

# 11. SEQUENCES AND SERIES

This chapter diverges wildly from everything we have done up to this point. Now more than ever it is important that you not miss any lecture. This chapter is much more about logic and applying theory than algorithmic calculation. For most of you this is not good news. However, don't despair. Just take it one day at a time and you'll get it. It will be easier if you have a good attitude about it. ( I speak from my own experience )

What is our goal? Our goal in this Chapter as well as the two that follow is to find a robust approximation scheme for functions. In particular, we will see how to rewrite most functions as a sort of infinite polynomial. We already took the first step towards this in calculus I, we replaced a function by its linearization. That is a first-order approximation. Next, you can replace a function by a quadratic polynomial, this would be a second-order approximation. If you continue without end you arrive at what is known as a power series. In practice we cannot go on forever on a computer calculation, however we can keep as many terms as we need to arrive at the precision that the problem requires. This Chapter is needed to build us up to the point of understanding how to carefully define a power series.

Historically speaking the idea of a power series approximation goes back several centuries and developments in calculus and series/sequences have been inextricably linked. Sequences form very important examples in the study of limits. Analysis ( careful mathematics built from limiting arguments ) matured historically because it demanded to arrive at a logically consistent treatment of sequences and series. The better part of the nineteenth century was filled with correcting minor mistakes in the arguments of Newton and Leibniz. Without getting too technical, what happened was that the early fathers of calculus used power series arguments without paying enough attention to what the proper domains should be for the series.

Details and domains matter more when you start getting to the edge of what is known. In the nineteenth century astronomy gathered observations of the motion of the planets that were very precise. However, the mathematics of Newton's Universal Law of Gravitation did not allow an exact solution. The problem of figuring out how all the planets pull on each other by the force of gravity is quite complicated. There is the Sun and all the planets, their motions are coupled. Approximations to the real forces have to be used just to make the mathematics workable. However, then you have to make sure the mathematical approximation is not creating error bigger than the error inherent in the measurements themselves. It took a herculean effort by an army of mathematicians and scientists to show that all the motions of the planets were explained beautifully by Newton's Theory. Well everything except for the perihelion of Mercury. Turns out they calculated correctly, Newton's theory was wrong. But, that is a story for another day.

Bottom line, power series are an indispensable tool for mathematical sciences.

## 11.1. SEQUENCE EXAMPLES

So what is a sequence? (by the way, you should read Stewart section 11.1, it's cleaner than these notes on certain points and he has lovely pictures)

Def<sup>n</sup> A sequence is a list of numbers  $\{a_1, a_2, a_3, \dots\} = \{a_n\}_{n=1}^{\infty} = \{a_n\}$ .  
 For each  $n \in \mathbb{N}$  we assign  $a_n \in \mathbb{R}$  (it's a fct. from  $\mathbb{N}$  to  $\mathbb{R}$ ).  
 Usually we take sequences starting from  $n=1$  however we also consider  
 $\{b_0, b_1, b_2, b_3, \dots\} = \{b_n\}_{n=0}^{\infty}$   
 $\{c_3, c_4, c_5, \dots\} = \{c_n\}_{n=3}^{\infty}$   
 the above to be sequences.

I should emphasize that a sequence is an ordered list of numbers.

### Examples 11.1.1 through 11.1.3

E1  $\{a_n\}_{n=0}^{\infty} = \{1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots\} = \{a_0, a_1, a_2, a_3, \dots\}$

What's a formula for  $a_k$ ? well inspection reveals:

$$a_k = \sqrt{k+1}$$

E2  $\{b_n\}_{n=1}^{\infty} = \{2, 2.5, 3.\bar{3}, 4.25, \dots\}$  then we can

see the formula for an arbitrary term in this sequence is

$$b_k = k + \frac{1}{k}$$

E3  $\{c_n\}_{n=1}^{\infty} = \{0, \frac{-1}{2}, \frac{2}{3}, \frac{-3}{4}, \dots\}$  what is  $c_n$ ? Well, the sequence is alternating ... the formula is

$$c_n = (-1)^{n+1} \left( \frac{n-1}{n} \right)$$

### Example 11.1.4 (Fibonacci Sequence)

Sequences naturally occur in computer science. Often those are defined recursively, some loop generates the next value in the sequence from the last. A recursively defined sequence may not have a nice global formula like we say in E1, E2, E3. The Fibonacci Sequence is one of the more famous recursively defined sequences:

$$f_1 \equiv 1, \quad f_2 \equiv 1, \quad f_3 = 1 + 1 = 2, \quad f_4 = 2 + 1 = 3, \quad f_5 = 3 + 2 = 5$$

Generally the pattern is  $f_n = f_{n-1} + f_{n-2}$  for  $n \geq 3$ . To summarize,

$$\{f_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$$

### Example 11.1.5 (Silly bonus point example)

I'll give you a bonus point if you can crack the definition of the following sequence and tell me the next element beyond those already listed:

$$\{19, 8, 18, 5, 11, 1, 14, 4, 6, 9, 15, 14, \dots\}$$

The next element not listed is fairly well suggested by what is already there, past that I suppose it could repeat, but in principle there are limitless options. Much like being given graph, we can't be certain what happens beyond the given viewing window.

**Remark:** since a sequence is just a function from  $\mathbb{N} \rightarrow \mathbb{R}$  it follows we can construct new sequences from old sequences in many of the same ways as we did for functions. If  $\{a_n\}_{n=1}^{\infty}$ ,  $\{b_n\}_{n=1}^{\infty}$  are sequences then  $\{a_n + b_n\}_{n=1}^{\infty}$  and  $\{a_n b_n\}_{n=1}^{\infty}$  are also sequences. We can also multiply a sequence by a number  $c$  to obtain a new sequence  $\{c_n\}_{n=1}^{\infty} \equiv \{ca_n\}_{n=1}^{\infty}$  where the formula for  $c_n$  is naturally  $c_n = ca_n$  for each  $n \geq 1$ . In contrast, composition of sequences almost never would make sense as the output of a sequence is real numbers and the outer function of the composite would need inputs of natural numbers.

**Big Picture Comment:** the concept of a sequence is much more general than our examples and this course portrays. Pretty much anything which can be listed in order forms a sequence. We insist that our list be filled with real numbers, but they could just as well be complex numbers, matrices, triangles, or clowns. A sequence in a space is a function from  $\mathbb{N}$  into the space. We will deal exclusively with the simple case of real-valued sequences in calculus II. (convergence is trickier in spaces other than  $\mathbb{R}$ )

## 11.2. CONVERGENCE OF SEQUENCES

Sequences can converge or diverge but not both. We say a sequence converges to  $L \in \mathbb{R}$  if as we go further out the sequence we get values closer to  $L$ . If this reminds you of our definition of  $\lim_{x \rightarrow \infty} f(x) = L$  then good, it is the same thing conceptually.

Def<sup>n</sup> A sequence  $\{a_n\}$  has limit  $L$  which is denoted  
 $\lim_{n \rightarrow \infty} a_n = L$  OR  $a_n \rightarrow L$  as  $n \rightarrow \infty$   
If we can make  $a_n$  arbitrarily close to  $L$  for sufficiently large  $n$ .  
When the limit exists we say  $\{a_n\}$  converges to  $L$  otherwise  
we say  $\{a_n\}$  diverges (or is divergent)

There is the definition and notation in words. Let me be a bit more exact. There is a technical formulation of this limit.

### Technical Definition of Limit of Sequence

Let  $\{a_n\}$  be a sequence then we say  $a_n \rightarrow L$  as  $n \rightarrow \infty$  iff for each  $\epsilon > 0$  there exists a  $M \in \mathbb{N}$  such that whenever  $M < n \in \mathbb{N}$  we find  $|a_n - L| < \epsilon$ .

For those of you who are keeping score this is verbatim the definition we gave before for  $f(x) \rightarrow L$  as  $x \rightarrow \infty$ . The only difference is that the sequence is tested at natural numbers whereas the function is tested at real numbers. Given this observation the following Theorem is quite unsurprising:

**Th<sup>n</sup>** If  $f(n) = a_n$  for each  $n \in \mathbb{Z}$ ,  $n \geq n_0$  then

$$\lim_{x \rightarrow \infty} f(x) = L \implies \lim_{n \rightarrow \infty} a_n = L$$

Additionally if the limit of  $f$  diverges then the limit of  $\{a_n\}$  diverges in the same way.

Stewart makes a fairly big deal about this in various examples. He says you cannot apply L'Hopital's Rule to a limit of a sequence. And technically he is correct, but the Theorem above shows that it is not wrong to think of extending the domain of the sequence to the real numbers. I will allow you to apply the Theorem by saying "I'm extending  $n$  to be a continuous variable" in the margin when you use L'Hopital's Rule. This saves some writing. I suppose I should mention that limits of sequences also share many of the same properties as limits of functions, we assume  $A, B \in \mathbb{R}$  in what follows:

### Properties of Limits:

Let  $a_n \rightarrow A$  and  $b_n \rightarrow B$  as  $n \rightarrow \infty$  then

- i)  $\lim_{n \rightarrow \infty} (a_n \pm b_n) = A \pm B$
- ii)  $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$
- iii)  $\lim_{n \rightarrow \infty} (c \cdot a_n) = c \cdot A \quad (c \in \mathbb{R})$
- iv)  $\lim_{n \rightarrow \infty} (a_n / b_n) = A / B \quad (B \neq 0)$
- v)  $\lim_{n \rightarrow \infty} (c) = c \quad (\text{dub.})$

**Example 11.2.1**

Find the limit of  $a_n = \frac{n+3}{n^2+5n+6}$ . I can think of about 4 or so somewhat distinct ways to solve this limit. Let's contrast the methods.

1. Use algebra:

$$\lim_{n \rightarrow \infty} \frac{n+3}{n^2+5n+6} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \frac{3}{n^2}}{1 + \frac{5}{n} + \frac{6}{n^2}} = \frac{0}{1} = 0$$

2. Use algebra:

$$\lim_{n \rightarrow \infty} \frac{n+3}{n^2+5n+6} = \lim_{n \rightarrow \infty} \frac{n+3}{(n+3)(n+2)} = \lim_{n \rightarrow \infty} \frac{1}{n+2} = 0$$

3. Use the largest power wins logic: (I'm fond of this one)

$$\lim_{n \rightarrow \infty} \frac{n+3}{n^2+5n+6} = \lim_{n \rightarrow \infty} \frac{n}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

4. Extending  $n$  to be a continuous variable we apply L'Hopital's Rule to type  $\frac{\infty}{\infty}$ :

$$\lim_{n \rightarrow \infty} \frac{n+3}{n^2+5n+6} = \lim_{n \rightarrow \infty} \frac{1}{2n+5} = 0$$

5. Eyeball it: as  $n \rightarrow \infty$  the denominator is huge compared to the numerator, just look at  $n=1000$  for example... the answer is zero.

When I am taking a limit as part of a larger problem and it is a simple limit like this one I do tend to use 5.) a fair amount. You should only attempt 5.) once you have mastered the other options. I do hope you gather an intuition about these things by the time we are done. For example, I hope you become fluent in the results below

LIMITS OF FUNCTIONS	$\Rightarrow$	LIMITS OF SEQUENCES
$\lim_{x \rightarrow \infty} (x^p) = \begin{cases} \infty & p > 0 \\ 1 & p = 0 \\ 0 & p < 0 \end{cases}$	$\Rightarrow$	$\lim_{n \rightarrow \infty} (n^p) = \begin{cases} \infty & p > 0 \\ 1 & p = 0 \\ 0 & p < 0 \end{cases}$
$\lim_{x \rightarrow \infty} (\ln(x)) = \infty$	$\Rightarrow$	$\lim_{n \rightarrow \infty} (\ln(n)) = \infty$
$\lim_{x \rightarrow \infty} (e^x) = \infty$	$\Rightarrow$	$\lim_{n \rightarrow \infty} (e^n) = \infty$
$\lim_{x \rightarrow \infty} (\tan^{-1}(x)) = \frac{\pi}{2}$	$\Rightarrow$	$\lim_{n \rightarrow \infty} (\tan^{-1}(n)) = \frac{\pi}{2}$

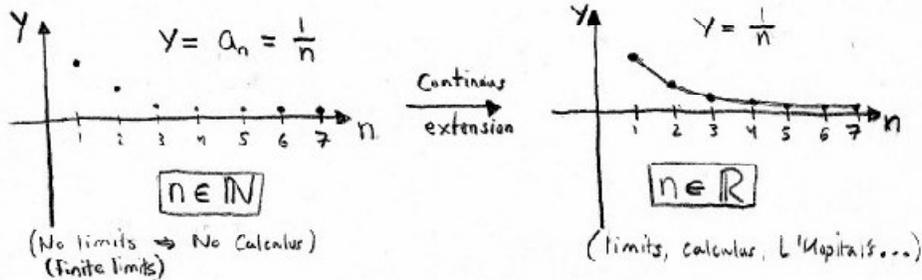
**Example 11.2.2 (the picture illustrates how we can extend a sequence to a function)**

$$\lim_{n \rightarrow \infty} (n e^{-n}) = \lim_{n \rightarrow \infty} \left( \frac{n}{e^n} \right)$$

$$\stackrel{f}{=} \lim_{\left(\frac{\infty}{\infty}\right)} \left( \frac{1}{e^n} \right) \quad \left( \text{Extending } n \text{ to be a continuous variable.} \right)$$

$$= \boxed{0}$$

Graphical Meaning of "Extending to continuous variable" for  $\frac{1}{n}$ ,



I know you have missed the squeeze theorem. Good news, its back:

Th<sup>m</sup> / Squeeze Th<sup>m</sup> for Sequences: If  $a_n \leq b_n \leq c_n$   
 for  $n \geq n_0$  and  $\lim_{n \rightarrow \infty} (a_n) = \lim_{n \rightarrow \infty} (c_n) = L$  then  

$$\lim_{n \rightarrow \infty} (b_n) = L$$

**Example 11.2.6**

**E6** Find  $\lim_{n \rightarrow \infty} \left( \frac{\sin(n)}{n} \right)$ .

Notice that:  $-1 \leq \sin(n) \leq 1$

And for  $n \geq 1$ ,  $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

Then clearly  $\pm \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ . Thus

By Squeeze Th<sup>m</sup> so must  $\frac{\sin(n)}{n} \rightarrow 0$  as  $n \rightarrow \infty$ .

$$\lim_{n \rightarrow \infty} \left( \frac{\sin(n)}{n} \right) = 0$$

Th<sup>o</sup> (Absolute Convergence for Sequences)

$$\text{If } \lim_{n \rightarrow \infty} |a_n| = 0 \text{ then } \lim_{n \rightarrow \infty} (a_n) = 0$$

Example 11.2.7

[E7] Calculate  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$  if possible.

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \right) = 0 \Rightarrow \boxed{\lim_{n \rightarrow \infty} \left( \frac{(-1)^n}{n} \right) = 0}$$

The example that follows is used often in later sections.

Example 11.2.8

[E8] • Important Example (not connected with Th<sup>o</sup>'s above)

$$\lim_{n \rightarrow \infty} (r^n) = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & -1 < r < 1 \\ \text{d.n.e} & r \leq -1 \end{cases}$$

When we study the geometric series this limit will help us stay out of trouble.

Increasing and Decrease in Sequences

Def<sup>n</sup> / A sequence  $\{a_n\}_{n_0}^{\infty}$  is increasing if  $a_n < a_{n+1}$  for all  $n \geq n_0$ .  
A sequence  $\{b_n\}_{n_0}^{\infty}$  is decreasing if  $b_n > b_{n+1}$  for all  $n \geq n_0$ .  
A sequence is monotonic if its either inc. or dec.

We can study the continuous extension of a sequence if it has a nice formula to extend:

Remark: If  $f(n) = a_n$  and  $f(x)$  is increasing on  $x \geq 1$  then its clear that  $\{a_n\}_{n=1}^{\infty}$  is an inc. sequence.

Def<sup>n</sup>/ A sequence  $\{a_n\}_{n=n_0}^{\infty}$  is bounded above if  $\exists M \in \mathbb{R}$  such that  $a_n \leq M$  for  $n \geq n_0$ . Likewise  $\{a_n\}_{n=n_0}^{\infty}$  is bounded below if  $\exists m \in \mathbb{R}$  such that  $a_n \geq m$  for  $n \geq n_0$ . When  $\{a_n\}_{n=n_0}^{\infty}$  is bounded above AND below it said to be bounded.

Don't get lost in the technicalities here, it's really very simple, a bounded sequence will fit inside some finite horizontal band if we look at large  $n$ . This doesn't mean it has to have a convergent limit. Sine and cosine are bounded but they certainly do not converge. We need something more to insure that a bounded sequence will converge.

**Example 11.2.9**

**E9**  $a_n = 3 + \frac{1}{n}$  is bounded. Why? Well notice that  $3 < 3 + \frac{1}{n} \leq 4$  for  $n \geq 1$ . Notice that  $a_n \rightarrow 3$  as  $n \rightarrow \infty$ . This is no accident,

**Th<sup>m</sup> / (Monotonic Sequence Th<sup>m</sup>)**  
Every bounded monotonic sequence is convergent

Notice that the sequence in E9 is monotonic because it is decreasing everywhere. Why is it decreasing? I recommend the following test:

**Decreasing Sequence Test (I use this in Ex. 11.3.14 and 11.3.15 and elsewhere)**

The advice is this: use differentiation to analyze increase/decrease. The steps that follow only apply to sequences which have formulas which extend nicely to functions of a continuous real variable. I wouldn't try my advice below for  $a_n = (-1)^n/n$  or  $a_n = 1/n!$

- 1.) Extend  $n$  to be a continuous variable then differentiate with respect to  $n$ .
- 2.) Analyze the derivative is it positive or negative for large  $n$ ?
  - a.) If  $da_n/dn > 0$  for all large  $n$  then the sequence is increasing.
  - b.) If  $da_n/dn < 0$  for all large  $n$  then the sequence is decreasing.
  - c.) If  $da_n/dn$  does oscillates between positive and negative values for large  $n$  then the sequence is not monotonic.

**Remark:** if a sequence fails to be monotonic we should not conclude that it diverges. See Example 11.2.7 for example.

### Example 11.2.10

**E10**  $n! = n(n-1)(n-2)\cdots(4)(3)(2)(1)$  "n factorial"

How does the sequence  $a_n = \frac{1}{n!}$  behave? Well notice that  $0 < \frac{1}{n!} \leq 1$  for  $n \geq 1$  so it's bounded,

$$\frac{1}{(n+1)!} = \frac{1}{(n+1)n(n-1)\cdots 3 \cdot 2 \cdot 1} < \frac{1}{n(n-1)\cdots 3 \cdot 2 \cdot 1} = \frac{1}{n!} \quad (\text{for } n > 1)$$

We see  $\frac{1}{n!} > \frac{1}{(n+1)!}$  the sequence  $\{\frac{1}{n!}\}$  is decreasing. Thus

$\{\frac{1}{n!}\}_{n=1}^{\infty}$  is a bounded monotonic sequence, it converges. In fact

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n!} \right) = 0 \quad \leftarrow \text{haven't argued why but it's not hard to believe.}$$

**In-class Exercise 11.2.10b:** Stewart gives a squeeze theorem argument to prove the boxed assertion. We almost have his proof here, what steps are we missing? Prove the boxed limit.

## 11.3. SERIES AND CONVERGENCE TESTS

The primary question we wish to answer is when does the sum of a sequence add up to a real number?

$$a_1 + a_2 + \cdots + a_n + \cdots = S$$

The sum of a sequence is called a series. We need to make sense of this more carefully,

**Def<sup>n</sup>** The  $n^{\text{th}}$  partial sum of  $\{a_n\}$  is  $S_n = \sum_{i=1}^n a_i$

**In-class Exercise 11.3.0:** The sequence of partial sums is  $\{s_1, s_2, s_3, \dots\}$ . Calculate the first three or four terms in the sequence of partial sums relative to the sequences

- Find the first 4 terms in the sequence of partial sums relative to the sequence  $a_n = \frac{1}{n}$  for  $n \geq 1$ .
- Find the first 4 terms in the sequence of partial sums relative to the sequence which has terms  $a_n = \frac{(-1)^n}{n}$  for  $n \geq 1$ .
- Find the first 4 terms in the sequence of partial sums relative to the sequence which has terms  $a_n = \frac{(-1)^n}{n}$  for  $n \geq 1$ .

Def<sup>n</sup>/ The series (relative to  $\{a_n\}$ ) is the limit of the partial sums  

$$S \equiv \lim_{n \rightarrow \infty} S_n \equiv a_1 + a_2 + a_3 + \dots$$

When this limit exists we say the series converges. Otherwise we say the series  $S$  diverges.

The question "does the series converge?" is possibly the most challenging question we ask calculus students. The majority of this chapter is dedicated to seeing how that question is answered by various tests. Before we get to the general tests we consider the nice examples of geometric and telescoping series. Many of these actually converge in a way which is easy to calculate and discuss. Before we get to that let me just list a few examples without proof.

**Examples 11.3.1 through 11.3.3 (we'll explain E1 and E2 later, E3 is too hard for us)**

[E1]  $\{a_n\} = \{\frac{1}{n}\}$  has a divergent series  $S = 1 + \frac{1}{2} + \frac{1}{3} + \dots$   
 this is called the harmonic series.

[E2]  $\{a_n\} = \{1\}$  gives div. series,  $S = 1 + 1 + 1 + \dots$

[E3]  $\{\frac{1}{n^2}\} = \{a_n\}$  is convergent  $S = 1 + \frac{1}{4} + \frac{1}{9} + \dots = \frac{\pi^2}{6}$

It is interesting and for most people a little surprising that E1 diverges while E3 converges. Probably E1 is the most important example besides the geometric series.

## **Geometric Series Test**

**GEOMETRIC SERIES:**

$$a + ar + ar^2 + \dots = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } -1 < r < 1$$

Proof: Strategy find  $S_n$  explicitly then let  $n \rightarrow \infty$  to find series,

$$\begin{aligned} S_n &= a + ar + \dots + ar^{n-1} \\ -rS_n &= ar + ar^2 + \dots + ar^n \\ \hline S_n - rS_n &= a - ar^n \quad \Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r} \end{aligned}$$

$$S = \lim_{n \rightarrow \infty} \frac{a(1-r^n)}{1-r} = a \cdot \lim_{n \rightarrow \infty} \left( \frac{1}{1-r} \right) - \frac{a}{1-r} \lim_{n \rightarrow \infty} (r^n)$$

$\downarrow 0 \quad (-1 < r < 1)$

$$\therefore \boxed{S = \frac{a}{1-r}} \quad \text{for } |r| < 1 \leftarrow (\text{interval of convergence})$$

**Example 11.3.4 (applying the geometric series result)**

**E4** Find the exact fraction that gives  $2.1\overline{6} = 2.161616$ . We can use the geometric series:

$$2.\overline{16} = 2 + \frac{16}{100} + \frac{16}{(100)^2} + \frac{16}{(100)^3} + \dots = 2 + \frac{16/100}{1 - \frac{1}{100}}$$

$$a = \frac{16}{100} \text{ and } r = \frac{1}{100}$$

$$\frac{16}{100(1 - \frac{1}{100})} = \frac{16}{100 - 1} = \frac{16}{99} = 0.\overline{16} \quad \therefore 2.\overline{16} = 2 + \frac{16}{99}$$

**In-class Exercise 11.3.5 (applying the geometric series result)**

Does the series  $\sum_{n=0}^{\infty} 3(2)^n e^{-n}$  converge or diverge? If it converges find the value to which it converges.

**Telescoping Series Examples**

**Examples 11.3.6 (the term "telescoping refers to the nice cancellation below)**

**E6** Calculate  $\sum_{n=1}^{\infty} (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) = 5$

$$S_n = (1 - \frac{1}{\sqrt{2}}) + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}) + (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}) + \dots + (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) = 1 - \frac{1}{\sqrt{n+1}}$$

$$S = \lim_{n \rightarrow \infty} (1 - \frac{1}{\sqrt{n+1}}) = 1 = 5$$

The fact that we can just calculate  $s_n$  by brute force is quite unusual in the big scheme of things however all the telescoping series work more or less like this example.

**Examples 11.3.7 (Telescoping Series)**

**E7**  $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = 5$  use partial-fractions  $\frac{4}{(4n-3)(4n+1)} = \frac{1}{4n-3} - \frac{1}{4n+1}$

$$S_n = (\frac{1}{4-3} - \frac{1}{4+1}) + (\frac{1}{8-3} - \frac{1}{8+1}) + (\frac{1}{12-3} - \frac{1}{12+1}) + \dots + (\frac{1}{4(n-1)-3} - \frac{1}{4(n-1)+1}) + (\frac{1}{4n-3} - \frac{1}{4n+1})$$
$$= (1 - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{9}) + (\frac{1}{9} - \frac{1}{13}) + \dots + (\frac{1}{4n-7} - \frac{1}{4n-3}) + (\frac{1}{4n-3} - \frac{1}{4n+1})$$
$$= 1 - \frac{1}{4n+1}$$

$$\therefore S = \lim_{n \rightarrow \infty} (1 - \frac{1}{4n+1}) = 1 = 5$$

**In-class Exercise 11.3.8:** Show that the series below converges and find its value.

$$s = \sum_{n=1}^{\infty} \left( \frac{2}{n^2 + 4n + 3} \right)$$

## N-th Term Test

Th<sup>m</sup>(6) If  $\sum_{n=1}^{\infty} a_n$  is convergent then  $\lim_{n \rightarrow \infty} a_n = 0$

(203)

Th<sup>m</sup>(7) (*n*<sup>th</sup> term divergence test)  
if  $\lim_{n \rightarrow \infty} a_n \neq 0$  then  $\sum_{n=1}^{\infty} a_n$  is divergent.

important!

Remark: this th<sup>m</sup> proves that  $1+1+1+\dots$  does not converge since  $a_n = 1$  and  $\lim_{n \rightarrow \infty} a_n = 1 \neq 0 \therefore \sum_{n=1}^{\infty} a_n$  diverges.

Remark: Th<sup>m</sup>(7) is the 1<sup>st</sup> and easiest of the divergence tests. We'll find more sophisticated tests in coming lectures, notice that

$$\lim_{n \rightarrow \infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

**In-Class Exercise 11.3.8b:** Does  $\sum_{n=0}^{\infty} \tan^{-1}(n)$  converge or diverge?

## New from Old Test

Th<sup>m</sup>(8) Given that  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are convergent and  $c \in \mathbb{R}$  then all of the following are also convergent series.

i)  $\sum_{n=1}^{\infty} c a_n = c \cdot \sum_{n=1}^{\infty} a_n$

ii)  $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

**Examples 11.3.9 and 11.3.10 (illustrate New from Old Test)**

**E9**  $\sum_{n=1}^{\infty} (ar^{n-1} + \frac{1}{n} - \frac{1}{n+1}) \stackrel{(*)}{=} \sum_{n=1}^{\infty} ar^{n-1} + \sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1})$

$$= \frac{a}{1-r} + 1$$

↑  
Geom. Series    See **E6**

bc these are convergent the (\*) equality is an ok step.

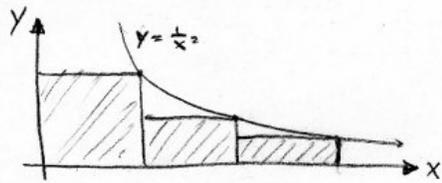
**E10**  $\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1}) \neq \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}$

telescopes  
to 1 in fact.

these are divergent  
thus Th<sup>m</sup>(8) cannot be used.

## Integral Test

Motivating Example 1:

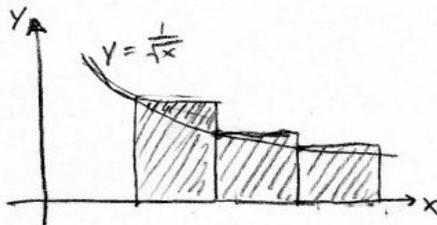


the boxed areas give  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

the curve  $y = \frac{1}{x^2}$  has the area under it given by  $\int_1^{\infty} \frac{1}{x^2} dx$   
(note we start at 1,  $\int_0^{\infty} \frac{1}{x^2} dx = \infty$ )

Remark: it seems reasonable to conclude that  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges since  $\int_1^{\infty} \frac{1}{x^2} dx = 1$  and is clearly bigger than the series, see graph above.

Motivating Example 2:



the boxed areas give  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

the curve  $y = \frac{1}{\sqrt{x}}$  has area under it  $\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$ .

Remark: if the  $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$  diverges then surely the larger object  $\sum \frac{1}{\sqrt{n}}$  should also diverge.

### THM (INTEGRAL TEST)

Suppose  $f$  is continuous, positive, decreasing fct on  $[1, \infty)$  with  $a_n = f(n)$ .  
Then the series  $\sum_{n=n_0}^{\infty} a_n$  is convergent  $\Leftrightarrow \int_{n_0}^{\infty} f(x) dx$  is convergent.

(a.) If  $\int_{n_0}^{\infty} f(x) dx$  converges  $\Rightarrow \sum_{n=n_0}^{\infty} a_n$  is convergent

(b.) If  $\int_{n_0}^{\infty} f(x) dx$  diverges  $\Rightarrow \sum_{n=n_0}^{\infty} a_n$  diverges.

Usually  $n_0 = 1$ .

(this is a weapon of last resort, most of the other tests are less work if they are applicable. What this test says is you can trade the given problem for an improper integral, it's only useful if you can integrate the formula for the series)

**In-Class Exercise 11.3.11a: (does the given series converge or diverge?)**

$$s = 1.5 + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \dots$$

### Example 11.3.11: (integral test example)

**E11** Determine whether  $\sum_{n=1}^{\infty} n e^{-n^2}$  converges or diverges.

Notice that we can perform the integration  $\int x e^{-x^2} dx$ ,

$$\begin{aligned} \int x e^{-x^2} dx &= \int e^u \left( \frac{-du}{2} \right) && \left\{ \begin{array}{l} u = -x^2 \\ du = -2x dx \end{array} \right. \\ &= \underline{\underline{-\frac{1}{2} e^{-x^2} + C}} \end{aligned}$$

Which makes it easy to compute the improper integral of interest,

$$\begin{aligned} \int_1^{\infty} x e^{-x^2} dx &= \lim_{t \rightarrow \infty} \left( \int_1^t x e^{-x^2} dx \right) \\ &= \lim_{t \rightarrow \infty} \left( \left. -\frac{1}{2} e^{-x^2} \right|_1^t \right) \\ &= \lim_{t \rightarrow \infty} \left( \underbrace{-\frac{1}{2} e^{-t^2}}_0 + \frac{1}{2} e^{-1} \right) \\ &= \boxed{\frac{1}{2e}} \quad \text{so it converges.} \end{aligned}$$

Therefore by integral test  $\sum_{n=1}^{\infty} n e^{-n^2}$  likewise converges.

### P-series Test

The "P-SERIES TEST", show  $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$

**In-class Exercise:** prove the P-series test is true.

### Example 11.3.11: (almost p-series test example)

Does  $s = \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \frac{1}{196} + \frac{1}{243} + \dots$  converge or diverge? Well this one is almost the  $p=3$  series since  $\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots$ . Let's say the  $p=3$  series converges to  $L$ , we know  $L$  is a real number by the P-series test. Then notice we can add and subtract  $1 + 1/8$  in order to see how the  $p=3$  series is related to the given series.

$$s = -1 - \frac{1}{8} + 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots = -\frac{9}{8} + L$$

Thus  $s = L - 9/8$ , it converges. (it doesn't matter that we don't know what  $L$  is precisely, we'll tackle the question of how to get a reasonably good approximation of  $L$  in a later section. "Converge or diverge?" is a question of existence)

## Alternating Series Test

**Th<sup>n</sup> / ALTERNATING SERIES TEST (A.S.T.):**

If you're given an alternating series,

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots, \quad b_n > 0$$

which satisfies the following two criteria

- i.)  $b_{n+1} \leq b_n$  (for  $n \geq n_0 \geq 1$ )
- ii.)  $\lim_{n \rightarrow \infty} (b_n) = 0$

Then the series converges.

Remark: I've heard some call this the wishful thinking test. If your series alternates and decreases so that terms get small then the series converges. Perhaps nonintuitively this is not the case when the series is not alternating.

### Example 11.3.13

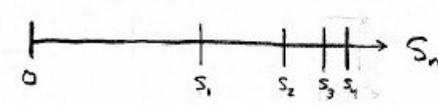
**E13**  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  is the "alternating harmonic series"

Notice  $b_n = \frac{1}{n} > 0$  and  $b_{n+1} \leq b_n$  since  $\frac{1}{n+1} \leq \frac{1}{n}$ .

Finally  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$   $\therefore$  By A.S.T.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  converges.

This is remarkable since the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges!

An interesting picture can help us understand why this happens. I graph the partial sums of the  $\sum_{n=1}^{\infty} \frac{1}{n}$  and  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  to compare,



Harmonic Series keeps growing

$$S_1 = 1$$

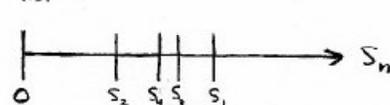
$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$\vdots$$

$$S_{1,78,482,301} \approx 20$$



Alternating Harmonic series converges

$$S_1 = 1$$

$$S_2 = 1 - \frac{1}{2}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

$$\vdots$$

$$\lim_{n \rightarrow \infty} (S_n) = \ln(2) = 0.693.$$

Wow! Look at how slow the harmonic series diverges. I should mention that the alternating harmonic series is said to be *conditionally convergent*. More on that later.

**Example 11.3.14 and 11.3.15**

Notice we have to check for decreasing  $b_n$ . If you claim to apply the AST then you must mention and/or check that  $b_n$  is both positive and decreasing. How much work is owed to prove it is decreasing depends on the formulas. These examples illustrate full-credit solutions. I do give partial credit for mildly illogical and/or incomplete proofs.

E14  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$  is an alternating series with  $b_n = \frac{n^2}{n^3+1} > 0$  (208)

Notice (Extending  $n$  continuously) we can show  $b_n$  decreases,

$$\begin{aligned} \frac{d}{dn} \left( \frac{n^2}{n^3+1} \right) &= \frac{2n(n^3+1) - n^2(3n^2)}{(n^3+1)^3} \\ &= \frac{-n^4 + 2n}{(n^3+1)^3} \\ &= \frac{n}{(n^3+1)^3} (2 - n^3) < 0 \quad (\text{provided } n \geq 2) \end{aligned}$$

One criteria down, one to go,

$$\lim_{n \rightarrow \infty} \left( \frac{n^2}{n^3+1} \right) = \lim_{n \rightarrow \infty} \left( \frac{1/n}{1+1/n^3} \right) = 0$$

Thus by A.S.T the series converges.

E15 Does  $\sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{1+2n}{n} \right)$  converge or diverge?

Clearly this is an alternating series,  $b_n = \frac{1+2n}{n}$  note then

$$\frac{d}{dn} \left( \frac{1+2n}{n} \right) = \frac{d}{dn} \left( \frac{1}{n} + 2 \right) = \frac{-1}{n^2} < 0 \Rightarrow b_n \text{ are decreasing}$$

But something is wrong, note  $\lim_{n \rightarrow \infty} \left( \frac{1+2n}{n} \right) \neq \lim_{n \rightarrow \infty} \left( \frac{2}{1} \right) = 2$ .

Thus the A.S.T. is inconclusive. Notice the  $n^{\text{th}}$  term test shows us this series diverges

$$\lim_{n \rightarrow \infty} \left( (-1)^{n+1} \left( \frac{1+2n}{n} \right) \right) \text{ d.n.e. (so its not equal to zero.)}$$

The sequence oscillates at  $\infty$  between 2 and -2.

This same logic can be applied whenever you find an alternating series with  $b_n \not\rightarrow 0$ . They all diverge.

**Remark:** I might lose a point on E15. What slight error did I make? E14 in contrast didn't neglect this detail.

## Ratio Test

Th<sup>n</sup> (RATIO TEST) We define  $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

(a.) If  $L < 1$  then  $\sum a_n$  is convergent

(b.) If  $L > 1$  or  $L = \infty$  then  $\sum a_n$  is divergent

Notice that the Ratio Test is inconclusive in the case  $L=1$ . This is especially important when we get to power series. The cases a. and b. determine almost the entire domain for the power series, however on the edges of the domain the ratio test returns  $L=1$  so we have to "check the endpoints" by one of the other tests.

### Examples 11.3.16 and 11.3.17 (Ratio Test)

**E16**  $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$  does it converge? Well consider

$$L = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1)^3}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^3} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{3n^3} \right|$$

$$= \frac{1}{3} \lim_{n \rightarrow \infty} \left| \left( \frac{n+1}{n} \right)^3 \right|$$

$$= \frac{1}{3} \lim_{n \rightarrow \infty} \left| \left( 1 + \frac{1}{n} \right)^3 \right|$$

$$= \frac{1}{3} < 1 \quad \therefore \quad \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n} \text{ converges by Ratio Test}$$

**E17**  $\sum_{n=1}^{\infty} \frac{2^n + 5}{3^n}$  Use Ratio Test to investigate convergence of this series,

$$L = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} + 5}{3^{n+1}} \cdot \frac{3^n}{2^n + 5} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{1}{3} \left( \frac{2^{n+1} + 5}{2^n + 5} \right) \right|$$

$$\stackrel{\frac{\infty}{\infty}}{\text{L'H}} \lim_{n \rightarrow \infty} \left| \frac{1}{3} \left( \frac{\ln(2) 2^{n+1}}{\ln(2) 2^n} \right) \right| \quad (\text{extending } n \text{ to be continuous variable})$$

$$= \frac{2}{3} < 1 \quad \therefore \quad \sum_{n=1}^{\infty} \frac{2^n + 5}{3^n} \text{ converges by Ratio Test.}$$

**In-class Exercise 11.3.17b:** find the value to which the series in E17 converges.

**Example 11.3.18** technically we are considering a whole bunch of series all at once. Each value of  $x$  gives a different series. It is interesting that each and every value of  $x$  yields a convergent series.

**E18** Show that  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges for any  $x$ .

We'll use the ratio test. Here  $a_n = \frac{x^n}{n!}$  so,

$$L = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| x \frac{n!}{(n+1)!} \right|$$

$$= |x| \lim_{n \rightarrow \infty} \left| \frac{n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1}{(n+1)n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1} \right|$$

$$= |x| \lim_{n \rightarrow \infty} \left( \frac{1}{n+1} \right)$$

$$= 0 < 1 \quad \therefore \text{By ratio test } \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ converges.}$$

**In-class Exercise 11.3.19** Calculate the following to 3 significant digits, you will need a calculator.

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

Identify this number and make a guess what the power series in E18 converges to for an arbitrary value for  $x$ . (this is with  $x = 1$ )

## COMPARISON TESTS

Compare to what? Well we know a number of basic examples at this point. Let's make a list and collect our thoughts up to this point.

Summary of Tests to determine convergence of $s = \sum_{n=1}^{\infty} a_n$	
1.)	<u><math>n^{\text{th}}</math> term test</u> said if $\lim_{n \rightarrow \infty} (a_n) \neq 0$ then $s$ diverges.
2.)	<u>geometric series</u> : if $\frac{a_{n+1}}{a_n} = r$ then $s = \frac{a}{1-r}$ if $ r  < 1$ if the ratio $r \geq 1$ the geometric series diverges.
3.)	<u>telescoping series</u> : We could actually calculate $S_n$ because of a bunch of nice cancellations, sometimes needs partial frac.
4.)	<u>integral test</u> : can use conv/div of improper integrals to infer conv/div. of the series.
5.)	<u>alternating series test</u> : if series alternates with decreasing terms whose limit is zero as $n \rightarrow \infty$ then $s$ converges
6.)	<u>ratio test</u> : $L \equiv \lim_{n \rightarrow \infty} \left  \frac{a_{n+1}}{a_n} \right $ then $L < 1$ means that the series converges, $L > 1$ says it diverges and $L = 1$ tells us nothing.

We also discussed the New from Old Test. The comparison tests allow us to treat examples which are similar to those we already analyzed. Roughly speaking, if some given series is a lot like one of the ones we have already categorized then the new one will fall into the same classification. We need to be careful about what I mean by "a lot like". The direction of the inequalities is crucially important in the test below.

**The Direct Comparison Test:** Let  $\sum a_n$  and  $\sum b_n$  be series with positive terms,

- If  $\sum b_n$  is convergent and  $a_n \leq b_n$  for all  $n$  then  $\sum a_n$  is also convergent.
- If  $\sum b_n$  is divergent and  $a_n \geq b_n$  for all  $n$  then  $\sum a_n$  is also divergent.

**Example 11.3.21 (Does the series below converge or diverge?)**

$$s = \sum_{n=1}^{\infty} \frac{1}{n^2 + 3}$$

This is a series with positive terms. We can compare this to the  $p=2$  series which we know converges (remember, you proved it). Observe that  $\frac{1}{n^2} \geq \frac{1}{n^2+3}$  for all  $n \geq 1$ .

Therefore, by the Comparison Test we find that  $s$  converges.

**Example 11.3.21 (Does the series below converge or diverge?)**

$$s = 1 + \sum_{n=2}^{\infty} \frac{1}{n - \ln(n)}$$

This is a series with positive terms. Notice that  $\frac{1}{n} > \frac{1}{n - \ln(n)}$  for all  $n \geq 2$ . This is true because  $\ln(n) > 0$  for  $n > 1$ . If we subtract a positive value from  $n$  then the resulting denominator will be smaller than  $n$  hence the quotient will be bigger. We can compare  $s$  to the  $p=1$  series. Identify the given series as the  $\sum a_n$  in the test, let  $a_1 = 1$  and  $a_n = \frac{1}{n - \ln(n)}$  for  $n \geq 2$ . We certainly have that  $a_n \geq b_n = \frac{1}{n}$  for all  $n \geq 1$ . Therefore,  $s$  diverges because it is bigger than the  $p=1$  (harmonic) series which is known to diverge (using the Direct Comparison Test)

**Remark:** there are endless examples that follow for this test. The Direct Comparison Test is called the "Direct Comparison Test" because it involves a direct comparison of two series. In contrast, the next test compares the two series in the limit.

**Limit Comparison Test:** Suppose that  $\sum a_n$  and  $\sum b_n$  are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where  $c \in \mathbb{R}$  such that  $c > 0$  then either both series converge or both diverge.

**Example 11.3.22 (Does the series below converge or diverge?)**

$$s = \sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^2}$$

This is a series with positive terms. Clearly it is similar to the convergent  $p=2$  series, let's compare the given series with the  $p=2$  series,

$$\lim_{n \rightarrow \infty} \frac{\frac{\tan^{-1}(n)}{n^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \tan^{-1}(n) = \frac{\pi}{2}$$

Therefore the series  $s$  converges by the Limit Comparison Test.

**Example 11.3.23** Does  $s = \sum_{n=1}^{\infty} \frac{n}{n^2+3}$  converge or diverge? Compare with  $p=1$  series

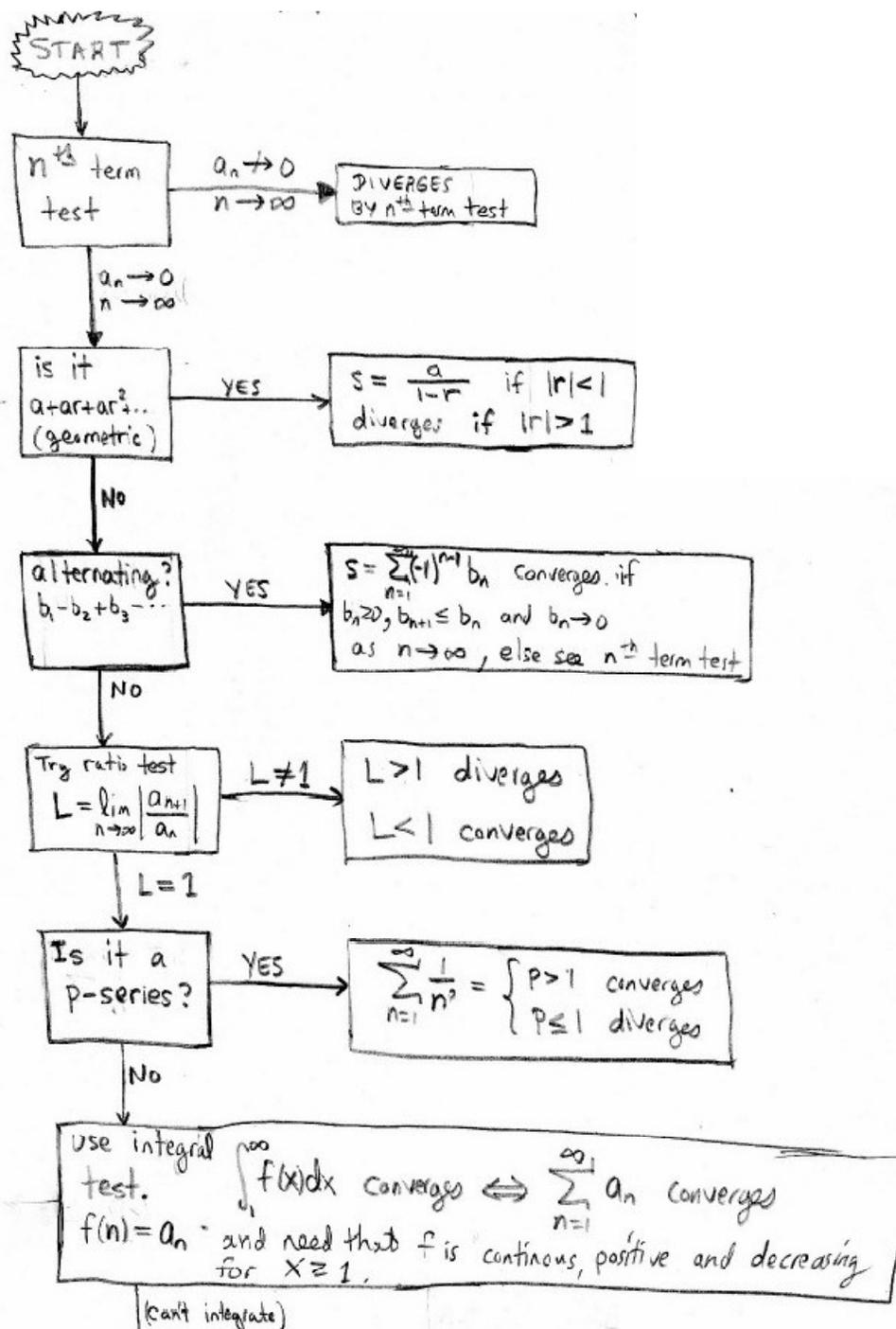
Via the limit comparison test,

$$\lim_{n \rightarrow \infty} \left( \frac{\frac{n}{n^2+3}}{\frac{1}{n}} \right) = \lim_{n \rightarrow \infty} \left( \frac{n^2}{n^2+3} \right) = 1 > 0$$

Thus by limit comp. test we know  $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$  diverges because  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

**In-class Exercise 11.3.24: Convergence/Divergence overall strategy flowchart**

Below is a flow chart that describes one possible strategy for answering the question "does the series converge or diverge?". Complete this flowchart to include all the tests we have used. Feel free to reorder my chart, this is just a rough draft.



**Remark:** there is also a “Root Test”, we will skip that in this course. If you are curious it’s in your text on page 754. It is in most texts, if you’re a math major you ought to at least read over it some time.

I have organized all of these topics in this single section because I wanted to emphasize the fact that they are part of a larger thought process. We still have a few loose ends to tie up. The next section is by far the most practical section.

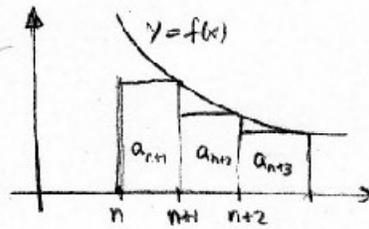
## 11.4. ERROR TESTS

The principle question we seek to answer in this section is: "How far off are we when we use a partial sum instead of the complete series?" It may not be possible to know, but in a few cases there are convenient tests. We discuss them here. The "tail" or "n-th remainder" of the series  $s = \sum_{n=1}^{\infty} a_n$  is defined below

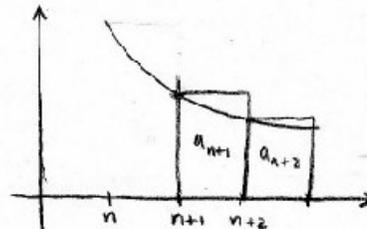
$$R_n = s - s_n = a_{n+1} + a_{n+2} + \dots$$

In other words,  $s = s_n + R_n$ . We can call  $R_n$  the error in  $s_n$  because it is precisely how far off the partial sum  $s_n$  is from the true value  $s$ .

### Integral Test Error Estimation



$$R_n = a_{n+1} + a_{n+2} + \dots \leq \int_n^{\infty} f(x) dx$$



$$R_n = a_{n+1} + a_{n+2} + \dots \geq \int_{n+1}^{\infty} f(x) dx$$

Th<sup>m</sup> / (ERROR ESTIMATE VIA INTEGRAL TEST). If  $\sum a_n$  converges by the integral test and  $R_n = s - s_n$  we have for  $f(n) = a_n$ ,

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

#### Example 11.4.1

**E1**  $s = \sum_{n=1}^{\infty} \frac{1}{n^2}$  approx.  $s$  by  $S_5$ , find a bound on the error in doing this.

Of course  $f(x) = \frac{1}{x^2}$  is decreasing, positive and with  $f(n) = \frac{1}{n^2}$

which converges  $\int_1^{\infty} \frac{1}{x^2} dx = \frac{-1}{x} \Big|_1^{\infty} = 1$ . Notice then

$$\int_{n+1}^{\infty} \frac{1}{x^2} dx = \frac{1}{n+1} \quad \text{and} \quad \int_n^{\infty} \frac{1}{x^2} dx = \frac{1}{n}$$

Thus by E.E. via I.T Th<sup>m</sup>  $\frac{1}{n+1} \leq R_n \leq \frac{1}{n}$  that is

for  $S_5$  we have  $n=5$  so  $\frac{1}{6} \leq \text{error} \leq \frac{1}{5}$ .

Recall  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.645$  (Beyond our course, I just quote result)

While  $S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{5269}{3600} = 1.464 \therefore R_n = 1.645 - 1.464 = 0.181$

$$\frac{1}{6} \leq R_n = 0.181 \leq \frac{1}{5} \quad \text{😊}$$

### Corollary to the Integral Test Error Estimate:

The following gives us a way to squeeze to the real series.

$$S_n + \int_{n+1}^{\infty} f(x) dx \leq S \leq S_n + \int_n^{\infty} f(x) dx$$

### Example 11.4.2

**E2** How does this work for **E1**? Well we found  $S_5 = 1.464$  while  $\int_{n+1}^{\infty} f(x) dx = \frac{1}{n+1}$  &  $\int_n^{\infty} f(x) dx = \frac{1}{n}$  thus for  $n=5$  we get

$$1.464 + \frac{1}{6} \leq S \leq 1.464 + \frac{1}{5}$$

$$1.631 \leq S \leq 1.664 \quad \text{[where } S = \frac{\pi^2}{6} \approx 1.64493 \text{ actually]}$$

What can we conclude,  $S \approx 1.6$  ← these digits are certain. If we let  $n=20$  we could do better,

$$S_{20} = \sum_{n=1}^{20} \frac{1}{n^2} = 1.59616$$

Apply cor. to Th<sup>m</sup>.

$$1.59616 + \frac{1}{21} \leq S \leq 1.59616 + \frac{1}{20}$$

$$1.64378 \leq S \leq 1.64616$$

$$S \approx 1.64$$

$n=20$  gives us 3 digits for sure.

### Alternating Series Error Test

Th<sup>m</sup> / (ALTERNATING SERIES ESTIMATION Th<sup>m</sup>) If  $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$  satisfies

- (a)  $0 < b_{n+1} \leq b_n$
- (b)  $\lim_{n \rightarrow \infty} b_n = 0$

Then  $|R_n| = |S - S_n| \leq b_{n+1}$ . In other words if an alternating series converges then  $S_n$  has an error smaller than the next term  $b_{n+1}$ .

**In-class Exercise 11.4.4:** Calculate the alternating harmonic series to 4 significant digits. Identify the number you find. Make an educated guess on what the actual value of the alternating harmonic series  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$  (your scientific calculator should help)

### Example 11.4.3

**E3** Find  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$  correct to 3 decimal places.  $b_n = \frac{1}{n!}$

$$0 \leq \frac{1}{(n+1)!} = \frac{1}{(n+1)n!} \leq \frac{1}{n!} \quad \therefore 0 \leq b_{n+1} \leq b_n$$

Additionally  $\frac{1}{n!} \leq \frac{1}{n}$  and  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$   $\therefore \frac{1}{n!} \rightarrow 0$  as  $n \rightarrow \infty$ .

Notice that

$$b_0 = 1, b_1 = 1, b_2 = 0.5, b_3 = 0.1\bar{6}, b_4 = 0.0166\bar{7}, b_5 = 0.00166\bar{6}$$

$$b_6 = 0.0001389, b_7 = 0.0000198$$

this is error in  $S_6$   
 $\therefore S_6$  will do.

$$S_6 = 0.368056 \text{ with error } b_7 = 0.000198$$

$$\therefore \boxed{S \approx 0.368}$$

### Principle of Least Astonishment Test (PLA)

Ignoring mathematical rigor for a moment let me speak pragmatically. For most examples if terms in the series are getting smaller and smaller then you can just study the digits in the partial sums. When a digit settles down and is no longer effected by additional terms being summed then you can with reasonable certainty assume that digit is correct. Of course you need to keep rounding in mind, and when I say "reasonable" I do not mean mathematical certainty. Sometimes mathematical certainty is not an option. In such cases you may be forced to this sort of heuristic reasoning.

$$s_{10} = 1.234544$$

$$s_{11} = 1.234703$$

$$s_{12} = 1.234769$$

$$s_{13} = 1.234774$$

$$s_{14} = 1.234770$$

Given the data above I would wager that  $s = 1.2347$  for certain. If I wanted more digits I'd want to calculate more to be on the safe side. That's a judgment call on my part.

Of course, I could be wrong, without any additional info it is entirely possible that the next term violates the pattern. It could be that  $s_{15} = 42$ . This kind of random divergence from the pattern above is insured by the various tests earlier in this section. In practice, we may not even have a formula from which the series is being generated. The series could come from some experimental measurement. We then just have to take it on faith that the pattern continues.

Often a mathematical pattern is assumed even though there is no physical derivation of the pattern. These sort of models in physics are termed “*phenomenological*”. Usually physicists are discontent with such models, one would like to explain why a certain equation describes a certain situation. One early instance of this was Kepler’s Laws. He gave a formula describing the motion of planets. However, Kepler gave no reason as to why this formula ought to apply. One of the great triumphs of Newtonian mechanics was to derive Kepler’s Laws as a consequence of Newton’s Laws of motion and Newton’s Universal Law of Gravitation.

This story continues to play out today. Some scientists will find a pattern, then later other scientists will give a reason for the pattern. At the base of it all a nagging question remains; why is there physical law at all? If the universe is random then why does it have such rich and beautiful physical law? There are other answers, but I believe the most logical answer to this question is the obvious one. The universe was created by an orderly being. God built the universe in such a way that not only could we enjoy the beauty of the cosmos at any level of detail. From our everyday experience, to the atomic level, to the subatomic level, it’s not random, it’s design.

## **11.5. CONDITIONAL AND ABSOLUTE CONVERGENCE**

Absolute convergence is stricter than convergence. We say a series  $\sum a_n$  **converges absolutely** iff the series  $\sum |a_n|$  converges. If the series  $\sum a_n$  converges and the series  $\sum |a_n|$  diverges then  $\sum a_n$  is defined to be **conditionally convergent**.

$$s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2) \quad \text{but,} \quad \sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad s \text{ conditionally convergent}$$

$$t = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \text{ converges} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad t \text{ absolutely convergent}$$

Notice the absolute value just kills the sign generating term  $(-1)^{n-1} = \pm 1$  in both of the examples above. Intuitively we should think of a conditionally convergent series as a series which almost diverges, it’s right on the edge. On the other hand, absolutely convergent series are in no such danger.

Basically, conditionally convergent series converge because of some fortunate cancellation. If we rearrange the terms in the series the rearranged series can converge to something different! Let me illustrate the danger of rearranging terms in a series which is not absolutely convergent. For example,

$$\begin{aligned}
 0 &= 1 - 1 \\
 &= 1 - 1 + 1 - 1 \\
 &= 1 - 1 + 1 - 1 + \cdots 1 - 1 + \cdots \\
 &= 1 + 1 - 1 + 1 + \cdots - 1 + 1 + \cdots \\
 &= 1 + 0 + 0 + \cdots 0 + \cdots \\
 &= 1
 \end{aligned}$$

Oops. Obviously the claimed equality above was not valid. How can we avoid such problems?

**Rearrangement Lemma:** If a series converges absolutely then any rearrangement of the series will converge to the same value.

Contrast this to the striking result due to Riemann,

**Riemann's Observation:** A conditionally convergent series can be rearranged so that it converges to any real number. In other words, rearranging an conditionally convergent series alters the result.

Look at Eqns. 6,7, and 8 of page 755. These show that you can rearrange the terms in the alternating harmonic series to make the rearranged series converge to  $\frac{3}{2} \ln(2)$ . I find it a bit disturbing, but it is exactly this sort of subtlety that show us why we must be careful with series calculations.

**Remark:** The Ratio Test actually gives us that the series converges absolutely when  $L < 1$ .

**Apology:** I am a novice on the matters discussed in this part of the course. If you would like to see a more sophisticated and breathtakingly deep set of notes on this material then I suggest you browse through

<http://kr.cs.ait.ac.th/~radok/math/mat6/startdiall.htm>

These are notes from a text by Courant. I peruse these and feel humbled by my abject ignorance. Bonus points certainly can be earned if you teach me something from those notes. If you've got the gumption, ask me we'll find a mutually agreeable example for you to dig into. By the way, <http://kr.cs.ait.ac.th/~radok/math/mat/startall.htm> has even more on all sorts of mathematics and physics.