MH 408 : TOPICS IN ANALYSIS : TEST 1 SOLUTION

(1.) Suppose $A \subseteq B$ and $C \subseteq D$. Let $x \in A \times C$ then $\exists a \in A$ and $c \in C$ such that $x = (a, c)$. However, $a \in A \subseteq B$ implies $a \in B$ and $c \in C \subseteq D$ implies $c \in D$. Thus $x = (a, c) \in B \times D$ and we've shown $A \times C \subseteq B \times D$.

(2.) Define $x \sim y$ for $x, y \in \mathbb{Z}$ iff $\exists j \in \mathbb{Z}$ for which $y - x = 4j$. Let $x \in \mathbb{Z}$ and note $x - x = 4(0)$ for $0 \in \mathbb{Z}$ hence $x \sim x$ and we find $\sim$ is reflexive. Next, suppose $x \sim y$ then $\exists j \in \mathbb{Z}$ such that $y - x = 4j \Rightarrow x - y = 4(-j)$ where $-j \in \mathbb{Z}$. Thus $y \sim x$ and we've shown $\sim$ is symmetric. Consider $x, y, z \in \mathbb{Z}$ such that $x \sim y$ and $y \sim z$. Hence $\exists j, k \in \mathbb{Z}$ for which $y - x = 4j$ and $z - y = 4k$. Consider,

$$z - x = z - y + y - x = 4k + 4j = 4(k + j)$$

and as $k + j \in \mathbb{Z}$ we've shown $x \sim z$ hence $\sim$ transitive. Therefore, $\sim$ is an <u>equivalence relation</u>.

equivalence classes for $\sim$ there are 4 distinct classes.

$$\begin{aligned}
[0] &= \{x \in \mathbb{Z} \mid x \sim 0\} \\
&= \{x \in \mathbb{Z} \mid x = 4j \text{ for some } j \in \mathbb{Z}\} \\
&= \{4j \mid j \in \mathbb{Z}\} \\
&= 4\mathbb{Z}.
\end{aligned}$$

Likewise, $[1] = 4\mathbb{Z} + 1$, $[2] = 2 + 4\mathbb{Z}$, $[3] = 3 + 4\mathbb{Z}$.

$4\mathbb{Z} = \{0, \pm 4, \pm 8, \dots\}$ $\quad 2 + 4\mathbb{Z} = \{\dots, 2, 6, 10, \dots\}$

$1 + 4\mathbb{Z} = \{\dots, 1, 5, 9, \dots\}$ $\quad 3 + 4\mathbb{Z} = \{\dots, 3, 7, 11, \dots\}$

(3.) $f(x) = \frac{x}{x-2} = \frac{x-2+2}{x-2} = 1 + \frac{2}{x-2}$

It follows range $(f(x)) = \mathbb{R} - \{1\} = (-\infty, 1) \cup (1, \infty)$.

We choose $B = (-\infty, 1) \cup (1, \infty)$. Now lets show

$f : (-\infty, 2) \cup (2, \infty) \longrightarrow (-\infty, 1) \cup (1, \infty)$ is a bijection.

Let $a, b \in \text{dom}(f)$ and suppose $f(a) = f(b)$ then $a, b \neq 2$ and,

$$\frac{a}{a-2} = \frac{b}{b-2} \Rightarrow a(b-2) = b(a-2)$$
$$\Rightarrow ab - 2a = ba - 2b$$
$$\Rightarrow -2a = -2b$$
$$\Rightarrow a = b \quad \therefore \underline{f \text{ is injective}}.$$

Easy way to prove onto is to calculate the inverse fnct.

$$y = \frac{x}{x-2} \Rightarrow y(x-2) = x$$
$$\Rightarrow yx - x = 2y$$
$$\Rightarrow x(y-1) = 2y$$
$$\Rightarrow x = f^{-1}(y) = \frac{2y}{y-1}$$

Notice, $\text{dom}(f^{-1})$ is $\mathbb{R} - \{1\}$. Since $\text{dom}(f^{-1})$ is range(f) this is a good check on our work.

Let $x \in B$ then $x \neq 1$ hence

$\frac{2x}{x-1} \in \mathbb{R}$. However, we need $\frac{2x}{x-1} \neq 2$

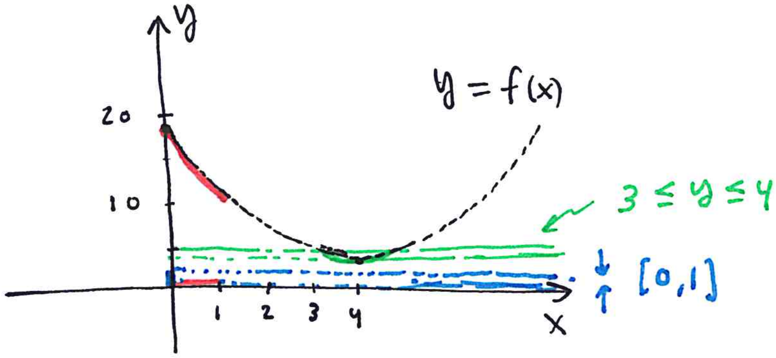
Notice $\frac{2x}{x-1} = 2 \Rightarrow 2x = 2(x-1) \Rightarrow 0 = -2$

hence $\frac{2x}{x-1} \neq 2$ and we see $\frac{2x}{x-1} \in \text{dom}(f)$.

Finally, observe $f\left(\frac{2x}{x-1}\right) = \dfrac{\frac{2x}{x-1}}{\frac{2x}{x-1} - 2} = \frac{2x}{2x - 2(x-1)} = \frac{2x}{2} = x$.

Thus f is surjective and hence $\underline{f \text{ is a bijection}}$.

(4.) $f(x) = 3 + (x-4)^2.$



(a.) $f([0,1]) = \{ 3 + (x-4)^2 \mid x \in [0,1] \}$

$= [f(1), f(0)]$

$= [3 + (-3)^2, 3 + (-4)^2]$

$= \underline{[12, 19]}.$

(b.) $f^{-1}([0,1]) = \{ x \in \text{dom}(f) \mid f(x) \in [0,1] \}$

$= \{ x \mid 0 \le 3 + (x-4)^2 \le 1 \}$

$= \{ x \mid -3 \le (x-4)^2 \le -2 \}$

$= \emptyset$ (could also see from graph since outputs of f(x) start at y = 3 and go up from there)

(c.)

$f^{-1}([3,4]) = \{ x \mid 3 \le 3 + (x-4)^2 \le 4 \}$

$= \{ x \mid 0 \le (x-4)^2 \le 1 \}$

$= \underline{[3, 5]}.$ these inputs produce points on y = f(x) between y = 3 and y = 4.

(5.) Let $f : X \to Y$ be a function. Suppose $A, B \subseteq X$.

Assume $A \subseteq B$. Let $y \in f(A)$ then by definition of image, $\exists a \in A$ such that $y = f(a)$. However, $a \in A \subseteq B$ implies $a \in B$ by definition of subset. Therefore $y = f(a)$ where $a \in B$ and by Def^{n} of image we note $y \in f(B)$. Thus $f(A) \subseteq f(B)$. //

(6.) Let P_n be claim $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Note $1^3 = \left[\frac{1(2)}{2}\right]^2$ hence P_1 true. Suppose inductively that P_n true for some $n \in \mathbb{N}$.

Examine

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \quad : \text{by induct. hypoth.}$$

$$= (n+1)\cdot(n+1)^2 + \left[\tfrac{1}{2} n(n+1)\right]^2$$

$$= \frac{(n+1)^2}{4}\left[4(n+1) + n^2\right]$$

$$= \frac{(n+1)^2}{4}\left[(n+2)^2\right]$$

$$= \left[\frac{(n+1)(n+1+1)}{2}\right]^2 \quad \therefore \; P_n \Rightarrow P_{n+1}$$

and we conclude by PMI that $\forall n \in \mathbb{N}$

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

(7.) $S = \left\{ \frac{2n}{n+3} \,\middle|\, n \in \mathbb{N} \right\} = \left\{ \frac{2}{4}, \frac{4}{5}, \frac{6}{6}, \frac{8}{7}, \frac{10}{8}, \dots \right\}$

Notice $\dfrac{2n}{n+3} = \dfrac{2(n+3)-6}{n+3} = 2 - \dfrac{6}{n+3} \leq 2$

for $n \in \mathbb{N}$. Thus 2 is an upper bound of S.

Claim $\frac{1}{2} \leq \frac{2n}{n+3}$ for all $n \in \mathbb{N}$. This claim can be verified as follows: if $n \in \mathbb{N}$ then $n \geq 1$ hence $n + 3 \leq n + 3n = 4n \Rightarrow 1 \leq \frac{4n}{n+3} \Rightarrow \frac{1}{2} \leq \frac{2n}{n+3}$.

Therefore $\frac{1}{2}$ serves as lower bound for S.

We can prove $\sup(S) = 2$ and $\inf(S) = \frac{1}{2}$.

Let $l \geq \frac{1}{2}$ be lower bound of S then since $\frac{1}{2} \in S$ we find $l \leq \frac{1}{2}$. Hence $l \leq \frac{1}{2}$ and $l \geq \frac{1}{2} \Rightarrow l = 1/2$.

Consequently, $\inf(S) = \frac{1}{2}$.

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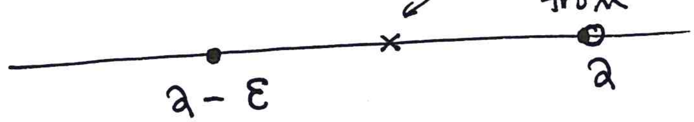
We've shown 2 is an upper bound for S. Let's use the ε-proposition to finish the argument. Let $\varepsilon > 0$ and examine,

need to produce something from S here.



We seek $n \in \mathbb{N}$ for which

$$2 - \varepsilon < 2 - \frac{6}{n+3} < 2$$

Note $\varepsilon/6 > 0$ hence $\exists\, n \in \mathbb{N}$ for which $\frac{\varepsilon}{6} > \frac{1}{n} > \frac{1}{n+3}$

by Archemedian Principle Thm

Thus $\varepsilon > \frac{6}{n+3} \Rightarrow -\varepsilon < \frac{-6}{n+3} \Rightarrow 2 - \varepsilon < 2 - \frac{6}{n+3} < 2$

$\therefore\ 2 - \frac{6}{n+3} \in S$ and this proves $\sup(S) = 2$. //