APPLICATION TO PHYSICS EXAMPLES

Let P denote <u>pressure</u>.

$$P = \frac{dF}{dA}$$

infinitesimal force dF
infinitesimal area dA

To calculate net-force on some region R due to force F from some sort of gas or liquid we simply find $dF = P\,dA \Rightarrow F = \int_R P\,dA$

The pressure due to water is proportional to the <u>depth</u>. For example, the water deeper down pushes more forcefully on a dam then the water near the surface. However, the water at a <u>fixed</u> depth pushes in a uniform fashion.

$$P = \rho g d = \delta d$$

density of water

acceleration of gravity

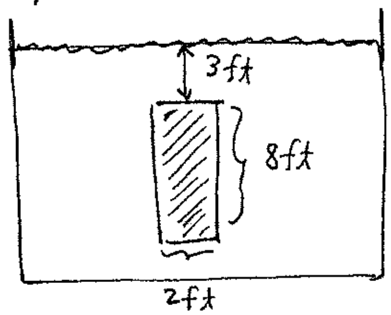
depth

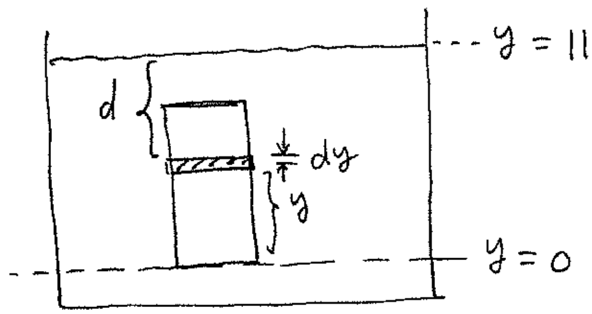
weight-density $\delta = 62.5 \frac{\text{lb}}{\text{ft}^3}$

$\rho = 1000 \frac{\text{kg}}{\text{m}^3}$ $\qquad g = 9.8 \frac{\text{m}}{\text{s}^2}$

<u>Remark</u>: these eq$^{\text{n}}$'s make sense in part because the density of water is independent of depth. Perhaps this is a bit surprising. In contrast, the equations for gas in the atmosphere would not have this feature, air is less dense at high altitudes.

1.) find hydrostatic force on window in aquarium pictured below,





- choose coordinates which are natural. Be sure to define them with picture or words.
- work out dA and P in terms of your coord. Need to express depth d to calculate P correctly.

$$dA = 2\,dy$$
$$d = 11 - y$$

$$dF = P\,dA = \delta\, d\, dA$$
$$dF = \delta(11-y)\,2\,dy \quad \text{for} \quad 0 \le y \le 8$$

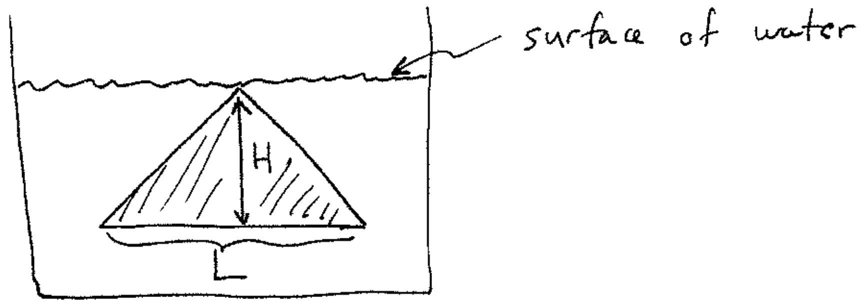
Thus we find hydrostatic force,

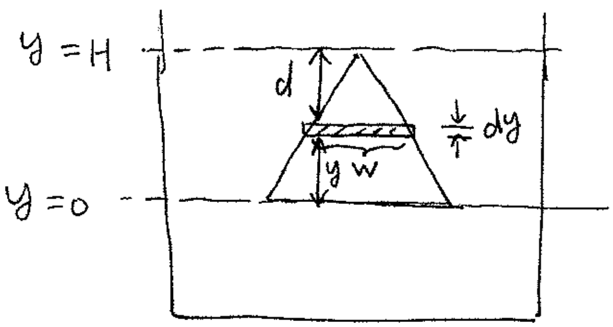
$$
\left.
\begin{aligned}
F &= \int_0^8 2\delta(11-y)\,dy \\
&= 2\delta\left[11y - \tfrac{1}{2}y^2\right]\Big|_0^8 \\
&= 2\delta\,[88 - 32] \\
&= 112\delta \\
&= 112(62.5)
\end{aligned}
\right\} \text{ Some units omitted}
$$

$= \boxed{7000 \text{ lbs}}$ ← answer <u>must</u> have correct units.

<u>CAUTION</u>: when omitting units beware of multipliers & inconsistently presented data (inches vs. ft etc.). In a science course such as PHYSICS you might be expected to <u>manifest</u> units in calculations & answers.

2.) Find force on pictured aquarium window in terms of given constants L and $\mathbf{H}$ and ρ, g.





- draw picture to introduce coordinates.
- set-up $dA = w\,dy$
  Here w is <u>linear</u> <u>fnct.</u> of y notice
  $w = L$ when $y = 0$
  $w = 0$ when $y = H$
  this suggests we write
  $$\underbrace{w = L - \left(\frac{L}{H}\right)y}_{\text{check it, } w(0) = L \checkmark,\ w(H) = 0 \checkmark}$$

$$dA = \left(L - \frac{Ly}{H}\right)dy$$

$$dF = P\,dA, \quad P = \rho g d, \quad d = H - y$$

$$\therefore \quad \underbrace{dF = \rho g L\left(1 - \frac{y}{H}\right)(H-y)\,dy}_{0 \le y \le H}$$

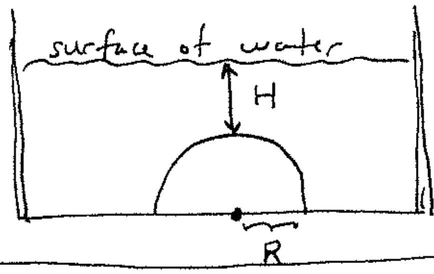
$$F = \int_0^H \rho g L\left(H - y - y + y^2/H\right)dy$$

$$= \rho g L\left(Hy - y^2 + \frac{1}{3H}y^3\right)\Big|_0^H$$

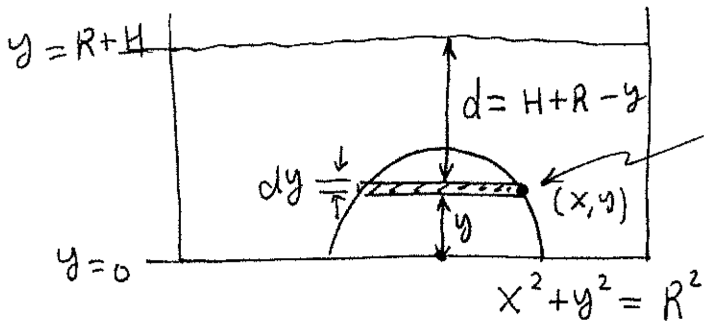
$$= \rho g L\left(H^2 - H^2 + \frac{1}{3H}H^3\right)$$

$$= \boxed{\frac{\rho g L H^2}{3}}$$

3.) Find force on half-circular window at depth H



give answer in terms of ρ, g, H and R.



$$dA = 2x\,dy = 2\sqrt{R^2-y^2}\,dy \quad *$$

for $0 \le y \le R$.

$$P = \rho g (H+R-y)$$

$$dF = P\,dA = \rho g (H+R-y)\cdot 2\sqrt{R^2-y^2}\,dy$$

$$F = 2\rho g \int_0^R (H+R-y)\sqrt{R^2-y^2}\,dy$$

$$= 2\rho g \left(\int_0^R (H+R)\sqrt{R^2-y^2}\,dy - \int_0^R y\sqrt{R^2-y^2}\,dy \right)$$

$$= \rho g (H+R) \int_0^R \underbrace{2\sqrt{R^2-y^2}\,dy}_{dA\,*} + \rho g \int_0^R \underbrace{-2y\,dy}_{du} \sqrt{\underbrace{R^2-y^2}_{u}}$$

$u(0) = R^2$
$u(R) = 0$

$$= \rho g (H+R)\left(\pi R^2/2\right) + \rho g \int_{R^2}^0 \sqrt{u}\,du$$

$$= \frac{1}{2}\pi R^2 (H+R)\rho g + \frac{2}{3}\rho g\, u^{3/2}\Big|_{R^2}^0$$

$$= \boxed{\left(\frac{1}{2}\pi R^2 (H+R) - \frac{2}{3}R^3\right)\rho g}$$

__Remark:__ you can compute $\int_0^R \sqrt{R^2-y^2}\,dy$ directly, but I chose to recall our knowledge that the area of a circle with radius R is πR^2.

Centroid, Center of Mass and Moments about origin in plane

Suppose $\rho = \frac{dm}{dA}$ is the mass per unit area, a.k.a. mass area density then $dm = \rho\, dA$ so,

$$M = \int_R \rho\, dA \quad : \text{ gives mass in region } R$$

Then, the <u>center of mass</u> (c.o.m.) is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{M} \underbrace{\int_R x \rho\, dA}_{M_x} \quad \& \quad \bar{y} = \frac{1}{M} \underbrace{\int_R y \rho\, dA}_{M_y}$$

The moments M_x, M_y are related to total mass in R of M by $\bar{x} = M_x / M$ and $\bar{y} = M_y / M$.

> **Defn/** <u>centroid</u> of R is the center of mass if we take $\rho = 1$ in which case $M = \int_R dA$ = Area of R.

In fact, if ρ is constant in R then the c.o.m. is in the same location as the centroid.

- When $R = \{ (x,y) \mid a \leq x \leq b,\ g(x) \leq y \leq f(x) \}$ then

$$M = \int_a^b \rho \left(f(x) - g(x) \right) dx$$

$$M_x = \int_a^b \rho x \left(f(x) - g(x) \right) dx$$

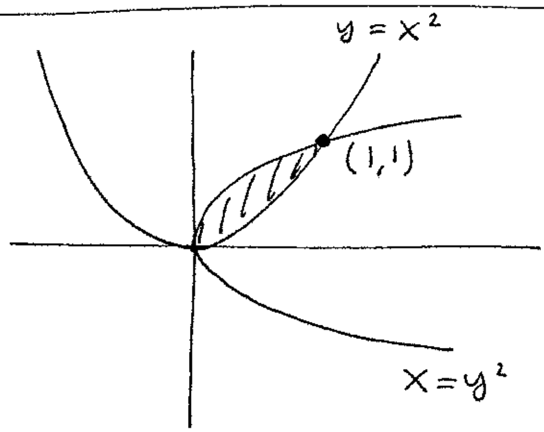
$$M_y = \int_a^b \rho \frac{1}{2} \left[(f(x))^2 - (g(x))^2 \right] dx$$

BEST EXPLAINED IN PHYSICS WITH PARTICLE CONCEPT...

FINAL COMMENT

- centroid is geometric center
- center of mass is balance point

4.) Find centroid of region bounded by $y = x^2$ & $x = y^2$



$0 \le x \le 1$

$x^2 \le y \le \sqrt{x}$

$$A = \int_0^1 (\sqrt{x} - x^2)\,dx$$
$$= \frac{2}{3} - \frac{1}{3}$$
$$= \frac{1}{3} = M$$

$$M_x = \int_0^1 x(\sqrt{x} - x^2)\,dx$$
$$= \int_0^1 (x^{3/2} - x^3)\,dx$$
$$= \frac{2}{5} - \frac{1}{4}$$
$$= \frac{3}{20}$$

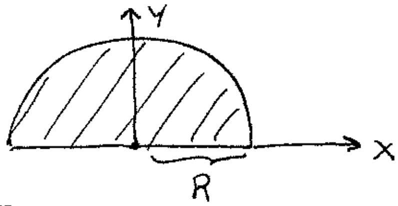
$$M_y = \int_0^1 \frac{1}{2}\left[(\sqrt{x})^2 - (x^2)^2\right]dx$$
$$= \frac{1}{2}\int_0^1 (x - x^4)\,dx$$
$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{5}\right]$$
$$= \frac{3}{20}$$

notice, by symmetry of the region we could have claimed this w/o calculation!

Thus $(\bar{x}, \bar{y}) = \left(\frac{M_x}{M}, \frac{M_y}{M}\right) = \left(\frac{3/20}{1/3}, \frac{3/20}{1/3}\right)$

the $\boxed{\text{centroid is at } (9/20, 9/20)}$

5.) Find the center of mass of pictured half disk of constant density ρ



$$M = \int_{\frac{1}{2}\text{disk}} \rho \, dA = \rho \int_{\frac{1}{2}\text{disk}} dA = \underline{\frac{\rho}{2}\pi R^2 = M}.$$

$$M_y = \int_{-R}^{R} \frac{\rho}{2}\left[\left(\sqrt{R^2 - x^2}\right)^2 \right] dx \quad \text{since } 0 \le y \le \sqrt{R^2 - x^2} \text{ for } -R \le x \le R$$

$$= \rho \int_0^R (R^2 - x^2)\, dx$$

$$= \rho\left(R^3 - \frac{1}{3}R^3\right)$$

$$= \frac{2}{3}R^3\rho$$

Notice $\bar{x} = 0$ by symmetry. Whereas

$$\bar{y} = \frac{M_y}{M} = \frac{\frac{2}{3}R^3\rho}{\frac{\pi}{2}R^2\rho} = \underline{\frac{4R}{3\pi} = \bar{y}}.$$

Thus $\boxed{(0,\ 4R/3\pi) \text{ is the c.o.m.}}$