

12. BASICS OF POWER SERIES

The first section in this Chapter defines the basic terminology and characteristics of power series. In short, a power series is a special kind of function. A power series is a function which is defined point-wise by a series-formula. When we first discussed functions we asked the question: “what is the domain of a function given the formula for a function?”. The answer to that question was a relatively simple, we simply needed to avoid division by zero and square roots of negative numbers. In the case of power series we can again ask what is the domain? Well, the domain is the set of all inputs for which the series-formula converges, which is just the same as saying it is everywhere the formula for the function makes sense. Now, we just have to work a little harder to get to the root of what constitutes a sensible series. Fortunately a theorem tells us the domain has to be a single interval (the “IOC”). It turns out that the ratio test will give the bulk of the domain in most examples but then the endpoints will need checking via the other various tests from Chapter 11. Don’t be discouraged by the first section, it has as much to do with the series’ actual application as the domain of functions has to do with their application. Careful understanding will help solidify other more pragmatic sections, so stick to it even if you don’t care for the first section.

The second section focuses on examples of power series which are generated by the geometric series. A theorem reveals that the integral and derivatives of a power series are again power series. This allows us to twist the geometric series result to cover other functions which are related through integration or differentiation to the basic geometric series formula. These tricks cover a fairly wide swath of examples. Of course Taylor’s Theorem in the next Chapter is more flexible and not nearly as sneaky, but Taylor’s Theorem requires much more work if applied to the problems we attack in this chapter.

12.1. WHAT IS A POWER SERIES?

A power series is a function which has a rather special formulation. A power series is a function which is defined point-wise by a series. Let us pin down some terminology:

Defⁿ A power series centered at zero has the form

$$f(x) = \sum_{n=0}^{\infty} C_n X^n = C_0 + C_1 X + C_2 X^2 + \dots$$

Where $C_i \in \mathbb{R} \forall i$, these numbers are called the coefficients of the power series.

A power series centered at a has the form

$$f(x) = \sum_{n=0}^{\infty} C_n (X-a)^n = C_0 + C_1 (X-a) + C_2 (X-a)^2 + \dots$$

Remark $f(x)$ is a function with domain constructed from all x such that $f(x) \in \mathbb{R}$ (that is where the series converges)

An easy source of examples for power series is the geometric series result.

Example 12.1.1

E1 When $C_i = 1 \forall i$ we have a geometric series with $a = 1$ and $r = x$

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \frac{1}{1-x}$$

this converges for each x in $(-1, 1)$

Example 12.1.2

E2 Many times we can use substitution to rewrite a power series,

$$1 + x + 1 + x^2 + 2x + 1 + x^3 + 3x^2 + 3x + 1 + \dots = \sum_{n=0}^{\infty} (x+1)^n$$

So this is again geometric series with $a = 1$ & $r = x+1$

$$1 + x + 1 + x^2 + 2x + 1 + \dots = \frac{1}{1-(x+1)} = \frac{-1}{x} = f(x)$$

which converges for $-1 < r = x+1 < 1 \Leftrightarrow -2 < x < 0 \leftarrow$ domain of f .

Comment: Notice that the geometric series convergence condition $|r| < 1$ tells us what the domain of the power series is in both of the examples above. Generally power series need not stem from the geometric series result. For example, later we'll see that the exponential function has the power series representation of $e^x = 1 + x + \frac{1}{2}x^2 + \dots$ and the domain of the series is in the whole real line. Many of the interesting examples of power series are generated via Taylor's Theorem. We'll talk about that eventually, but for now we will make the most of the geometric series result. It turns out that these

geometric series type calculations are much easier than the Taylor's Theorem arguments for the same series. The viewpoint put forth in E1 and E2 will not be supplanted by easier arguments later. Taylor's Theorem is extremely general but that generality also comes at a weighty price, you'll have to take arbitrarily many derivatives to describe the series exactly.

Th^o(3) When $f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n$ is a power series then \exists 3 possibilities

- i.) The series converges only when $x = a$
- ii.) The series converges $\forall x \in \mathbb{R}$
- iii.) $\exists R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

In (iii) $R \equiv$ the radius of convergence in this case we may have (for $a \geq 0$)
 $\text{dom}(f) = [a-R, a+R], (a-R, a+R), (a-R, a+R], [a-R, a+R)$
 We call the $\text{dom}(f)$ the interval of convergence. (I.O.C.)

This Theorem tells us that the domain of a power series cannot be things like $(0, 1) \cup (2, 3)$. We call the set where the power series converges the **Interval of Convergence (IOC)**. The distance from the center of the IOC to its edges is called the **Radius of Convergence (ROC or R)**. If we know the radius and the center of a series then we have a very good idea of what the IOC looks like. In fact, we need only worry about the endpoints. Their inclusion or exclusion leads to the 4 cases listed in the Theorem.

Example 12.1.3a

Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{n} (2x-4)^n$. We begin with the ratio test,

$$L = \lim_{n \rightarrow \infty} \left| \frac{(2x-4)^{n+1}}{n+1} \cdot \frac{n}{(2x-4)^n} \right| = |2x-4| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) = |2x-4|$$

Then $L < 1 \Leftrightarrow |2x-4| < 1 \Rightarrow |x-2| < \frac{1}{2} \Rightarrow R = \frac{1}{2}$

The endpoints need to be checked

$$x = 2 - \frac{1}{2} \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, } (p=1)$$

$$x = 2 + \frac{1}{2} \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges by alt-series test as } \frac{1}{n} \text{ is positive, decreases and has limit zero as } n \rightarrow \infty.$$

Hence

$$\text{I.O.C} = \left(\frac{3}{2}, \frac{5}{2} \right]$$

aka $\frac{3}{2} < x \leq \frac{5}{2}$ is the interval of convergence.

We can see a pattern emerging from the examples.

FINDING THE INTERVAL OF CONVERGENCE

Step 1: Use the Ratio Test to find the interval for which the series converges (absolutely), usually $a - R < x < a + R$

Step 2: If $(a - R, a + R)$ is a finite interval then check the endpoints for convergence/divergence. This usually will entail a n^{th} term test or an alternating series test.

Step 3: Using the Th^m(3) we're done. (If Step 2 has both endpts. divergent then I.O.C is $(a - R, a + R)$)

Remark: this not a comprehensive advice, sometimes it'll be easier & sometimes you'll need to think or do some algebra before applying the steps.

Example 12.1.3b

E3 Find the I.O.C for $\sum_{n=0}^{\infty} \frac{\sqrt{n} x^n}{3^n} = \sum_{n=0}^{\infty} a_n = S(x)$. STEP 1: RATIO TEST (216)

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1} x^{n+1} 3^n}{3^{n+1} \sqrt{n} x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1}}{3\sqrt{n}} x \right|$$

$$= \frac{1}{3} |x| \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1}}{\sqrt{n}} \right| \text{ since } \frac{\sqrt{n+1}}{\sqrt{n}} = \frac{n+1}{\sqrt{n^2+n}} = \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Thus $L = \frac{1}{3} |x|$ so $L < 1$ if $\frac{1}{3} |x| < 1$ aka $|x| < 3$ and $L > 1$ if $|x| > 3$. Thus $S(x)$ converges on $(-3, 3)$.

STEP 2: Endpoints, test for conv/div at $x = \pm 3$
Consider thus

$$S(3) = \sum_{n=0}^{\infty} \frac{\sqrt{n}}{3^n} 3^n = \sum_{n=0}^{\infty} \sqrt{n} : \text{diverges since } \sqrt{n} \rightarrow \infty \text{ as } n \rightarrow \infty \text{ (using } n^{\text{th}} \text{ term test)}$$

$$S(-3) = \sum_{n=0}^{\infty} \frac{\sqrt{n}}{3^n} (-3)^n = \sum_{n=0}^{\infty} (-1)^n \sqrt{n} : \text{diverges by } n^{\text{th}} \text{ term test again } (-1)^n \sqrt{n} \not\rightarrow 0 \text{ as } n \rightarrow \infty.$$

Example 12.1.4 (sometimes we get the endpoints for free)

(a.) $\sum 2^n(x+1)^n$ this is a geometric series with $a=1$ & $r=2(x+1)$
it converges iff $|r| < 1 \Rightarrow |2(x+1)| < 1$
 $|x+1| < \frac{1}{2}$

Thus the $R = \frac{1}{2}$ and $-\frac{3}{2} < x < -\frac{1}{2}$ is the I.O.C.

Example 12.1.5 (this was a test question from a previous course)

Find the IOC and ROC for the power series defined below.

PROBLEM FIVE Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x}{3}\right)^{n-1}$.

$$L = \lim_{n \rightarrow \infty} \left| \frac{(x/3)^n}{n+1} \frac{n}{(x/3)^{n-1}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \left(\frac{x}{3}\right) \left(\frac{n}{n+1}\right) \right|$$

$$= \left| \frac{x}{3} \right| \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right) \rightarrow 1$$

$$= \left| \frac{x}{3} \right| < 1 \Rightarrow -1 < \frac{x}{3} < 1 \Rightarrow -3 < x < 3$$

will converge
 $x = \pm 3$ don't know
yet.

$$f(3) = \sum_{n=1}^{\infty} \frac{1}{n}, \text{ p-series } p=1 \Rightarrow \text{diverges.}$$

$$f(-3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}, \quad b_n = \frac{1}{n} > 0, \quad b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$$

and $\frac{1}{n} \rightarrow 0$ as $n \rightarrow \infty$ \therefore converges by A.S.T.

So we find I.O.C = $[-3, 3)$ and $R = 3$

12.2. POWER SERIES VIA GEOMETRIC SERIES

Given a function can we find a power series representation for that function? In this section we will see how to find power series to represent $f(x) = \frac{a}{1-r}$ for appropriate identifications of a and r as functions of x . We use the geometric series result

$$a + ar + ar^2 + \dots = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

(I start the summation at zero because I like to work with r^n as opposed to r^{n-1} , this is equivalent to the formula $\sum_{k=1}^{\infty} ar^{k-1}$ which also gives $a + ar + ar^2 + \dots$)

Example 12.2.1

The equation below follows from the geometric series result with $a=1$ and $r=x$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{n=0}^{\infty} x^n$$

I have expressed the power series in the “dot-dot-dot” notation and the “sigma” or “summation” notation. I require you to understand both, but it may be conceptually easier to tackle the “dot-dot-dot” notation to begin. Then once that’s settled come back and deal with the sigma notation. I do expect you to master both.

Example 12.2.2

Find the power series representation of $f(x) = \frac{x^2}{1+x}$. Identify that $a = x^2$, $r = -x$

$$\begin{aligned} f(x) &= \frac{x^2}{1+x} = \sum_{n=0}^{\infty} x^2(-x)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{n+2} \\ &= x^2 - x^3 + x^4 - x^5 + \dots \end{aligned}$$

This calculation could also be seen as x^2 times the power series for $\frac{1}{1+x}$ which is simply $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$.

Example 12.2.3

Find the power series for $f(x) = \frac{4}{2-x}$. To begin we must make it look like $\frac{a}{1-r}$. We need to factor out the 2 as follows: $\frac{4}{2-x} = \frac{2}{1-x/2}$. Now we identify that $a = 2$ and $r = x/2$ hence the geometric series result yields the power series expansion below:

$$f(x) = \frac{2}{1-x/2} = 2 + 2(x/2) + 2(x/2)^2 + \dots = \boxed{2 + x + \frac{1}{2}x^2 + \dots}$$

Or in the summation notation,

$$f(x) = \frac{2}{1-\frac{x}{2}} = \sum_{n=0}^{\infty} 2\left(\frac{x}{2}\right)^n = \boxed{\sum_{n=0}^{\infty} 2^{1-n}x^n}$$

Remark: you can gather that there are many more examples one can glean from the geometric series result. The Theorem below will extend the reach of the geometric series to all sorts of new examples. I classify the examples which use the geometric series and the Theorem below as “geometric series tricks” because they’re tricky.

Th^m (Term by term calculus on power series). If $s(x) = \sum C_n(x-a)^n$ has radius of conv. $R > 0$ then $s(x)$ is differentiable (or smooth) on $(a-R, a+R)$ and

a.) $\frac{d}{dx} \left[\sum_{n=0}^{\infty} C_n(x-a)^n \right] = \sum_{n=0}^{\infty} \frac{d}{dx} [C_n(x-a)^n] = \sum_{n=1}^{\infty} C_n n(x-a)^{n-1}$

b.) $\int \left[\sum_{n=0}^{\infty} C_n(x-a)^n \right] dx = \sum_{n=0}^{\infty} \int C_n(x-a)^n dx = C + \sum_{n=0}^{\infty} C_n \frac{(x-a)^{n+1}}{n+1}$

The radii of convergence for a. & b. are also R although the I.O.G. may differ.

Remarks one must have a power series for this to work eg. $\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ converges while $\sum_{n=1}^{\infty} \frac{n! \cos(n!x)}{n^2}$ diverges

The series in the remark are known as Fourier Series, those are sums of sines and cosines. Power series are sums of power functions. Power series have pretty nice properties in comparison to some other types of series. We often cover Fourier series in the differential equation course at LU.

Example 12.2.4 (geometric series plus tricks)

E4 Identify the function $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ $-1 \leq x \leq 1$. Let's use the th^m to get geom. series,

$f'(x) = 1 - x^2 + x^4 - \dots = \frac{a}{1-r} = \frac{1}{1+x^2}$ (where $a=1$, $r=-x^2$.)

Now we can integrate:

$\int f'(x) dx = \int \frac{dx}{1+x^2} \Rightarrow f(x) = C + \tan^{-1}(x)$

Now $f(0) = 0 \Rightarrow C + \tan^{-1}(0) = 0 \therefore C = 0$, hence $f(x) = \tan^{-1}(x)$

Example 12.2.5 (geometric series plus tricks)

E5 Find a power series rep. of $\ln(1+x) = f(x)$. Try diffⁿ

$$f'(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n \quad \left(\begin{array}{l} \text{geom. series} \\ \text{with } a=1 \\ \& r=-x \end{array} \right)$$

Now integrate term by term,

$$f(x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Find C by eval. f at $x=0$, $f(0) = \ln(1) = \boxed{0 = C}$ thus

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \boxed{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots = \ln(1+x)}$$

The pattern to Example 12.2.4 and 12.2.5 is the same, differentiate, apply geometric series result, integrate, fix integration constant, answer. The next example reverses the pattern.

Example 12.2.6 (geometric series plus tricks)

E6 Find power series expansion of $f(x) = \frac{1}{(1+x)^2}$

$$\int \frac{1}{(1+x)^2} dx = \frac{-1}{1+x} + C, \text{ note } f(0) = 1 \Rightarrow C = 2.$$
$$\int f(x) dx = 2 - \frac{1}{1+x} = 2 - \left(\sum_{n=0}^{\infty} (-x)^n \right) \quad : \text{ geom. series } \begin{array}{l} a=1 \\ r=-x \end{array}$$
$$f(x) = \frac{d}{dx} \int f(x) dx = \frac{d}{dx} \left[2 - \sum_{n=0}^{\infty} (-x)^n \right] = - \sum_{n=1}^{\infty} n(-x)^{n-1} = \boxed{1 - 2x + 3x^2 + \dots = \frac{1}{(1+x)^2}}$$

Example 12.2.7 (geometric series plus tricks)

E7 Power series can also help with some formidable integrals!

$$\int \frac{\tan^{-1}(x)}{x} dx = f(x)$$
$$f'(x) = \frac{\tan^{-1}(x)}{x} = \frac{1}{x} \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right) = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \dots$$
$$\int f'(x) dx = f(x) = C + x - \frac{x^3}{9} + \frac{x^5}{25} - \dots$$

Thus $\boxed{\int \frac{\tan^{-1}(x)}{x} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{(x)^{2n+1}}{(2n+1)^2}}$

Example 12.2.8 (geometric series plus tricks)

E8 Find power series expansion of $f(x) = \ln(1+x^2)$. Notice,

$$f'(x) = \left(\frac{1}{1+x^2}\right) 2x$$

Now we can see how to apply geometric series result, $a = 2x$, $r = -x^2$

$$f'(x) = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (2x)(-x^2)^n = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}$$

Now recover $f(x)$ by integrating, $f(x) = \int f'(x) dx$

$$\begin{aligned} f(x) &= \int \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} dx \\ &= \sum_{n=0}^{\infty} (-1)^n 2 \int x^{2n+1} dx \\ &= \sum_{n=0}^{\infty} (-1)^n 2 \left(\frac{x^{2n+2}}{2n+2} \right) + C \quad \text{note } f(0) = 0 \Rightarrow \underline{C=0} \end{aligned}$$

In-Class Exercise 12.2.9 (geometric series tricks summary)

Create a flow-chart that illustrates the two basic patterns we have used for these examples. Come up with an additional example which requires 3 differentiations before the geometric series can be applied. Include that case in your flow chart. I'll get the chart started on the white board.

Remark: we have said precious little about the IOC and ROC in this section. We could still ask those questions for each and every example: what is the IOC and ROC for Ex. 12.2.1-12.2.8. The calculus for power series Theorem leaves the ROC untouched, but typically differentiation will delete endpoints while integration may add endpoints to the IOC. In this course we would have to analyze the series in a case by case basis. The logic I would use is that the geometric series result tells us the IOC modulo the endpoints. To determine the endpoints I would check them separately. In practice, I rarely ask about the endpoints. The ROC and center of the IOC is much more important for actual applications.