§III.2 (3.2) Freitag (Power Series)

Prop. III.2.1 For each power series $\underbrace{a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots}_{S(z)}$

$\exists!\ r \in [0,\infty] = [0,\infty) \cup \{\infty\}$ s.t.

a) the series $S(z)$ converges normally in the open disk $U_r(a) = \{z \in \mathbb{C} \mid |z-a| < r\}$.

b) the series $S(z)$ diverges for each $z \in \mathbb{C}$ with $|z-a| > r$.
($z \notin \overline{U_r(a)}$).

Cor. III.2.1.1 A power series defines an analytic fuct point wise on its open-disk of convergence.
($f : U_r(a) \to \mathbb{C}$ analytic $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$.)

$\boxed{\text{E65}}$ $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$. Convergent on $|z| \le 1$.
$\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges by $p=2$ series test.
($\sum_{n=0}^{\infty} f_n(z)$ converges uniformly on U if we have M_n with $\sum_{n=0}^{\infty} M_n$ converges & for each $z \in U$ $|f_n(z)| \le M_n$ $\forall n \in \mathbb{N} \cup \{0\}$.)

$\boxed{\text{E66}}$ $\sum_{n=0}^{\infty} z^n$ converges on $|z| < 1$.
(geometric series with radius z diverges on $|z| = 1$)

$\boxed{\text{E67}}$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = \frac{\text{what? (!)}}{\text{Log}(1+z)}$
has $r=1$ and converges at all z with $|z| = 1$ except $z = -1$
($p=1$ harmonic series).

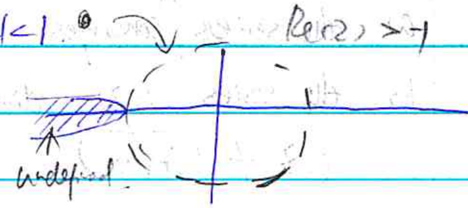
about what?

$g(z) = \mathrm{Log}(1+z)$ need $1+z \in \mathbb{C}$ $(\mathrm{Re}(1+z) \not\leq 0)$.

$g'(z) = \dfrac{1}{1+z}$ $\quad$ ($z \in \mathbb{C}$ s.t. if $\mathrm{Im}(z+1)=0$ then $\mathrm{Re}(z+1)>0$)

$= \dfrac{1}{1-(-z)}$ $\quad$ $\mathrm{Re}(z)+1>0$

$= \sum_{n=0}^{\infty} (-z)^n$ $\quad |z|<1$ $\quad$ $\mathrm{Re}(z) > -1$



$\dfrac{dg}{dz} = \sum_{n=0}^{\infty} (-1)^n z^n$

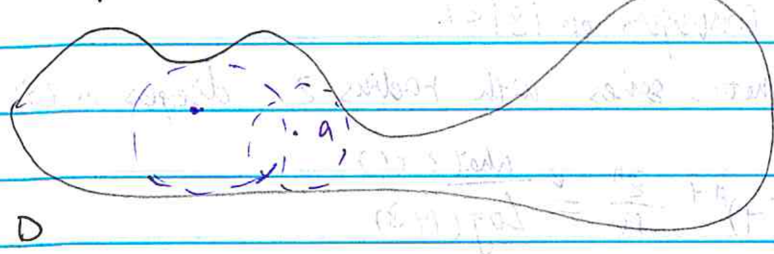
$\Rightarrow g(z) = C + \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n+1} z^{n+1}$

Note $g(0) = \mathrm{Log}(1) = 0 = C+0$ $\therefore C=0$

$\therefore \mathrm{Log}(z+1) = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n+1} z^{n+1}$

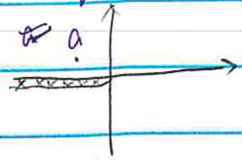
let $j = n+1$, $n = j-1$ $\quad \hookrightarrow \sum_{j=1}^{\infty} \dfrac{(-1)^{j-1}}{j} z^j = \mathrm{Log}(z+1)$

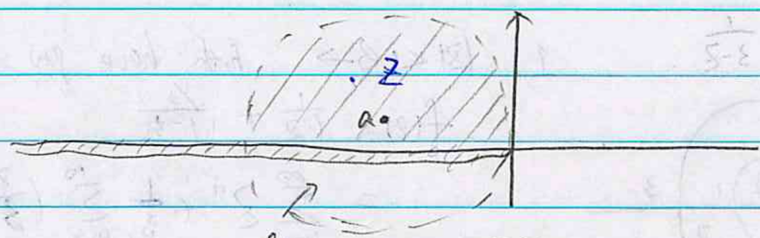
Th^m/ If $f: D \subseteq \mathbb{C} \to \mathbb{C}$ is analytic then at each $a \in D$.
$\exists\, r>0$ s.t.
$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ for each $z \in U_r(a)$.



$\boxed{E68}$ $\log(z) = \mathrm{Log}(a) + \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{a^n n} (z-a)^n$. $\quad$ ($a \in (0,\infty)$ $\cancel{a \in \mathbb{R}}$). $\quad$ ratio.

has conv. radius $r = |a|$. $\quad$ geometric





$g(z) = \log(z) + 2\pi i(z)$

[Z6]

$g(z) = \log(z)$ want center somewhere.

$g'(z) = \frac{1}{z} = \boxed{\frac{1}{a+z-a}} = \frac{1/a}{1-\left(\frac{z-a}{a}\right)} = \frac{\frac{1}{a}}{1-\left(\frac{a-z}{a}\right)}$ $\quad \left|\frac{a-z}{a}\right| < 1$, $|z-a| < a$

$= \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{z-a}{a}\right)^n (-1)^n$

[Z7.]

$f_1(z) = \frac{\sin(z)}{z}$ $\quad \mathrm{dom}(f_1) = \mathbb{C} - \{0\} = \mathbb{C}^\circ$

$f_2(z) = \frac{z}{z-1}$ $\quad \mathrm{dom}(f_2) = \mathbb{C} - \{1\}$.

$f_3(z) = \sin\left(\frac{1}{z}\right)$ $\quad \mathrm{dom}(f_3) = \mathbb{C}^\circ$

$\hat{f}_1(0) = \lim_{z\to 0}\left(\frac{\sin z}{z}\right) = 1.$ (L'Ret)

$f_1(z) = 1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 + \cdots$

$\left(\sin(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots \right.$

$= 1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 + \cdots$

A removable singularity can → analytic

$\frac{z}{z-1} = \frac{z-1+1}{z-1} = \frac{z-1}{z-1} + \frac{1}{z-1} = 1 + \frac{1}{z-1}$

$\therefore f_2(z) = \frac{z}{z-1} = 1 + \frac{1}{z-1}$ (Laurent series pole of order 1)

$g_2(z) = (z-1)f_2(z)$ ← removeable singularity at $z=1$.

$f_3(z) = \sin\left(\frac{1}{z}\right)$

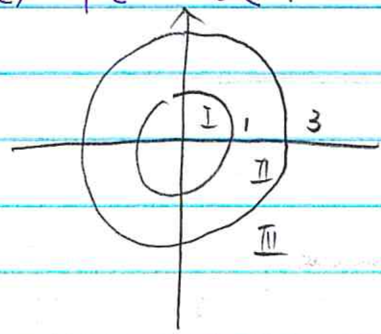
essential singularity $z=0$. $\quad$ take all but 1. near the point to $\mathbb{C}$-plane

$f_3(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1}$

$= \left(\sum_{m=0}^{\infty}\frac{(-1)^m}{(-2m+1)!}z^{-1+2m}\right)$ (?)

$= \frac{1}{z} - \frac{1}{6}\frac{1}{z^3} + \frac{1}{120}\frac{1}{z^5} - \frac{1}{7!}\frac{1}{z^7} + \cdots$

$\boxed{271}$ $f(z) = \frac{1}{1-z} + \frac{1}{3-z}$.



I) $|z| < 1, 3 \to$ both have pov Series

$$f(z) = \frac{1}{1-z} + \frac{1/3}{1-z/3}$$

$$= \sum_{n=0}^{\infty} z^n + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(1+\frac{1}{3^{n+1}}\right)z^n. \quad \left(\begin{array}{l}\text{Talyer Series}\\ \text{Concentral at } z=0\end{array}\right)$$

no singular point

II) $1 < |z| < 3$. $\to |z/3| < 1$, $\to |z| > 1 \to \frac{1}{|z|} < 1$.

$\cancel{|z/3|}$

$$f(z) = \frac{1}{z(\frac{1}{z}-1)}$$

$$= \frac{1}{-z(\frac{-1}{z}+1)} + \frac{1/3}{1-\frac{z}{3}}$$

$$= \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$= -\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} + \sum_{n=0}^{\infty}\frac{1}{3^{n+1}}z^n \quad \text{have singu}$$

III). $f(z) = \frac{1}{1-z} + \frac{1}{3-z} = \frac{1}{1-}$

$$= \frac{1}{-z(1-\frac{1}{z})} + \frac{1}{-z(1-3/z)}$$

$$= \frac{-1}{z}\left(\sum_{n=0}^{\infty}(z^{-1})^n + \sum_{n=0}^{\infty}(3z^{-1})^n\right)$$

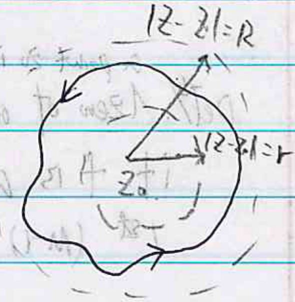
$$= -\sum_{n=0}^{\infty}\left((1+3^n)\left(\frac{1}{z}\right)^{n+1}\right)$$

Th^m/. Let f be analytic in the annulus $r < |z - z_0| < R$.
then f can be expressed as

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}$$

where both Σ converge in the annulus, and conv. uniformly in any closed sub-annulus $(r < r' \le |z - z_0| \le R' < R)$.

Then the coeff. are given by:



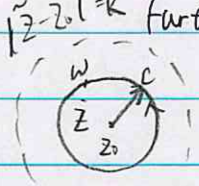
$$a_j = \frac{1}{2\pi i} \oint_c \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{j+1}}$$

for $j = 0, \pm 1, \pm 2, \ldots$

Th^m/. Given a convergent Laurent Series on an annulus there exist a function $f(z) = \sum_{-\infty}^{\infty} b_n (z - z_0)^n$ which is analytic on the annulus.

Th^m/ If f is analytic in a disk $|z - z_0| < R$. Then the Taylor Series. $\left(\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \right)$ Converges to $f(z)$ $\forall z$ in the disk.

$|z - z_0| = R$ Furthermore, the convergence is uniform in any closed subdisk.



$$f(z) = \frac{1}{2\pi i} \oint_c \frac{f(w)\, dw}{w - z} = \frac{1}{2\pi i} \oint \sum_{n=0}^{\infty} \frac{f(w) (z - z_0)^n\, dw}{(w - z_0)^{n+1}} \longrightarrow \text{back.}$$

$|z - z_0| < |w - z_0|$

$< R$

$= |z - z_0|$

$$\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = \frac{1}{(w - z_0)\left[1 - \left(\frac{z - z_0}{w - z_0}\right)\right]} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right.$$

$\cancel{w<z}$.
$$\left[\frac{1}{w - z} = \frac{1}{w - z_0 - (z - z_0)} = \frac{-1}{(z - z_0)\left[1 - \left(\frac{w - z_0}{z - z_0}\right)\right]} = \frac{-1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{w - z_0}{z - z_0} \right)^n \right]$$

xx

$$\left| \frac{z - z_0}{w - z_0} \right| < 1 \longleftrightarrow |z - z_0| < |w - z_0|$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi i}\oint \sum_{n=0}^{\infty} \frac{f(w)(z-z_0)^n\,dw}{(w-z_0)^{n+1}}$$

$$= \sum_{n=0}^{\infty}\Bigg(\underbrace{\frac{1}{2\pi i}\oint \frac{f(w)\,dw}{(w-z_0)^{n+1}}}_{\frac{f^{(n)}(z_0)}{n!}}(z-z_0)^n\Bigg)$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}$$

Def^{n}. a point z_0 is a zero of order m for fnct. f.
if f is analytic z_0 and if the
1st $(m-1)$ derivatives vanish at z_0. but $f^{(m)}(z_0) \neq 0$.

$\boxed{E_2}$ $f(z) = z^{41}\sin(z)$
$z_0 = 0$ is a zero of order $\underline{42}$

$$f(z) = z^{41}\left(z - \tfrac{1}{6}z^3 + \tfrac{1}{120}z^5 + \cdots\right)$$

$$= z^{42} - \tfrac{1}{6}z^{44} + \tfrac{1}{120}z^{46} + \cdots = f(0) + f'(0)z + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \cdots$$

$$= z^{42}(1 - \tfrac{1}{6}z^2 + \cdots$$

Clear $f^{(n)}(0) = 0$ for $n = 1, 2, \cdots 41$.
but $f^{(42)}(0) = (42)! \neq 0$.

(~~Th^{m}. f has zero of order m at z_0 iff~~ (analytic at z_0))

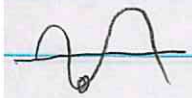
Th^{m} Let f be analytic at z_0. Then f has zero of order m at z_0.
iff $f(z) = (z-z_0)^m g(z)$. where g is analytic at z_0 and $g(z_0) \neq 0$.

Def^{n}. Let f have an isolated singularity at z_0, ~~let~~
let $f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j$ ⟸ Laurent expansion of f centered at z_0, $0 < |z-z_0| < R$.

$$f(z) = \left(\frac{z-z_0}{z-z_0}\right) g(z) \quad \text{for } g \text{ analytic at } z_0.$$

1). If $a_j = 0$ for all $j<0$. then z_0 is a removeable singularity.

2). If $a_{-m} \neq 0$. for some positive $m \in \mathbb{Z}$ but $a_j = 0$. for all $j < -m$, we say z_0 is a pole of order m for f.



$y = -\sin(x)\left[\frac{x-\pi/2}{x-\pi/2}\right]$

↑ dig a hole.

3). If $a_j \neq 0$ for an infinite # of negative j-valued, we say z_0 is an essential singularity of f.

$f(z) = \frac{1}{z^{10}} + \frac{1}{z^2} + 1 + \cdots$ ($z_0 = 0$, pole of order 10).

$$f(z) = \sin\left(\frac{1}{z-z_0}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z-z_0)^{-n}.$$

Def^n. If f has an isolated singularity z_0. then the coefficient a_{-1} of $\frac{1}{z-z_0}$ in the Laurent expansion $f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j$.

$$Res(f; z_0) = a_{-1} = Res(z_0)$$

Coeff of -1 $(z-z_0)$.

Th^m (Cauchy's Residue Th^m).

If Γ is a simple, closed, pos. oriented contour and f is analytic inside and on Γ except at $z_1, z_2, \cdots z_n$ inside Γ then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res(z_k)$$

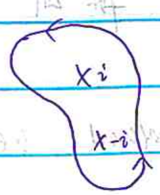
[E]3

$$f(z) = \frac{1}{z^2}\sin(z) = \frac{1}{z^2}\left(z - \frac{1}{6}z^3 + \cdots\right) = \frac{1}{z} - \underbrace{\frac{1}{6}z + \cdots}_{\text{Taylor piece}}$$

$Res(0) = 1$. coeff $(\frac{1}{z}) = 1$

$$\oint_\Gamma f(z)\,dz = 2\pi i.$$

$\boxed{Z74}$ $\int_\Gamma \frac{dz}{1+z^2}$ $\quad f(z) = \frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$



$1 = A(z-i) + B(z+i)$

$\cancel{1 = Az - iA + Bz + iB}$

$z = i \quad 1 = 2iB \to B = \frac{1}{2i} = -\frac{i}{2}$

$z = -i \quad 1 = -2i \to A = \frac{1}{-2i} = \frac{i}{2}$

$f(z) = \frac{i}{2}\left(\frac{1}{z+i}\right) + \left(-\frac{i}{2}\right)\left(\frac{1}{z-i}\right) = \frac{i}{2}\left(\frac{1}{z+i}\right) - \frac{i}{2}\left(\frac{1}{z-i}\right)$

$Res(-i) = \frac{i}{2} \quad\quad Res(i) = -\frac{i}{2}$

$\int_\Gamma \frac{dz}{1+z^2} = 2\pi i\, Res(-i) + 2\pi i\, (Res\, i) = 0$

$\boxed{Z75}$ $I = \int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{5+4\cos\theta} = \int_C \frac{-\frac{1}{4}(z-\frac{1}{z})^2 \frac{dz}{iz}}{5+4(\frac{1}{2}(z+\frac{1}{z}))} = \int_C \frac{\frac{i}{4}(z-\frac{1}{z})^2 \frac{dz}{z}}{5+2(z+\frac{1}{z})}$

$C: |z| = 1$

$z = e^{i\theta} \quad dz = ie^{i\theta}d\theta = iz\,d\theta \quad d\theta = \frac{dz}{iz}$

$\frac{1}{z} = e^{-i\theta}$

$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z})$

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - \frac{1}{z})$

$-\frac{i}{4}\oint_C \frac{(z-\frac{1}{z})^2\, dz}{5z+2z^2+z} = \frac{i}{4}\oint \frac{(z^2-2+\frac{1}{z^2})\,dz}{2z^2+5z+2}$

$= \frac{i}{4}\oint_C \frac{z^2(z^2-2+\frac{1}{z^2})\,dz}{z^2(z+2)(2z+1)} = \frac{i}{4}\oint_C \left[\frac{\overset{A}{z^4} - \overset{B}{2z^2} + \overset{C}{1}}{z^2(z+2)(2z+1)}\right] dz$ $\quad$ A B removable singh.

$\frac{1}{2} \quad \frac{2}{1}$ $\quad = \boxed{\frac{\pi}{4}}$ $\quad Res(-\frac{1}{2}) = \frac{3}{4} \quad Res(0) = -\frac{5}{4}$