)^3 \leq 3^{n+1}$ and we've shown $P_n \Rightarrow P_{n+1}$ for $n \geq 4$.

Therefore $n^3 \leq 3^n$ $\forall n \in \mathbb{N}$ by PMI.//

$\boxed{P24}$ Exercise 1.3.5 / Given $a \neq 1$, prove that

$$\underbrace{1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}}_{P_n} \quad \text{for all } n \in \mathbb{N}.$$

Observe that $\dfrac{1-a^2}{1-a} = \dfrac{(1-a)(1+a)}{1-a} = 1+a \quad \therefore P_1$ true.

Suppose $P_n$ true for some $n \in \mathbb{N}$ with $n > 1$,

$$1+a+a^2+\cdots+a^n+a^{n+1} = \frac{1-a^{n+1}}{1-a} + a^{n+1} \quad : \text{induction hypothesis.}$$

$$= \frac{1 - a^{n+1} + a^{n+1}(1-a)}{1-a} \quad : \text{made common denominator}$$

$$= \frac{1 - a^{n+1+1}}{1-a}$$

Thus $P_n \Rightarrow P_{n+1}$ and we conclude $P_n$ true $\forall n \in \mathbb{N}$ by PMI. //

$\boxed{P25}$ Ex. 1.3.7 / Let $a \geq -1$. Prove $(1+a)^n \geq 1+na \quad \forall n \in \mathbb{N}$

Observe $(1+a)^1 = 1+a \geq 1 + 1\cdot a$ hence the $n=1$ case holds true.

Suppose that $(1+a)^n \geq 1+na$ for some $n \in \mathbb{N}$. Consider,

$$\begin{aligned}
(1+a)^{n+1} &= (1+a)(1+a)^n && : \text{def}^{\text{n}} \text{ of exponents.} \\
&\geq (1+a)(1+na) && : \text{by induction hypothesis.} \\
&= 1 + (n+1)a + na^2 && : \text{algebra.} \\
&\geq 1 + (n+1)a && : \text{since } na^2 \geq 0
\end{aligned}$$

Thus $(1+a)^n \geq 1+na \Rightarrow (1+a)^{n+1} \geq 1+(n+1)a$ for $n \geq 1$.

and we conclude $(1+a)^n \geq 1+na \quad \forall n \in \mathbb{N}$ by PMI. //

P26 Ex. 1.4.2 / Prove Prp. 1.4.1 part (c.) and (d.).

PROPOSITION 1.4.1 (c)

For $x, y, z \in \mathbb{R}$, if $x \neq 0$ and $xy = xz$, then $y = z$.

Proof: Suppose $x \neq 0$ and assume $xy = xz$ for some $y, z \in \mathbb{R}$.

Notice $xy - xz = xz - xz = 0$ by Axiom 1d.

Then $xy - xz = x(y - z) = 0$ by Axiom 2e.

Since $x \neq 0$ by Axiom 2d, $\exists x^{-1}$ such that $x^{-1}x = \underline{1}$ hence

$x^{-1}x(y-z) = x^{-1}(0) \Rightarrow 1 \cdot (y - z) = 0$ by $\underbrace{\text{Prop. 1.4.1 e}}_{*}$.

Then by Axiom 2b and 2c we find $y - z = 0$. Finally, using Axiom 1a and 1c and 1d to add $z$ to both sides we find $y = z$. //

-(I'll omit part d.)- * : does not depend on part c. See page 20 in text

$\boxed{P27}$ Ex. 1.4.4 / Prove Prop. 1.4.2 parts (a), (b), (c.)

Let $x, y, M \in \mathbb{R}$ and suppose $M > 0$.

(a.) $|x| \geq 0$,

(b.) $|-x| = |x|$,

(c.) $|xy| = |x||y|$

Proof: If we use $|x| = \sqrt{x^2}$ then (a.) is immediately clear since $\sqrt{y} \geq 0$ for all $y \geq 0$. Consider (b.),

$$|-x| = \sqrt{(-x)^2} = \sqrt{(-1 \cdot x)^2} = \sqrt{(-1)^2(x^2)} = \sqrt{x^2} = |x|.$$

Likewise,

$$|xy| = \sqrt{(xy)^2} = \sqrt{x^2y^2} = \sqrt{x^2}\sqrt{y^2} = |x||y|. //$$

//

Remark: the other way to prove things about $|x|$ is to break into cases $|x| = x$ for $x \geq 0$
$|x| = -x$ for $x < 0$

for example, either $x \geq 0$ then $|x| = x \geq 0$ or $x < 0$ then $|x| = -x > 0$ $\therefore$ $|x| \geq 0$ in all cases.

$\boxed{P28}$ Ex. 1.4.2 already did for $\boxed{P26}$

P29 Exercise 1.5.1

Prove $A \subseteq \mathbb{R}$ is bounded iff $\exists M \in \mathbb{R}$ such that $|x| \leq M \ \forall x \in A$.

See my lectures ☺

P30 Exercise 1.5.3 a, b, c (partial sol$^{n}$, I don't justify claims here)

(a.) $S = \{1, 5, 17\}$ is bounded below by 1 and above by 17.
Moreover, $\inf(S) = 1$ and $\sup(S) = 17$.

(b.) $S = [0, 5)$ has $0 \leq x < 5 \ \forall x \in S$ thus $S$ is bounded below by 0 and above by 5. We can prove $\inf(S) = 0$ and $\sup(S) = 5$.

(c.) $S = \left\{ 1 + \frac{(-1)^n}{n} \ \middle| \ n \in \mathbb{N} \right\} = \{0, 3/2, 2/3, 5/4, 4/5, \dots\}$

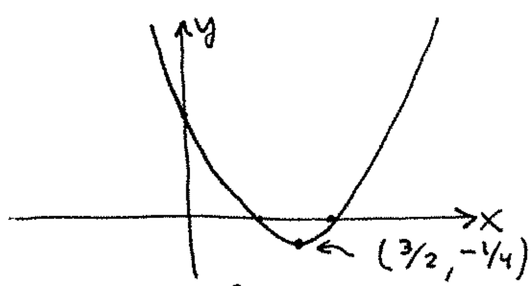
Bounded below by 0 and above by $3/2$.
We can argue $\inf(S) = 0$ and $\sup(S) = 3/2$.

P31 Exercise 1.5.3 d, e, f

(d.) $S = (-3, \infty)$ has $\inf(S) = -3$ and $\sup(S) = \infty$.

(e.) $S = \{x \in \mathbb{R} \mid \underbrace{x^2 - 3x + 2 = 0}_{(x-1)(x-2)=0}\} = \{1, 2\} \therefore \sup(S) = 2$, $\inf(S) = 1$.

(f.) $S = \{x^2 - 3x + 2 \mid x \in \mathbb{R}\} = [-1/4, \infty)$

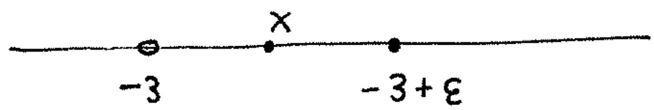



$\sup(S) = \infty$
$\inf(S) = -1/4$

$$y = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 2$$
$$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$$

P31 using the ε- Prop. 1.5.1 and Archimedean Principle   didn't need it

(d.) $S = (-3, \infty)$ by Defn $x \in (-3, \infty)$ has $x > -3$ thus $-3$ is a lower bound for $S$. Let $\varepsilon > 0$ then we seek to find $x \in S$ as below:




Simply use $x = \dfrac{-3+(-3+\varepsilon)}{2} = -3 + \varepsilon/2$. Notice $-3 < -3 + \varepsilon/2 \therefore -3 + \varepsilon/2 \in (-3, \infty)$. Thus $\inf(S) = -3$ by the ε-inf-proposition

my name for it.

(e.) I leave proof to you (it's easy)

(f.) Notice $x^2 - 3x + 2 = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$ and $\left(x - \frac{3}{2}\right)^2 \geq 0$ and $\left(x - \frac{3}{2}\right)^2 = 0$ iff $x = \frac{3}{2}$.

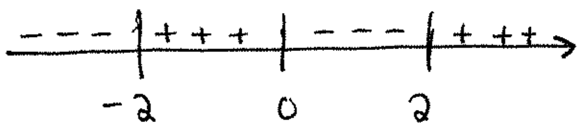
In fact, $\left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \geq \frac{-1}{4} \therefore \frac{-1}{4}$ is lower bound for $S = \{x^2 - 3x + 2 \mid x \in \mathbb{R}\}$. Let $l \geq \frac{-1}{4}$ be lower bound of $S$, then since $\frac{-1}{4} \in S$ we have $l \leq \frac{-1}{4}$ hence $l = -1/4$ and it follows $\inf(S) = \frac{-1}{4}$.

P32) Exercise 1.5.3 g, h

(g.) $\{x \in \mathbb{R} \mid \underbrace{x^3 - 4x < 0}\} = S = (-\infty, -2) \cup (0, 2)$

$x(x^2 - 4) < 0$ $\quad\quad$ $\sup(S) = 2.$

$x(x-2)(x+2) < 0$ $\quad\quad$ $\inf(S) = -\infty.$




(h.) $\{x \in \mathbb{R} \mid 1 \le |x| < 3\} = S = \emptyset$

If $x > 0$ then $1 \le |x| = x \le 3$

If $x < 0$ then $|x| = -x$ and $1 \le -x < 3$ ~~$\Rightarrow -3 > x \ge -1$~~

~~$-1 \le x < -3$~~

~~no such x exists!~~

WRONG!

Thus $S = [1, 3)$

and so,

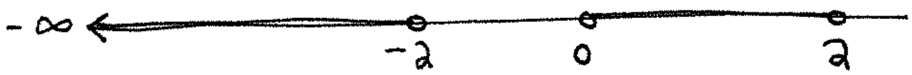
$\sup(S) = 3$

$\cancel{\inf(S) = 1.}$

SORRY!

P32 continued

(g.) I've already shown $2$ is an upper bound for $S = (-\infty, -2) \cup (0, 2)$




Let $\varepsilon > 0$. If $\varepsilon > 2$ then $2 - \varepsilon < 0 < \varepsilon \in S$.
If $\varepsilon < 2$ then $2 - \varepsilon > 0$ thus we may picture


use midpoint $m = \dfrac{2 - \varepsilon + 2}{2} = 2 - \dfrac{\varepsilon}{2}$ for which

$$2 - \varepsilon < 2 - \frac{\varepsilon}{2} < 2$$

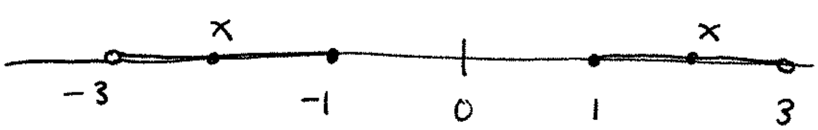
Hence $2 - \varepsilon/2 \in S$ and we've shown $2'$ holds for the <u>ε-sup-prop</u>.

$\therefore \ \sup(S) = 2$.

(h.) $S = \{ x \in \mathbb{R} \mid \underbrace{1 \le |x| < 3}_{} \}$ (I think my old sol$^{\underline{n}}$ has error)

$|x|$ = distance from $0$ to $x$

$S$ is set of points distance at least 1 or 1 and less than 3 from the origin.




$$S = (-3, -1] \cup [1, 3)$$

then by arguments similar to those I offered for (g.)

$$\inf(S) = -3$$
$$\sup(S) = 3.$$

[P33] Exercise 1.5.4

Let $A, B \subseteq \mathbb{R}$ with $A, B \neq \emptyset$ and $A, B$ both bounded above. Let $A+B = \{a+b \mid a \in A \text{ and } b \in B\}$. Prove that $A+B$ is bounded above and $\sup(A+B) = \sup(A) + \sup(B)$

Proof: Since $A, B$ are bounded above $\exists M_A, M_B$ for which $M_A \geq a \ \forall a \in A$ and $M_B \geq b \ \forall b \in B$. Let $x \in A+B$ then $x = a+b$ for some $a \in A$ and $b \in B$, then

$$x = a+b \leq M_A + M_B$$

hence $M = M_A + M_B$ serves as an upperbound for $A+B$.

I'll use Prop. 1.5.1, notice we've already shown $M = M_A + M_B$ satisfies (1')

Next, let $\varepsilon > 0$ and notice $\varepsilon/2 > 0$ hence by Prop. 1.5.1 $\exists a \in A$ and $b \in B$ such that $M_A - \varepsilon/2 < a$ and $M_B - \varepsilon/2 < b$ thus $\exists\, a+b \in A+B$ with $M_A + M_B - \varepsilon < a+b$ which shows (2') for $M$ w.r.t. $A+B$ $\therefore$ $\sup(A+B) = M_A + M_B = \sup(A) + \sup(B)$. //

[P34] Let $\emptyset \neq A \subseteq \mathbb{R}$. Define $-A = \{-a \mid a \in A\}$.

(a.) Suppose $A$ is bounded below by $L$ then $L \leq a \ \forall a \in A$. Thus $-a \leq -L$ for each $a \in A$. If $x \in -A$ then $x = -a \leq -L$. thus $-A$ is bounded above by $-L$.

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(b.) Suppose $A$ is bounded below. Then $-A$ is bounded above hence by the completeness axiom $\sup(-A) \in \mathbb{R}$. Therefore if $M$ is any upper bound of $-A$ we have $\sup(-A) \leq M$.

Notice $-a \leq \sup(-A) \ \forall a \in A$ thus $\underbrace{-\sup(-A)}_{\text{lower bound for } A}$ $\leq a \ \forall a \in A$.
Let $\underline{L \geq -\sup(-A)}_{*}$ be lower bound of $A$ then $-L$ is upper bound of $-A$ hence $\sup(-A) \leq -L \Rightarrow \underline{L \leq -\sup(-A)}_{**}$

P34 continued

By $*$ and $**$ we find $L = -\sup(-A)$ is the greatest lower bound of $A$

That is, $\inf(A) = -\sup(-A)$. //

P35 Exercise 1.5.8 (also P36 oops ☺)

Let $\phi \neq A, B \subseteq \mathbb{R}$ and suppose $A$ & $B$ are bounded below. Prove $A \subseteq B$ implies $\inf(A) \geq \inf(B)$

Suppose $A \subseteq B$ and $A \neq \phi$, $B \neq \phi$ are subsets of $\mathbb{R}$ which are bounded below. From 1.5.5. (b) we note $\inf(A), \inf(B) \in \mathbb{R}$.

Let $x \in A$ then $x \in B$ thus $\inf(B) \leq x$ thus $\inf(B)$ serves as a lower bound for $A$. Consequently $\inf(B) \leq \inf(A)$ as $\inf(A)$ is larger than any other lower bound of $A$.

P37 Exercise 1.6.1(a) $S = \left\{ \frac{3n}{n+4} \mid n \in \mathbb{N} \right\} = \left\{ \frac{3}{5}, \frac{6}{6}, \frac{9}{7}, \frac{12}{8}, \frac{15}{9}, \dots \right\}$

It appears $\frac{3}{5} \leq \frac{3n}{n+4} < 3$ and I expect we can prove $\inf(S) = 3/5$ and $\sup(S) = 3$.

P37 continued

Consider $\frac{3n}{n+4} = \frac{3(n+4)-12}{n+4} = 3 - \frac{12}{n+4}$

If $n \geq 1$ then $n+4 \geq 5$ thus $\frac{1}{n+4} \leq \frac{1}{5} \Rightarrow \frac{-12}{n+4} \geq \frac{-12}{5}$

hence $\frac{3n}{n+4} = 3 - \frac{12}{n+4} \geq 3 - \frac{12}{5} = \frac{15-12}{5} = \frac{3}{5}$

Therefore $\frac{3}{5}$ bounds $S = \{ \frac{3n}{n+4} \mid n \in \mathbb{N} \}$ below. Moreover $\frac{3}{5} \in S$. Suppose $l \geq \frac{3}{5}$ is an ~~upper~~ lower bound for $S$. Since $\frac{3}{5} \in S$ we have $l \leq \frac{3}{5}$ thus $l = \frac{3}{5}$ and we conclude $\inf(S) = 3/5$.

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Notice for $n \in \mathbb{N}$, $\frac{3n}{n+4} < \frac{3n}{n} = 3$ $\therefore$ 3 is upperbound of $S$.

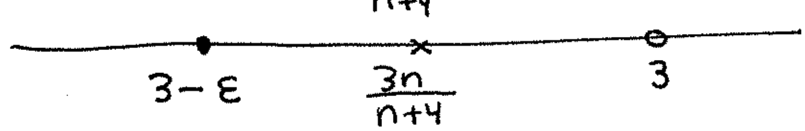
Let $\varepsilon > 0$ then $\varepsilon/12 > 0$ thus $\exists n \in \mathbb{N}$ for which

$$\frac{\varepsilon}{12} > \frac{1}{n} > \frac{1}{n+4} \quad \text{by Archimedean Principle}$$

Thus $\frac{12}{n+4} < \varepsilon \Rightarrow \frac{-12}{n+4} > -\varepsilon \Rightarrow 3 - \frac{12}{n+4} > 3 - \varepsilon$

But $3 - \frac{12}{n+4} = \frac{3n}{n+4} \in S$. To illustrate:

$$3 - \varepsilon < \frac{3n}{n+4} < 3$$




Thus by Prop. 1.5.1 we've shown 1' and 2' and it follows that $\sup(S) = 3$.

$\boxed{P38}$ $\boxed{\text{Ex. } 1.6.2}$ Let $r \in \mathbb{Q}$ such that $0 < r < 1$. Prove $\exists n \in \mathbb{N}$ such that $\frac{1}{n+1} < r \leq \frac{1}{n}$

By Th^m 1.6.2 (d.) we know for any $x \in \mathbb{R}$, $\exists m \in \mathbb{Z}$ such that $m-1 \leq x < m$. Consider $x = \frac{1}{r} \in \mathbb{R}$ as $r \neq 0$ then $\exists m \in \mathbb{Z}$ s.t. $m-1 \leq \frac{1}{r} < m$. Let $m-1 = n$ and notice $n \leq \frac{1}{r} < n+1$.

We can prove $n \geq 1$. Then,

$$n \leq \frac{1}{r} \Rightarrow r \leq \frac{1}{n} \quad \& \quad \frac{1}{r} < n+1 \Rightarrow \frac{1}{n+1} < r$$

Therefore, $\frac{1}{n+1} < r \leq \frac{1}{n}$. It remains to prove $n \in \mathbb{N}$.

We already have $n \in \mathbb{Z}$. Recall $n = m-1$ and we know $m-1 \leq \frac{1}{r} < m \Rightarrow n \leq \frac{1}{r} < n+1$

But, $0 < r < 1$ so $1 < \frac{1}{r} \Rightarrow \frac{1}{r} - 1 > 0$

Yet $\frac{1}{r} - 1 < n$ thus $0 < \frac{1}{r} - 1 < n \Rightarrow n \geq 1$

$\therefore n \in \mathbb{N}$. //

P39 Ex. 1.6.3

Let $x \in \mathbb{R}$. Prove for every $n \in \mathbb{N}$, $\exists r \in \mathbb{Q}$ such that $|x-r| < \frac{1}{n}$.

Consider $x \in \mathbb{R}$.

Let $n \in \mathbb{N}$ and consider, $\frac{1}{n} > 0$ hence:


thus $x - \frac{1}{n} < x + \frac{1}{n}$ and by Thm 1.6.3., $\exists r \in \mathbb{Q}$ such that $x - \frac{1}{n} < r < x + \frac{1}{n}$. Therefore,

$$x - r - \frac{1}{n} < 0 < x - r + \frac{1}{n}$$

Hence $x - r < \frac{1}{n}$ and $-\frac{1}{n} < x - r$ $\therefore$ $|x-r| < \frac{1}{n}$. //

P40 Ex. 1.6.4 Prove that if $x \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$ then $x + y \in \mathbb{R} - \mathbb{Q}$. What can you say about $xy$?

Suppose $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ $(y \in \mathbb{R})$. Suppose $x + y \in \mathbb{Q}$ towards a $\rightarrow\leftarrow$. If $x + y = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ where $n \neq 0$ then observe $x \in \mathbb{Q}$ hence $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ where $q \neq 0$. Thus,

$$y = \frac{m}{n} - x = \frac{m}{n} - \frac{p}{q} = \frac{mq - pn}{nq} \in \mathbb{Q}$$

Yet $y \notin \mathbb{Q}$ hence $\rightarrow\leftarrow$ and we conclude $x + y \notin \mathbb{Q}$. That is $x + y \in \mathbb{R} - \mathbb{Q}$ ($x + y$ is irrational).

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$xy$ may or may not be irrational.

$x = 0 \in \mathbb{Q}$ and $\overset{x}{0}\,(\overset{y}{\pi}) = 0$ where $\pi \in \mathbb{R} - \mathbb{Q}$.

$x = 1 \in \mathbb{Q}$ and $\underset{x}{1}\,(\underset{y}{\pi}) = \pi$

So $xy = 0 \in \mathbb{Q}$ possible & $xy = \pi \in \mathbb{R} - \mathbb{Q}$ also possible.