

1/7/2 Fiber Bundles

→ Bleecker: Gauge Theory & Variational Principles

Outline:

Hodge dual, Lie Groups, Fiber Bundles — Vector Bundles
Principal Fiber Bundles,
Connections on Fiber Bundles, Associated bundles, Other Topics.

Def 1.1 Let V and W be finite dim'l vector spaces. For $p > 1$

a mapping $\alpha: V^p \rightarrow W$ is a W -valued p -form iff

(i) α is multilinear: $\forall x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_p \in V$

then $x \mapsto \alpha(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_p)$ is linear

(ii) α is skew-symmetric: $\alpha(x_1, \dots, x_p) = \text{Sgn}(\sigma)\alpha(x_{\sigma(1)}, \dots, x_{\sigma(p)})$
 (interchanging entries flips the sign)

Denote: $\Lambda^p(V^*, W) = \{\alpha: V^p \rightarrow W \mid \alpha \text{ is a } p\text{-form}\}$

If $W = \mathbb{R}$ we write $\Lambda^p(V^*, \mathbb{R}) = \Lambda^p(V^*)$, $\Lambda^0 V = \mathbb{R}$, $\Lambda^1 V^* = V$

$\omega^1, \dots, \omega^p \in \Lambda(V^*)$ define $\omega^1 \otimes \dots \otimes \omega^p: V^p \rightarrow \mathbb{R}$ by

$$(\omega^1 \otimes \dots \otimes \omega^p)(v_1, \dots, v_p) = \omega^1(v_1) \dots \omega^p(v_p)$$

Notation: $\bigotimes^p V^* = \text{all } p\text{-multilinear maps into the reals}$

Def 1.3 $\omega^1 \wedge \dots \wedge \omega^p = \frac{1}{p!} \sum_{\sigma \in S_p} \text{Sgn}(\sigma) \omega^{\sigma(1)} \otimes \dots \otimes \omega^{\sigma(p)}$

Def 1.2 For $\alpha \in \Lambda^k V^*$, $\beta \in \Lambda^\ell V^*$ $k, \ell > 0$

let $\alpha \wedge \beta \in \Lambda^{k+\ell} V^*$ be $(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \frac{1}{k! \ell!} \sum_{\sigma \in S_{k+\ell}} \text{Sgn}(\sigma)$

$$\alpha(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \beta(v_{\sigma(p+1)}, \dots, v_{\sigma(p+k)})$$

If $k=0$ or $\ell=0$ then $\alpha \wedge \beta = \alpha \beta$

Properties:

$$\alpha \in \Lambda^p, \beta \in \Lambda^q, \gamma \in \Lambda^r$$
$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma.$$
$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$
$$c \wedge \alpha \wedge \beta = c\alpha \wedge \beta = \alpha \wedge c\beta = c(\alpha \wedge \beta)$$

Ex: $(e^1 \wedge e^2 \wedge e^3) \wedge (e^4 \wedge e^5) = (-1)^3 e^4 \wedge (e^1 \wedge \dots) = (-1)^6 e^4 \wedge e^5 \wedge e^1 \wedge \dots$

Theorem 1.4 If v_1, \dots, v_n is a basis of V then

$\{v^{i_1} \wedge \dots \wedge v^{i_p} \mid 1 \leq i_1 < \dots < i_p \leq n\}$ is a basis of $\Lambda^p V^*$
where $v^i \in V^*$ is defined by $v^i(v_j) = \delta_{ij}$ (dual basis)
and $\dim(\Lambda^p V^*) = \binom{n}{p}$

$$\begin{aligned}
 \text{Let } \tilde{e}_i &= A_i^k v_k \quad g(\tilde{e}_i, \tilde{e}_j) = A_i^k A_j^\ell g(v_k, v_\ell) = A_k^T A_j^\ell G_k \\
 &= (A^T G A)_{ij} = \delta_{ij} D_{ii} \text{ non-deg} \Rightarrow D_{ii} \neq 0 \quad D_{ii} = d; \\
 \therefore g(\tilde{e}_i, \tilde{e}_j) &= d_i \delta_{ij} \quad e_i = \frac{\tilde{e}_i}{\sqrt{|d_i|}} \Rightarrow g(e_i, e_j) = \pm \delta_{ij} \checkmark
 \end{aligned}$$

Sources:

+ Curtis & Miller
Bleeker

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Def 1.7 Let V be a vector space of dimension n . Then μ is a volume on V if $\mu \in \Lambda^n V^* \ni \mu \neq 0$.

→ We say μ is induced by g (a metric on V) iff \exists a g -orthonormal basis $\{e_i\} \ni \mu(e_1, \dots, e_n) = 1$ (sometimes we say a "metric volume")

Remark: If μ is a volume and $\{v_i\}$ a basis for V then

$$\mu(v_1, \dots, v_n) \neq 0$$

proof

if not $\exists \{v_i\}$ a basis $\ni \mu(v_1, \dots, v_n) = 0$ for $\lambda_i \in \mathbb{R}$ we have

$$\mu(\sum \lambda_1^{i_1} v_{i_1}, \dots, \sum \lambda_n^{i_n} v_{i_n}) = \sum \lambda_1^{i_1} \dots \lambda_n^{i_n} \mu(v_{i_1}, \dots, v_{i_n}) = 0$$

now if $i_j = i_k$ for $j \neq k$ then $\mu(v_{i_1}, \dots, v_{i_n}) = 0$ but if i_1, \dots, i_n are distinct then $\mu(v_{i_1}, \dots, v_{i_n}) = (-1)^{\sigma} \mu(v_1, \dots, v_n) = (-1)^{\sigma} \cdot 0 = 0$
for some permutation $\sigma \Rightarrow \mu(\sum \dots) = 0 \therefore \mu = 0 \rightarrow$

Remark(2): \exists a basis $\{\bar{v}_i\} \ni \mu(\bar{v}_1, \dots, \bar{v}_n) = 1$

b/c $\mu(v_1, \dots, v_n) \neq 0$ call it λ . let $v_i = \bar{v}_i$ for $i > 1$ and $\bar{v}_1 = v_1/\lambda$
then $\mu(\bar{v}_1, \dots, \bar{v}_n) = \mu(v_1, \dots, v_n)/\lambda = \lambda/\lambda = 1 \checkmark$

Remark(3): Let $\mu = \mu_g$ is a metric volume and that $\{e_i\}$ is a g -orthonormal basis $\ni \mu(e_1, \dots, e_n) = 1$. If $\{v_i\}$ is any basis then $\mu(v_1, \dots, v_n) = \pm \sqrt{\det G} \quad G_{ij} = g(v_i, v_j)$

proof

$$\begin{aligned} \text{Let } v_i &= A_i^j e_j \text{ then } \mu(v_1, \dots, v_n) = A_1^{j_1} \dots A_n^{j_n} \mu(e_{j_1}, \dots, e_{j_n}) \\ &= \sum_{\sigma \in S_n} A_1^{(\sigma(1))} \dots A_n^{(\sigma(n))} (-1)^{\sigma} = \underline{\det(A)} \end{aligned}$$

$$\begin{aligned} \text{Let } G_{ij} &= g(v_i, v_j) = A_i^k A_j^l g(e_k, e_l) = A_i^k A_j^l D_{kl} \quad (= G_{ij}^i) \\ \Rightarrow G_j^i &= \tilde{A}_k^i D_{kl} A_j^l \quad \therefore G = A^T D A \quad \therefore \det(G) = \det(A^T D A) \end{aligned}$$

$$\begin{aligned}
 & (\det(A)) \quad (\pm 1) \\
 & \downarrow \\
 & = \det(A^T) \det(D) \det(A) = \pm 1 \det(A)^2 \\
 & \therefore |\det(G)| = \det(A)^2 \Rightarrow \det(A) = \pm \sqrt{|\det(G)|} \quad \checkmark
 \end{aligned}$$

Def 1.8 If μ is a volume a basis $\{V_i\}$ is positively oriented iff $\mu(V_1, \dots, V_n) > 0$ (o.w. negatively oriented)

Def 1.9 If g is a metric on V then \exists an induced metric \hat{g} on V^* defined by $\hat{g}(\alpha, \beta) = g(\alpha^*, \beta^*)$ where

$$\alpha(x) = g(\alpha^*, x) \text{ & } \beta(x) = g(\beta^*, x) \quad \forall x,$$

(g is non-degenerate so α^*, β^* are well-defined)

$$\begin{aligned}
 \hookrightarrow \alpha(x) &= g(\alpha^*, x) = g^b(\alpha^*)(x) \quad \therefore \alpha = g^b(\alpha^*) \\
 \Rightarrow \alpha^* &= (g^b)^{-1}\alpha
 \end{aligned}$$

Component Notation $\alpha = \alpha_i e^i$ then $g_{ij} e^i \otimes e^j = g$ so
 $\alpha^* = \alpha^{*j} e_j$ where $\alpha_i = g_{ij} \alpha^{*j}$ $\alpha^{*j} = g^{ji} \alpha_i$

More generally, if $\alpha, \beta \in \Lambda^p V^*$ then

$$\alpha = \alpha_{i_1 \dots i_p} v^{i_1} \otimes \dots \otimes v^{i_p}$$

$$\beta = \beta_{j_1 \dots j_p} v^{j_1} \otimes \dots \otimes v^{j_p}$$

$$\text{define: } \tilde{g}: \Lambda^p V^* \times \Lambda^p V^* \rightarrow \mathbb{R} \quad \tilde{g}(\alpha, \beta) = \frac{1}{p!} \alpha_{i_1 \dots i_p} \beta_{j_1 \dots j_p} \hat{g}(v^{i_1}, v^{j_1}) \dots \hat{g}(v^{i_p}, v^{j_p})$$

$$\begin{aligned}
 \text{or } \tilde{g}(\alpha, \beta) &= \frac{1}{p!} \alpha_{i_1 \dots i_p} \beta_{j_1 \dots j_p} g^{i_1 j_1} \dots g^{i_p j_p} \\
 &= \frac{1}{p!} \alpha^{j_1 \dots j_p} \beta_{i_1 \dots i_p} = \sum_{j_1 < \dots < j_p} \alpha^{j_1 \dots j_p} \beta_{i_1 \dots i_p} \\
 &\stackrel{\text{strictly inc.}}{=} \alpha^{i_1 \dots i_p} \beta_{j_1 \dots j_p}
 \end{aligned}$$

Definition of Hodge Dual $*\beta$

$$\alpha \wedge (*\beta) = \hat{g}(\alpha, \beta) \mu_g$$

Next time:

$$\text{if } \beta = \beta_{i_1 \dots i_p} v^{i_1} \wedge \dots \wedge v^{i_p} \text{ then } *\beta = \pm (\det G)^{\frac{1}{2}} \epsilon_{j_1 \dots j_p i_1 \dots i_p} \beta^{j_1 \dots j_p} v^{i_1 \dots i_p}$$

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Correction:

$$\alpha \wedge \beta(v_1, v_2, \dots, v_p, v_{p+1}, \dots, v_q)$$

$$= \frac{1}{p!} \frac{1}{q!} \sum_{\sigma} (-1)^{\sigma} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(p)}) \beta(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})$$

$$\text{So } \omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^r = \sum_{\sigma} (-1)^{\sigma} (\omega^1 \otimes \dots \otimes \omega^r) \\ (\text{no } \frac{1}{r!} \text{ in front of the RHS})$$

Theorem 1.10: For $\beta \in \Lambda^p V^*$, there exists an unique $*\beta \in \Lambda^{n-p} V^*$ such that

$$\alpha \wedge * = \hat{\phi}(\alpha, \beta) \mu_g \quad (\forall \alpha \in \Lambda^p V^*)$$

where μ_g is a volume induced by a metric g on V . The mapping from $\Lambda^p V^*$ to $\Lambda^{n-p} V^*$ defined by $\beta \mapsto *\beta$ is an isomorphism. This $*$ is called the Hodge dual.

Proof: For $\gamma \in \Lambda^p V^*$, let $\phi_\gamma: \Lambda^p V^* \rightarrow \mathbb{R}$ for $\alpha \in \Lambda^p V^*$, $\phi_\gamma(\alpha)$ is the unique number $\Rightarrow \alpha \wedge \gamma = [\phi_\gamma(\alpha), \mu_g]$

$$\text{Since } (\alpha_1 + \alpha_2) \wedge \gamma = \phi_\gamma(\alpha_1 + \alpha_2), \mu_g$$

$$\alpha_1 \wedge \gamma = \phi_\gamma(\alpha_1), \mu_g$$

$$\alpha_2 \wedge \gamma = \phi_\gamma(\alpha_2), \mu_g$$

$$\text{So } (\alpha_1 + \alpha_2) \wedge \gamma = [\phi_\gamma(\alpha_1) + \phi_\gamma(\alpha_2)], \mu_g$$

$$\Rightarrow \phi_\gamma(\alpha_1 + \alpha_2) = \phi_\gamma(\alpha_1) + \phi_\gamma(\alpha_2)$$

$$\phi_\gamma(c\alpha_1) = c\phi_\gamma(\alpha_1)$$

Proof cont'd: Therefore $\phi \in (\Lambda^p V^*)^*$

Let $\hat{\phi}: \Lambda^{n-p} V^* \rightarrow (\Lambda^p V^*)^*$ be defined by $\hat{\phi}(\gamma) = \phi_\gamma$.

$$\begin{aligned}\phi_{\gamma_1 + \gamma_2}(\alpha) &= \alpha \wedge (\gamma_1 + \gamma_2) = \phi_{\gamma_1}(\alpha) + \phi_{\gamma_2}(\alpha) \\ &= (\phi_{\gamma_1} + \phi_{\gamma_2})(\alpha)\end{aligned}$$

so this is how we add in the dual space $= \hat{\phi}(\gamma_1 + \gamma_2)(\alpha)$

It is easy to show $\phi_{c\gamma} = c\phi_\gamma$ as well.

so $\hat{\phi}$ is also linear.

If $\hat{\phi}(\gamma) = 0$, then $\phi_\gamma = 0 \Rightarrow \phi_\gamma(\alpha) = 0$
 $\Rightarrow \alpha \wedge \gamma = 0 \quad \forall \alpha$. Thus, by Exercise

1.3, $\gamma = 0$. $\therefore \hat{\phi}(\gamma) = 0 \Rightarrow \gamma = 0$

\Rightarrow the Kernel of $\hat{\phi} = \{0\}$.

$$\dim \Lambda^{n-p} V^* = \binom{n}{n-p} = \binom{n}{p} = \dim \Lambda^p V^* = \dim (\Lambda^p V^*)^*$$

Recall the identity that $\dim V = \dim(\ker t) + \dim(t(V))$

For $\delta \in (\Lambda^p V^*)^*$, let $\#\delta \in \Lambda^p V^* \ni$

$\delta = \hat{g}(\#\delta, \cdot)$. Note that this means

$$\hat{g}^\flat(\#\delta) = \delta$$

$$\#\delta = (\hat{g}^\flat)^{-1}(\delta)$$

\hat{g} is the metric on V
 \hat{g}^\flat metric on $\Lambda^p V^*$

$$W = \Lambda^p V^*$$

$$\hat{g}^\flat: W \rightarrow W^*$$

Proof of existence of the Hodge Dual (cont'd)

Define $*\beta = \hat{\phi}^{-1}(\#^{-1}(\beta)) \quad \beta \in \Lambda^p V^*$

$$\text{so } \hat{\phi}(*\beta) = \#^{-1}(\beta)$$

$$\begin{aligned} \tilde{g}(\alpha, \beta) &= \tilde{g}(\beta, \alpha) = \tilde{g}(\# \hat{\phi}(*\beta), \alpha) \\ &= \hat{\phi}(*\beta)(\alpha) \\ &= \phi_{*\beta}(\alpha) \end{aligned}$$

$$\alpha \wedge *\beta = \phi_{*\beta}(\alpha) u_g = \tilde{g}(\alpha, \beta) u_g$$

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Lemma: Assume V is a VS, g is a metric and $\{v_1, v_2, \dots, v_n\}$ is a basis of V . If $u = u_g$ is a volume induced by g , then

$$u = \pm |\det(g_{ij})| (v^1, v^2, \dots, v^n)$$

each $v^i \in V^*$

+ used when $\{v_1, v_2, \dots, v_n\}$ are positively oriented, - otherwise.

Note: when working with manifolds, $v_i = \partial_i$
 $v^i = dx^i$, $V = T_p M$.

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Lemma Assume V is a vector space, g is a metric on V and $\{v_i\}$ is a basis for V . If $\mu = \mu_g$ is a volume induced by g then $\mu_g = \pm |\det G|^{\frac{1}{2}} v^1 \wedge \dots \wedge v^n$ where $G = (G_{ij})$ and $G_{ij} = g(v_i, v_j)$.
 + is used when $\{v_i\}$ is pos. oriented otherwise we use -.

proof

Let $\{e_i\}$ be orthonormal (basis for V) Write $v_i = a_i^j e_j$ then $v^i = (a^i)_j e^j$. $G_{ij} = g(v_i, v_j) = a_i^k a_j^l g(e_k, e_l) = a_i^k a_j^l D_{kl}$
 $\Rightarrow G_{ij}^i = (a^i)_k D_{kl} a_j^l \therefore G = A^t D A \Rightarrow \det(G) = \det(A)^2 \det(D)$
 $\Rightarrow |\det(A)| = \sqrt{|\det(G)|}$
 $v^1 \wedge \dots \wedge v^n = (a^1)_j, \dots (a^n)_j e^{j_1} \wedge \dots \wedge e^{j_n} = \sum_{\sigma \in S_n} (-1)^\sigma a_{\sigma(1)}^1 \dots a_{\sigma(n)}^n e^{1 \wedge \dots \wedge n} = \pm \det(A) \mu_g \therefore \mu_g = \pm \det(A) v^1 \wedge \dots \wedge v^n = \pm \sqrt{|\det(G)|} v^1 \wedge \dots \wedge v^n$

$$v^1 \wedge \dots \wedge v^n (v_1, \dots, v_n) = \sum_{\sigma \in S_n} (-1)^\sigma v^{\sigma(1)} \otimes \dots \otimes v^{\sigma(n)} (v_1, \dots, v_n) \\ = 0 \text{ unless } \sigma(i) = i \Rightarrow \sigma = \text{id} = + \\ \therefore \text{pos. oriented} \Rightarrow \mu_g (v_1, \dots, v_n) = |\det(G)|^{\frac{1}{2}} \cdot 1 > 0$$

Thm 1.11 Let V be a vector space, g a metric on V , $\mu = \mu_g$ a volume induced by g . Let $\{v_i\}$ be a basis of V . $\beta \in \Lambda^p V^*$

$$\beta = \beta_{j_1 \dots j_p} v^{j_1} \wedge \dots \wedge v^{j_p} \text{ then } \star \beta = \pm |\det(G)|^{\frac{1}{2}} \beta^* \epsilon_{j_1 \dots j_p k_1 \dots k_{n-p}} v^{j_1} \wedge \dots \wedge v^{j_p} \wedge v^{k_1} \wedge \dots \wedge v^{k_{n-p}}$$

$$G_{ij} = g(v_i, v_j) \text{ and } \epsilon_{h_1 \dots h_n} = \begin{cases} 0 & h_k = h_\ell, k \neq \ell \\ 1 & \{h_1, \dots, h_n\} \text{ is even perm of } \{1, \dots, n\} \\ -1 & \text{otherwise} \end{cases}$$

proof

Let $\alpha, \beta \in \Lambda^p V^*$ and write $\alpha = \alpha_{i_1 \dots i_p} v^{i_1} \wedge \dots \wedge v^{i_p}$ $\beta = \beta_{j_1 \dots j_p} v^{j_1} \wedge \dots \wedge v^{j_p}$

$$\text{Let } \star \beta = \beta^{i_1 \dots i_p} \epsilon_{j_1 \dots j_p i_1 \dots i_{n-p}} v^{i_1} \wedge \dots \wedge v^{i_{n-p}}$$

Note that this sum gives non-zero terms when $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_p\} = \{1, \dots, n\}$
thus $\alpha \wedge (\hat{*}\beta) = \sum_{i, j, s \text{ inc.}} \alpha_{i_1, \dots, i_p} \beta^{j_1, \dots, j_p} \epsilon_{j_1, \dots, j_p; s_1, \dots, s_{n-p}} v^{i_1} \wedge \dots \wedge v^{i_p} \wedge v^{s_1} \wedge \dots \wedge v^{s_{n-p}}$

$[i_1 < \dots < i_p \text{ determines } s_1 < \dots < s_{n-p} \text{ (if } i_k = s_\ell \text{ then } v^{i_k} = 0\text{)} -$
but $s_1 < \dots < s_{n-p}$ determines $j_1 < \dots < j_p$ (if $s_k = j_\ell$ then $\epsilon_{j_k} = 0$)]

$$= \sum_{i_1 < \dots < i_p} \alpha_{i_1, \dots, i_p} \beta^{i_1, \dots, i_p} \epsilon_{i_1, \dots, i_p; s_1, \dots, s_{n-p}} v^{i_1} \wedge \dots \wedge v^{i_p} \wedge v^{s_1} \wedge \dots \wedge v^{s_{n-p}}$$

for each $i_1 < \dots < i_p \exists \sigma \in S(n) \sigma(1) = i_1, \dots, \sigma(p) = i_p, \sigma(p+1) = s_1, \dots, \sigma(n) = s_{n-p}$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma(n) < \dots < \sigma(p)}} \alpha_{\sigma(1), \dots, \sigma(p)} \beta^{\sigma(1), \dots, \sigma(p)} (-1)^{\sigma} v^{i_1} \wedge \dots \wedge v^n$$

$$\therefore \alpha \wedge \hat{*}\beta = \sum_{i_1 < \dots < i_p} \alpha_{i_1, \dots, i_p} \beta^{i_1, \dots, i_p} v^{i_1} \wedge \dots \wedge v^n$$

$$\hat{g}(\alpha, \beta) = \alpha^{i_1, \dots, i_p} \beta_{i_1, \dots, i_p} \Rightarrow \alpha \wedge \hat{*}\beta = \hat{g}(\alpha, \beta) v^{i_1} \wedge \dots \wedge v^n$$

$$\alpha \wedge (\text{Idet}(G))^{\frac{k}{2}} \hat{*}\beta = \hat{g}(\alpha, \beta) \text{Idet}(G)^{\frac{k}{2}} v^{i_1} \wedge \dots \wedge v^n$$

$$\text{pos. oriented} = \hat{g}(\alpha, \beta)/\mu_g \quad \text{neg.} = -\hat{g}(\alpha, \beta)/\mu_g$$

$$\therefore \hat{*}\beta = \pm \text{Idet}(G)^{\frac{k}{2}} (\hat{*}\beta) + \begin{matrix} \text{pos. orient} \\ - \text{neg. orient} \end{matrix}$$

Cook

Handout: pg 17-20
pg 17 starts: Theorem 1.13

- Also Bleeker pg 10-13

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Thm 1.12 V is a vector space with metric g , $\dim(V) = n$. With $\mu = \mu_g$ a metric volume. Then for $\alpha \in \Lambda^k V^*$

$$**\alpha = (-1)^{(n-k)k+s} \alpha$$

$s = \text{index of } g$ ie $g = \begin{pmatrix} + & - & - \\ - & - & - \end{pmatrix}$ s minuses.

proof (Later -- Handout)

Ex: $\eta = \begin{pmatrix} +1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$ F a 2-form then $**F = (-1)^{(4-2)2+3} F = -F$

Def ω is a differential k -form on M means

$$\omega: M \rightarrow \Lambda^k(T^*M) \quad \omega(p) \in \Lambda^k T_p^*M \quad \forall p \in M$$

Also ω has to be smooth. (ie if (U, x) is a chart about p $p \mapsto \omega_p(\frac{\partial}{\partial x^{i_1}}|_p, \dots, \frac{\partial}{\partial x^{i_k}}|_p)$ is smooth for $p \in U$.)

$$\omega = \frac{1}{k!} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} = \omega_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$$

$$\text{ie } \omega_{i_1 \dots i_k} = \omega(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}})$$

also

$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

no $\frac{1}{k!}$ term!

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{\sigma \in S_k} (-1)^{\sigma} (dx^{i_{\sigma(1)}} \otimes \dots \otimes dx^{i_{\sigma(k)}})$$

$$f \in \Omega^0(T^*M) \Rightarrow f: M \rightarrow \mathbb{R} \\ (= \Omega^0(M))$$

$$df(Y) = Y(f) \text{ in coords } df = \frac{\partial f}{\partial x^i}|_p dx^i$$

$$\omega = \frac{1}{k!} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$d\omega = \frac{1}{k!} d\omega_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{1}{k!} \frac{\partial \omega_{i_1 \dots i_k}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Cartan's Formula:

$$d\omega(\bar{X}_1, \dots, \bar{X}_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \bar{X}_i (\omega(\bar{X}_1, \dots, \hat{\bar{X}}_i, \dots, \bar{X}_{k+1}))$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} \omega([\bar{X}_i, \bar{X}_j], \bar{X}_1, \dots, \hat{\bar{X}}_i, \dots, \hat{\bar{X}}_j, \dots, \bar{X}_{k+1})$$

↑ delete
↑ delete

Def $O \subseteq M$ is open iff $\forall (x, U)$ chart on M then
 $x(O \cap U)$ is open in \mathbb{R}^m (M is an m -dim manifold)
 $F \subseteq M$ is closed iff $M - F$ is open
 $A \subseteq M$ then the closure of A , $\bar{A} = \bigcap \{F \text{ closed} \mid A \subseteq F\}$

$$\alpha \in \Omega^k(M)$$

$$\text{Support } (\alpha) = \overline{\{p \mid \alpha(p) \neq 0\}}$$

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Def $K \subseteq M$ (a manifold) is compact iff every open cover has a finite subcover.

If M, N are manifolds, $f: M \rightarrow N$ is smooth, then $\forall \alpha \in \Omega^k(N)$

$$(f^*\alpha)_p(v_1, \dots, v_k) = \alpha_{f(p)}(df_p(v_1), \dots, df_p(v_k))$$

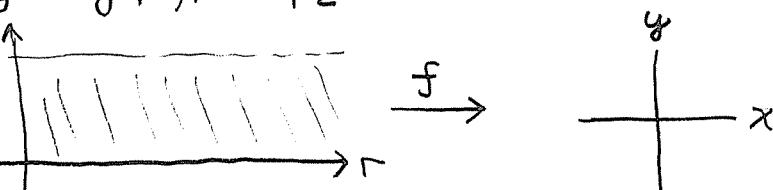
then $f^*\alpha \in \Omega^k M$, if $\alpha: M \rightarrow \mathbb{R}$ $f^*\alpha = \alpha \circ f$

Properties:

- ① $d(f^*\alpha) = f^*(d\alpha)$
- ② $(g \circ f)^*(\alpha) = g^*(f^*(\alpha)) \in \Omega^k P$ for $\alpha \in \Omega^k M$ $M \xrightarrow{f} N \xrightarrow{g} P$
- ③ $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$

Ex: \mathbb{R}^2 $x(p_1, p_2) = p_1$ $f(r, \theta) = (r \cos \theta, r \sin \theta)$

$$\theta \quad y(p_1, p_2) = p_2$$



Pullback of volume form:

$$\begin{aligned} f^*(dx \wedge dy) &= f^*(dx) \wedge f^*(dy) = d(f^*x) \wedge d(f^*y) = d(x \circ f) \wedge d(y \circ f) \\ &= d(r \cos \theta) \wedge d(r \sin \theta) = (\cos \theta dr - r \sin \theta d\theta) \wedge \\ &\quad (\sin \theta dr + r \cos \theta d\theta) = \cos \theta r \cos \theta dr \wedge d\theta - r \sin \theta \sin \theta dr \wedge d\theta \\ &= r \cos^2 \theta dr \wedge d\theta + r \sin^2 \theta dr \wedge d\theta = r(dr \wedge d\theta) \end{aligned}$$

$\int_D \beta$ is an n -form defined on a bounded open set D of \mathbb{R}^n
Let x^1, \dots, x^n be coordinates on \mathbb{R}^n

$\beta = b dx^1 \wedge \dots \wedge dx^n$ where $b: D \rightarrow \mathbb{R}$ is smooth.

Assume $\text{supp } \beta \subseteq D$ ($\subseteq \mathbb{R}^n$) then $\int_D \beta = \int_D b(x^1, \dots, x^n) dx^1 \dots dx^n$

Def A nowhere zero form $\nu \in \Omega^n M$ where $\dim(M) = n$,
is called a volume.

(M, ν) is called an oriented manifold.

(1) Let $\alpha \in \Omega^n(M)$ $\exists K = \text{supp}(\alpha)$ is compact. The compactness ensures that \exists a finite number of charts (U_i, χ_i) which cover K (ie $K \subseteq \bigcup U_i$), $(\chi_i U_i)$ is bounded, $\exists \rho_i \in C^\infty(M)$ $\exists \text{supp}(\rho_i) \subseteq U_i$; $0 \leq \rho_i(p) \leq 1 \forall p \in M$ and $\sum_{i=1}^n \rho_i(p) = 1 \forall p \in K$

1/25/2

$\int_D \beta = \int_D b$ where $\beta = b (dx^1 \wedge \dots \wedge dx^n)$ where D open & $\text{supp}(b) \subseteq D$

i.e. $\int_{\mathbb{R}^n} b(p) d_p x^1 \wedge \dots \wedge d_p x^n = \iint \dots \int b(x^1, \dots, x^n) dx^1 \dots dx^n$

Let $\alpha \in \Omega^m M$ where $\dim(M) = n$, $K = \text{supp}(\alpha)$ is compact

$\exists \{(U_i, x_i)\}_{i=1}^N$ charts $\ni K \subseteq \bigcup_{i=1}^N U_i$ and $x_i(U_i)$ bounded.

also $\exists \rho_i : M \rightarrow [0, 1]$, $\rho_i \in C^\infty(M)$, $\text{supp}(\rho_i) \subseteq U_i$ and

$$\sum_{i=1}^N \rho_i(x) = 1 \quad \forall x \in K \quad \begin{array}{l} \text{need pos. oriented... } (x^{-1})^* \alpha = \text{non-neg.} \\ \text{mult. of volume} \end{array}$$

$$\int_M \alpha = \sum_{i=1}^N \int_{x_i(U_i)} (x_i^{-1})^*(\rho_i \alpha) \left(= \sum_{i=1}^N \int_{\mathbb{R}^n} (x_i^{-1})^*(\rho_i \alpha) \text{ since } \rho_i \alpha = 0 \text{ on } x_i(U_i)^c \right)$$

Theorem (Baby Stoke's) Let M be an oriented n -manifold

and $\alpha \in \Omega^{n-1} M$ have compact support Then $\int_M d\alpha = 0$

proof

$K = \text{supp}(\alpha)$ Choose appropriate (U_i, x_i, ρ_i) (as above)

$$\alpha = \sum_{i=1}^N \rho_i \alpha \quad \int_M d\alpha = \sum_{i=1}^N \int_M d(\rho_i \alpha) = \sum_{i=1}^N \int_{M \setminus U_i} (x_i^{-1})^*(d\rho_i \alpha)$$

$$= \sum_{i=1}^N \int_{x_i(U_i)} d((x_i^{-1})^*(\rho_i \alpha)) \quad (\text{differential \& pullback commute})$$

α ($n-1$) form $\Rightarrow \rho_i \alpha$ is $(n-1)$ -form $\Rightarrow (x_i^{-1})^*(\rho_i \alpha)$ is an $(n-1)$ -form

$\therefore d[(x_i^{-1})^*(\rho_i \alpha)]$ is an n -form

Choose $i \in \{1, \dots, N\}$ let $\beta = (x_i^{-1})^*(\rho_i \alpha)$ then $\beta = \sum b_j dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$

$$d\beta = \sum_{i,j} \frac{\partial b_j}{\partial x^i} (dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^n)$$

$$= \sum_j \frac{\partial b_j}{\partial x^i} (-1)^{j-1} dx^1 \wedge \dots \wedge dx^n \quad \text{since } i \neq j \text{ then } dx^i \wedge dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^n \rightarrow 0$$

$$\therefore \int_{\mathbb{R}^n} d\beta = \sum_j (-1)^{j-1} \int_{\mathbb{R}^n} \frac{\partial b_j}{\partial x^i} dx^1 \dots dx^n = \sum_j (-1)^{j-1} \boxed{\int_{\mathbb{R}^n} \frac{\partial b_j}{\partial x^i} dx^i dx^1 \dots}$$

use FTC $\int_{\mathbb{R}^n} \frac{\partial b_j}{\partial x^i} dx^i dx^1 \dots \hat{dx^j} \dots dx^n$ if we go out far enough
(b/c of compact support) $\int_{C_j} \frac{\partial b_j}{\partial x^i} dx^i = 0 \Rightarrow \int_{\mathbb{R}^n} d\beta = 0$

$$\Rightarrow \int_M d\alpha = \sum_{i=1}^N 0 = 0 //$$

If V_N & V_M are volumes
 $f^*V_M = (\text{pos, mult.})V_N$

Thm $f: N \rightarrow M$ is orientation preserving diffeomorphism

$$\text{then } \int_N f^* \alpha = \int_M \alpha$$

proof

→ see Bleecker

1/28/2

Def The co-differential $\delta: \Omega^p M \rightarrow \Omega^{p-1} M$ defined by
 $\delta\alpha = (-1)^{n(p+1)+s+1} * (d(*\alpha))$ where $\dim(M) = n$ & $s = \text{sig. of metric}$

Note: $\delta^2 \alpha = \pm * d * \pm 1 * d * \alpha = \pm 1 * d^2 * \alpha \xrightarrow{\text{d}^2=0} 0$

Recall: $g^b: V \rightarrow V^*$ def. by $g^b(v)(w) = g(v, w)$
 $g^*: V^* \rightarrow V$ def. by $g^*(\alpha) = v$ where $g(v, w) = \alpha(w)$
 $\Rightarrow (g^b)^* = g^*$

Def The gradient of $f: M \rightarrow \mathbb{R}$ is denoted ∇f and def by
 $\nabla f \in \mathfrak{X}(M)$ (vector field) $\exists df = g(\nabla f, \cdot)$ or $\nabla f = g^*(df)$

Symp. G. $M = T^*Q$ $\omega = dq_i \wedge dp^i$ symplectic form (non-deg.)
 $f: M \rightarrow \mathbb{R}$ $df = \omega(\mathbf{Z}_f, \cdot)$ or $\mathbf{Z}_f = \omega^*(df)$ Hamiltonian...

$\nabla f = a^i \frac{\partial}{\partial x^i}$ where (U, x^i) a chart

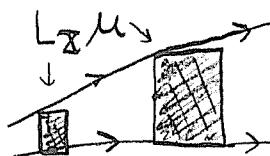
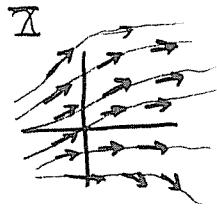
$$\frac{\partial f}{\partial x^j} = df\left(\frac{\partial}{\partial x^j}\right) = g\left(a^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = a^i g_{ij}$$
$$\Rightarrow g^{kj} \frac{\partial f}{\partial x^j} = a^i g_{ij} g^{ik} \Rightarrow g^{kj} \frac{\partial f}{\partial x^j} = a^k$$

$$\therefore \nabla f = \left(g^{ij} \frac{\partial f}{\partial x^j}\right) \frac{\partial}{\partial x^i}$$

Ex: \mathbb{R}^n $g_{ij} = \delta_{ij}$ coords x^i def everywhere then $\nabla f = \sum_i \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$

Ex: \mathbb{R}^4 $\eta = \begin{pmatrix} 1 & -1 & -1 & -1 \end{pmatrix}$ $\nabla f = \frac{\partial f}{\partial t} \frac{\partial}{\partial t} - \frac{\partial f}{\partial x} \frac{\partial}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial}{\partial y} - \frac{\partial f}{\partial z} \frac{\partial}{\partial z}$
if $x^1 = ct$, $x^2 = x$, $x^3 = y$, $x^4 = z$

Def The divergence of \mathbf{Z} (a vector field) is $\text{Div } \mathbf{Z}: M \rightarrow \mathbb{R}$
def by $\text{Div}(\mathbf{Z}) = -\delta(g^b \mathbf{Z})$



scale the volume element

or equivalently, X has a flow $\{\varphi_t\}$ ie $\varphi_t: M \rightarrow M \ni$
 $\varphi_0 = \text{id}_M \quad \frac{d}{dt}(\varphi_t(x)) = X(\varphi_t(x))$

Then $L_X \mu = (\text{Div } X)\mu$ where L is the Lie der. & μ a vol.

Let (U, x^i) be a chart. $g^b(X) = g^b(X^i \frac{\partial}{\partial x^i}) = g_{ij} X^j dx^i$

$$\text{Div } X = -\delta(X^i g_{ij} dx^j) = (-1)^{2n+s+2} * d * (X^i g_{ij} dx^j)$$

$$= (-1)^s * d \left\{ \left[\det G^{\frac{1}{2}} g^{jk} X^i g_{ij} \epsilon_{i|s_1 \dots s_{n-1}} \right] dx^{s_1} \wedge \dots \wedge dx^{s_{n-1}} \right\}$$

$$= (-1)^s * \frac{\partial}{\partial x^k} \left[\dots \right] dx^k \wedge dx^{s_1} \wedge \dots \wedge dx^{s_{n-1}}$$

$$= (-1)^s * \left\{ \frac{\partial}{\partial x^k} \left[\det G^{\frac{1}{2}} X^i \right] \epsilon_{i|s_1 \dots s_{n-1}} dx^k \wedge dx^{s_1} \wedge \dots \wedge dx^{s_{n-1}} \right\}$$

$$= (-1)^s * \left\{ \sum_{i=1}^n \frac{\partial}{\partial x^i} \left[\det G^{\frac{1}{2}} X^i \right] \epsilon_{i|s_1 \dots s_{n-1}} dx^i \wedge dx^{s_1} \wedge \dots \wedge dx^{s_{n-1}} \right\}$$

$$= (-1)^s * \left\{ \sum_{i=1}^n \frac{\partial}{\partial x^i} \left[\det G^{\frac{1}{2}} X^i \right] dx^i \wedge \dots \wedge dx^n \right\} \xrightarrow{i \in \{1, \dots, n\} - \{s_1, \dots, s_{n-1}\}}$$

$$= (-1)^s \sum_i \frac{\partial}{\partial x^i} \left[\det G^{\frac{1}{2}} X^i \right] (-1)^{\frac{\text{odd}}{n(n+1)+s+1}} = (-1)^{2s} \sum \dots$$

$$\therefore \text{Div } X = \det G^{\frac{1}{2}} \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}$$

1/30/2

Bjorn Felsager - Geometry / Physics ← develops much physics in differential form notation. Lots of exercises mostly worked out.

Curl of a vector field?

$$\tilde{g}^{\#} * (d(g^b(\bar{x})))$$

(M, g) oriented (U, x^i) a chart
 $\dim(M) = n$

$$g^b(\bar{x}^i \partial_i) = \bar{x}_j dx^j \quad (\bar{x}_j = g_{ij} \bar{x}^i)$$

$$d(g^b(\bar{x})) = \partial_i \bar{x}_j dx^i \wedge dx^j = \sum_{i < j} \left[\frac{\partial}{\partial x^i} (\bar{x}^k g_{kj}) - \frac{\partial}{\partial x^j} (\bar{x}^k g_{ki}) \right] dx^i \wedge dx^j$$

$$* d(g^b(\bar{x})) = (-1)^{n(n+1)+s+1} |\det G|^{1/2} g^{kc} g^{jb} [-]_{kj} \epsilon_{lab|s_1 \dots s_{n-2}} dx^{s_1} \wedge \dots \wedge dx^{s_{n-2}}$$

$\tilde{g}^{\#}$ converts $* d(g^b(\bar{x}))$ at each pt in M to an element of $(\Lambda^{n-2} M)^*$

$$\dim(\Lambda^{n-2} M) = \binom{n}{n-2} = \frac{1}{2} n(n-1) \quad \text{If } n=3 \text{ then } (\Lambda^1 M)^* = (\Lambda^1 M)^* \\ \rightarrow n \neq 3 \text{ doesn't make sense!}$$

$n=3$

$$\hat{g}^{\#} * (dg^b(\bar{x})) = (-1)^s |\det G|^{1/2} g^{sc} g^{kb} [-]_{kj} \epsilon_{lab|c} \frac{\partial}{\partial x^c}$$

($\$G$ diag.)

$$= (-1)^s |\det G|^{1/2} g^{aa} g^{bb} g^{cc} [-]_{ab} \epsilon_{lab|c} \frac{\partial}{\partial x^c} \quad (\text{sum over } a, b, c)$$

($\$$ Euclidean)

$$= \sum_{a,b,c} \left[\frac{\partial}{\partial x^a} (\bar{x}^b) - \frac{\partial}{\partial x^b} (\bar{x}^a) \right] \epsilon_{lab|c} \frac{\partial}{\partial x^c}$$

↪ Why we won't use Curl.

Def The Laplace-Beltrami operator on M is the mapping:

$$\Delta : \Omega^p M \rightarrow \Omega^p M$$

is defined by

$$\Delta = \delta d + d\delta$$

(an "elliptic differential operator")

* Warner: proof 1-harmonic form per cohomology class

Def $\Delta(\alpha) = 0$ then we say α is harmonic

Note: $f \in \Omega^0 M$ $\Delta f = \delta df + d\delta f = \delta df$

$$\hookrightarrow \Delta f = \delta(g^b(g^{\#}df)) = -\text{Div}(g^{\#}(df)) = -\text{Div}(\nabla f)$$

(Laplacian of an operator)

$$= -\text{Div}\left(g_{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x^j}\right) = -\sum_i \frac{\partial}{\partial x_i}\left(\det G \sum_j g^{ij}\frac{\partial f}{\partial x^j}\right)$$

$$\text{if } g_{ij} = \delta_{ij} \text{ then } -\sum_i \frac{\partial}{\partial x_i}\left(\delta^{ij}\frac{\partial f}{\partial x^j}\right) = -\sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x^i}\right)$$

Thm If $\alpha \in \Omega^k M$, $\beta \in \Omega^{k+1} M$ then $\tilde{g}(\delta\alpha, \beta)\mu_g = \tilde{g}(\alpha, \delta\beta)\mu_g + d(\alpha \wedge * \beta)$

proof

note that $\delta\beta = (-1)^{n(k+1)+s+1} * (d * \beta)$ Let $p = [n-(k+1)]+1 (= n-k)$
 then $*\beta \in \Omega^{p-1} M \Rightarrow d(*\beta) \in \Omega^p M$ So $**d(*\beta) =$
 $(-1)^{p(n-p)+s} d(*\beta)$ $*(\delta\beta) = (-1)^{n(k+2)+s+1} **d * \beta$
 $= (-1)^{n(k+2)+s+1} (-1)^{p(n-p)+s} d(*\beta) \quad n(k+2)+s+1+(n-k)k+s$
 $= 2nk + 2n + 2s + 1 - k^2 \equiv 1 - k^2 \equiv k+1 \pmod{2}$

$$\rightarrow *\delta\beta = (-1)^{k+1} d(*\beta)$$

$$\begin{aligned} \tilde{g}(\delta\alpha, \beta)\mu_g - \tilde{g}(\alpha, \delta\beta)\mu_g &= \delta\alpha \wedge (*\beta) - \alpha \wedge (*\delta\beta) \\ &= \delta\alpha \wedge (*\beta) + (-1)^k \alpha \wedge (d * \beta) = d(\alpha \wedge * \beta) // \end{aligned}$$

2/1/2

$$\omega = \frac{1}{k!} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$\text{where } \omega_{i_1 \dots i_k} = \omega(\partial_{x^{i_1}}, \dots, \partial_{x^{i_k}})$$

Lemma Let $\alpha \in \Omega^k M$, $\alpha = \frac{1}{k!} \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ in a local chart (U, x) . Then, $d\alpha = \frac{1}{(k+1)!} (d\alpha)_{j_1 \dots j_{k+1}} dx^{j_1} \wedge \dots \wedge dx^{j_{k+1}}$ where $(d\alpha)_{j_1 \dots j_{k+1}} = d\alpha(\partial_{j_1}, \dots, \partial_{j_{k+1}}) = \frac{(-1)^k}{k!} \sum_{\sigma \in S_{k+1}} (-1)^\sigma \alpha_{j_{\sigma(1)} \dots j_{\sigma(k+1)}}$ where $\alpha_{j_{\sigma(1)} \dots j_{\sigma(k+1)}} = \partial_{j_{\sigma(1)}} \alpha_{j_{\sigma(2)} \dots j_{\sigma(k)}}$

proof

$$d\alpha = \frac{1}{k!} d\alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{1}{k!} \partial_j \alpha_{i_1 \dots i_k} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

then

$$\begin{aligned} d\alpha(\partial_{j_1}, \dots, \partial_{j_{k+1}}) &= \frac{1}{k!} \partial_j \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_{k+1}} (\partial_{j_1} \dots \partial_{j_{k+1}}) \\ &= \frac{(-1)^k}{k!} \partial_{i_{k+1}} \alpha_{i_1 \dots i_k} \underbrace{dx^{i_1} \wedge \dots \wedge dx^{i_{k+1}}}_{(\partial_{j_1} \dots \partial_{j_{k+1}})} \\ &= \frac{(-1)^k}{k!} \alpha_{i_1 \dots i_k, j_{k+1}} \delta_{j_1 \dots j_{k+1}}^{i_1 \dots i_{k+1}} \quad \text{define } \delta_{j_1 \dots j_{k+1}}^{i_1 \dots i_{k+1}} \\ &\quad \text{note } \delta_{j_1 \dots j_{k+1}}^{i_1 \dots i_{k+1}} = 0 \text{ if } i_l = j_{l'} \text{ for some } l \neq l' \\ &\therefore \exists \sigma \in S_{k+1} \ni j_{\sigma(1)} = i_1, \dots, j_{\sigma(k+1)} = i_{k+1} \text{ if } \delta_{j_1 \dots j_{k+1}}^{i_1 \dots i_{k+1}} \neq 0 \\ &\quad \text{and } \delta_{j_1 \dots j_{k+1}}^{j_{\sigma(1)} \dots j_{\sigma(k+1)}} = (-1)^\sigma \\ &= \frac{(-1)^k}{k!} \sum_{\sigma \in S_{k+1}} (-1)^\sigma \alpha_{j_{\sigma(1)} \dots j_{\sigma(k)}, j_{\sigma(k+1)}} = \frac{(-1)^k}{k!} \alpha_{[j_1 \dots j_k, j_{k+1}]} \end{aligned}$$

Thm Let $\beta \in \Omega^{k+1} M$ and (U, x^i) a pos. oriented chart (w/r resp. M_g induced by g on M) Then

$$(\delta \beta)_{i_1 \dots i_k} = \frac{(-1)^{k+1}}{k! |\det G|^{1/2}} \partial_i (\det G)^{1/2} \beta^{i_1 \dots i_k}$$



may skip

$$*(L^*(\beta)) = L^*(\beta) \text{ where } \beta \in \Lambda^k W$$

where (V, g) (W, h) are vector spaces w/r metrics

$L: V \rightarrow W$ is a linear isometry (orientation pres.)

2/4/2

Thm $\beta \in \Omega^{k+1} M$, (U, χ) is a chart of M . If g is a metric
 M_g is a compatible volume and (x^i) has positive orientation
then $(\delta\beta)^{j_1 \dots j_{k+1}} = \frac{(-1)^{k+1}}{k! |\det G|^{\frac{k}{2}}} \partial_j (\det G^{\frac{k}{2}} \beta^{j_1 \dots j_{k+1}}) \quad (*)$

where $\beta = \frac{1}{(k+1)!} \beta_{i_1 \dots i_{k+1}} dx^{i_1} \wedge \dots \wedge dx^{i_{k+1}}$

Proof

If ρ and η are differential forms write $\rho \approx \eta$ iff $\rho - \eta$ is exact. We have $\forall \alpha \in \Omega^k M \quad \tilde{g}(\alpha, \delta\beta) M_g \approx \tilde{g}(\alpha, \beta) M_g$

$$= \frac{1}{(k+1)!} (\det G)^{j_1 \dots j_{k+1}} \beta^{j_1 \dots j_{k+1}} M_g = \frac{(-1)^k}{(k+1)! k!} \sum_{\sigma \in S_{k+1}} (-1)^{\sigma} \alpha_{j_{\sigma(1)} \dots j_{\sigma(k)}, j_{\sigma(k+1)}} \cdot \beta^{j_1 \dots j_{k+1}} M_g$$

$$= \frac{(-1)^k}{(k+1)! k!} \sum_{\sigma \in S_{k+1}} (-1)^{\sigma} \alpha_{j'_1 \dots j'_k, j'_{k+1}} \beta^{j'_{\sigma(1)} \dots j'_{\sigma(k+1)}} M_g \quad \text{(use Lemma, change } j'_i = j_{\sigma(i)})$$

$$\beta^{j'_{\sigma(1)} \dots j'_{\sigma(k+1)}} = (-1)^{\sigma} \beta^{j'_1 \dots j'_{k+1}} \quad (\text{b/c } \beta \text{ is skew})$$

$$= \frac{(-1)^k}{(k+1)! k!} \sum_{\sigma \in S_{k+1}} \alpha_{j_1 \dots j_k, j_{k+1}} \beta^{j_1 \dots j_{k+1}} M_g = \frac{(-1)^k}{k!} \alpha_{j_1 \dots j_k, j_{k+1}} \beta^{j_1 \dots j_{k+1}} M_g$$

$$= \frac{(-1)^k}{k!} \alpha_{j_1 \dots j_k, j_{k+1}} \beta^{j_1 \dots j_{k+1}} |\det G|^{\frac{k}{2}} d^n x \quad \text{use } f(\tilde{g}) = \tilde{g}(f) - (\delta f) g$$

$$= \frac{\partial}{\partial x^{j_{k+1}}} \left(\frac{(-1)^k}{k!} \alpha_{j_1 \dots j_k} \beta^{j_1 \dots j_{k+1}} |\det G|^{\frac{k}{2}} \right) d^n x - \frac{(-1)^k}{k!} \alpha_{j_1 \dots j_k} \partial_{j_{k+1}} (\beta^{j_1 \dots j_{k+1}} |\det G|^{\frac{k}{2}})$$

$$\gamma^i = \frac{(-1)^k}{k!} \alpha_{j_1 \dots j_k} \beta^{j_1 \dots j_{k+1}} |\det G|^{\frac{k}{2}}, \text{ define } (\bar{\delta}\beta)^{j_1 \dots j_k} \equiv (\text{RHS } *)$$

$$\tilde{g}(\alpha, \delta\beta) M_g \approx \partial_j \gamma^i d^n x - \underbrace{\frac{(-1)^k}{k!} \alpha_{j_1 \dots j_k} \partial_{j_{k+1}} (\beta^{j_1 \dots j_{k+1}} |\det G|^{\frac{k}{2}}) d^n x}_{\text{integrate}}$$

$$\hookrightarrow \tilde{g}(\alpha, \delta\beta - \bar{\delta}\beta) M_g \approx \partial_j \gamma^i d^n x \quad \text{integrate} \quad \alpha_{j_1 \dots j_{k+1}} |\det G|^{\frac{k}{2}} (\bar{\delta}\beta)^{j_1 \dots j_k} d^n x$$

$$\int_U \tilde{g}(\alpha, \delta\beta - \bar{\delta}\beta) M_g = \int_U \alpha^{j_0} + \int_U (\partial_j \gamma^i) d^n x \quad \text{for some } h \text{ (diff for Baby Stokes)} \quad \text{let } d^n x_j = dx^1 \wedge \dots \wedge dx^{i_n} \dots \wedge dx^{k+1}$$

$$\partial_j \gamma^i d^n x = d(\gamma^i(d^n x)_j)$$

$$\begin{gathered} \tilde{g}(\alpha, \delta\beta - \bar{\delta}\beta) = 0 \\ \Rightarrow \qquad \qquad \qquad \Rightarrow \\ \therefore \int_U \tilde{g}(\alpha, \delta\beta - \bar{\delta}\beta) \mu_g = 0 \quad \forall \alpha \quad \delta\beta - \bar{\delta}\beta = 0 // \end{gathered}$$

$\int_U \tilde{g}(\alpha, \beta) \mu_g$ is a metric on k-forms, ($\alpha, \beta \in \Omega^k M$)

- Jelsager : Particles, Fields, and Geometry
↪ Excellent bridge between Math & Physics.