13. POWER SERIES TECHNIQUES

Taylor polynomials can be generated for a given function through a certain linear combination of its derivatives. The idea is that we can approximate a function by a polynomial, at least locally. In calculus I we discussed the tangent line approximation to a function. We found that the linearization of a function gives a good approximation for points close to the point of tangency. If we calculate second derivatives we can similarly find a quadratic approximation for the function. Third derivatives go to finding a cubic approximation about some point. I should emphasize from the outset that a Taylor polynomial is a polynomial, it will not be able to exactly represent a function which is not a polynomial. In order to exactly represent an analytic function we'll need to take infinitely many terms, we'll need a power series.

The Taylor series for a function is formed in the same way as a Taylor polynomial. The difference is that we never stop adding terms, the Taylor series is formed from an infinite sum of a function's derivatives evaluated at the series' center. There is a subtle issue here, is it possible to find a series representation for a given function? Not always. However, when it is possible we call the function analytic. Many functions that arise in applications are analytic. Often functions are analytic on subdomains of their entire domain, we need to find different series representations on opposite sides of a vertical asymptote. What we learned in the last chapter still holds, there is an interval of convergence, the series cannot be convergent on some disconnected domain. But, for a given function we could find a Taylor series for each piece of the domain, ignoring certain pathological math examples.

We calculate the power series representations centered about zero for most of the elementary functions. From these so-called Maclaurin series we can build many other examples through substitution and series multiplication.

Sections 13.4 and 13.5 are devoted to illustrating the utility of power series in mathematical calculation. To summarize, the power series representation allows us to solve the problem as if the function were a polynomial. Then we can by-pass otherwise intractable trouble-spots. The down-side is we get a series as the answer typically. But, that's not too bad since a series gives us a way to find an approximation of arbitrarily high precision, we just keep as many terms as we need to obtain a the desired precision. We discussed that in the last chapter, we apply it here to some real world problems.

Section 13.5 seeks to show how physicists think about power series. Often, some physical approximation is in play so only one or two of the terms in the series are needed to describe physics. For example, $E = mc^2$ is actually just the first term in an infinite power series for the relativistic energy. The binomial series is particularly important to physics. Finally, I mention a little bit about how the idea of series appears in modern physics. Much of high energy particle physics is "perturbative", this means a series is the only description that is known. In other words, modern physics is inherently approximate when it comes to many cutting-edge questions.

13.1. TAYLOR POLYNOMIALS

The first two pages of this section provide a derivation of the Taylor polynomials. Once the basic formulas are established we apply them to a few simple examples at the end of the section.

N=1. Recall the linearization to $y = f(x)$ at $(a, f(a))$ is $\boxed{L(x) = f(a) + f'(a)(x-a)\ .}$
We found this formula on the basis of three assumptions:

$$\begin{aligned} L(x) &= mx + b \\ f(a) &= L(a) \\ f'(a) &= L'(a) \end{aligned}$$

It's easy to see that $f'(a) = m$ and $f(a) = ma + b \implies b = f(a) - f'(a)a$ hence $L(x) = f'(a)x + f(a) - f'(a)a = f(a) + f'(a)(x-a)$ as I claimed.

N=2. How can we generalize this to find a quadratic polynomial which approximates $y = f(x)$ at $(a, f(a))$? I submit we would like the following conditions to hold:

$$\begin{aligned} P(x) &= Ax^2 + Bx + C \\ f(a) &= P(a) \\ f'(a) &= P'(a) \\ f''(a) &= P''(a) \end{aligned}$$

We can calculate,

$$\begin{aligned} f''(a) = P''(a) = 2A \quad &\implies \quad A = \frac{1}{2}f''(a) \\ f'(a) = P'(a) = 2Aa + B \quad &\implies \quad B = f'(a) - f''(a)a \\ f(a) = P(a) = Aa^2 + Ba + C \quad &\implies \quad C = f(a) - \frac{1}{2}f''(a)a^2 - (f'(a) - f''(a)a)a \end{aligned}$$

The formula for C simplifies a bit; $C = f(a) - f'(a)a + \frac{1}{2}f''(a)a^2$. Plug back into $P(x)$:

$$\begin{aligned} P(x) &= \frac{1}{2}f''(a)x^2 + (f'(a) - f''(a)a)x + f(a) - f'(a)a + \frac{1}{2}f''(a)a^2 \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x^2 - 2ax + a^2) \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \end{aligned}$$

I anticipated being able to write $P(x) = L(x) + \cdots$, as you can see it worked out.

$$\boxed{P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2\ .}$$

N=3. If you think about it a little you can convince yourself that an n-th order polynomial can be written as a sum of powers of $(x-a)$. For example, an arbitrary cubic ought to have the form:

$$Q(x) = A_3(x-a)^3 + A_2(x-a)^2 + A_1(x-a) + A_0$$

Realizing this at the outset will greatly simplify the calculation of the third-order approximation to a function. To find the third order approximation to a function we would like for the following condtions to hold:

$$\begin{aligned} Q(x) &= A_3(x-a)^3 + A_2(x-a)^2 + A_1(x-a) + A_0 \\ f(a) &= Q(a) \\ f'(a) &= Q'(a) \\ f''(a) &= Q''(a) \\ f'''(a) &= Q'''(a) \end{aligned}$$

The details work out easier with this set-up,

$$\begin{aligned} f(a) &= Q(a) = D \\ f'(a) &= Q'(a) = C \\ f''(a) &= Q''(a) = 2B \\ f'''(a) &= Q'''(a) = 3(2)A = (3!)A \end{aligned}$$

Therefore,

$$\boxed{Q(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3 \ .}$$

These approximations are known as *Taylor polynomials.* Generally, the n-th Taylor polynomial centered at $x=a$ is found by calculation n-derivatives of the function and evaluating those at $x=a$ and then you assemble the polynomial according to the rule:

$$\boxed{T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \ .}$$

You can check that we have:

$$\begin{aligned} T_1(x) &= f(a) + f'(a)(x-a) \\ T_2(x) &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 \\ T_3(x) &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 \end{aligned}$$

Example 13.1.1

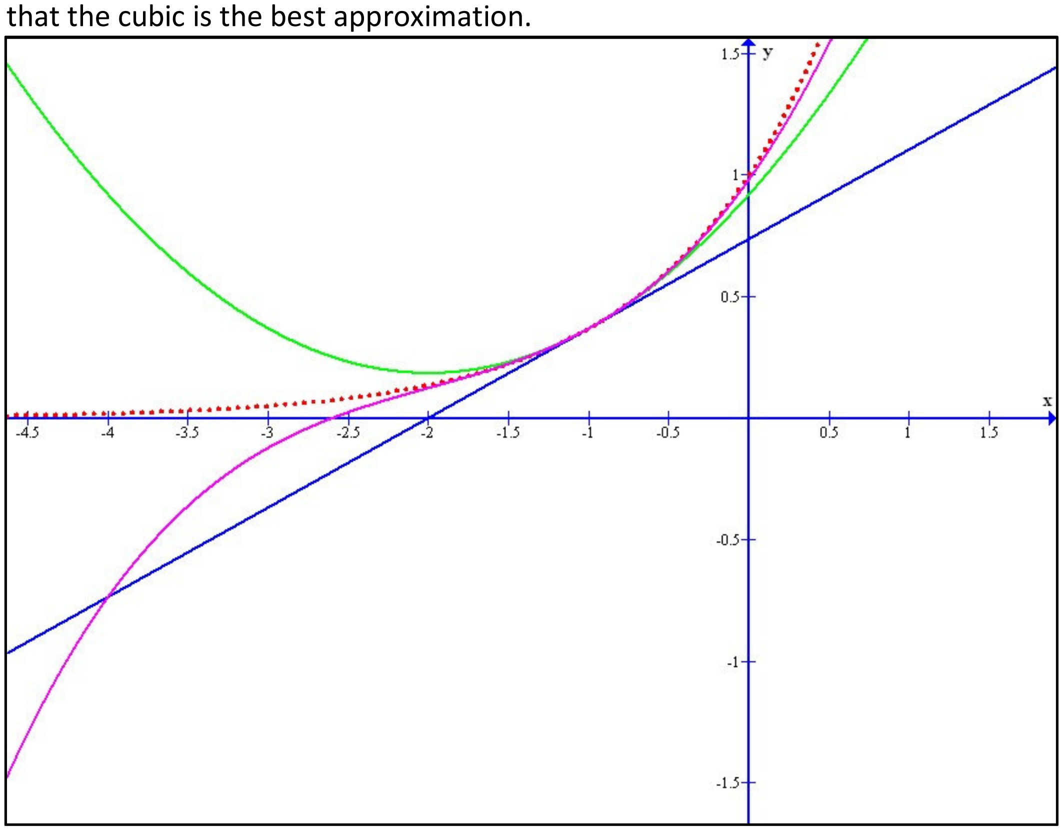
Let $f(x) = e^x$. Calculate the the first four Taylor polynomials centered at $x = -1$. Plot several and see how they compare with the actual graph of the exponential.

$$\begin{aligned} f(x) = e^x &\implies f(-1) = e^{-1} \\ f'(x) = e^x &\implies f'(-1) = e^{-1} \\ f''(x) = e^x &\implies f''(-1) = e^{-1} \\ f'''(x) = e^x &\implies f'''(-1) = e^{-1} \end{aligned}$$

Thus,

$$\begin{aligned} T_0(x) &= \frac{1}{e} \\ T_1(x) &= \frac{1}{e} + \frac{1}{e}(x+1) \\ T_2(x) &= \frac{1}{e} + \frac{1}{e}(x+1) + \frac{1}{2e}(x+1)^2 \\ T_3(x) &= \frac{1}{e} + \frac{1}{e}(x+1) + \frac{1}{2e}(x+1)^2 + \frac{1}{6e}(x+1)^3 \end{aligned}$$

The graph below shows $y = f(x)$ as the dotted red graph, $y = T_1(x)$ is the blue line, $y = T_2(x)$ is the green quadratic and $y = T_3(x)$ is the purple graph of a cubic. You can see that the cubic is the best approximation.



Example 13.1.2

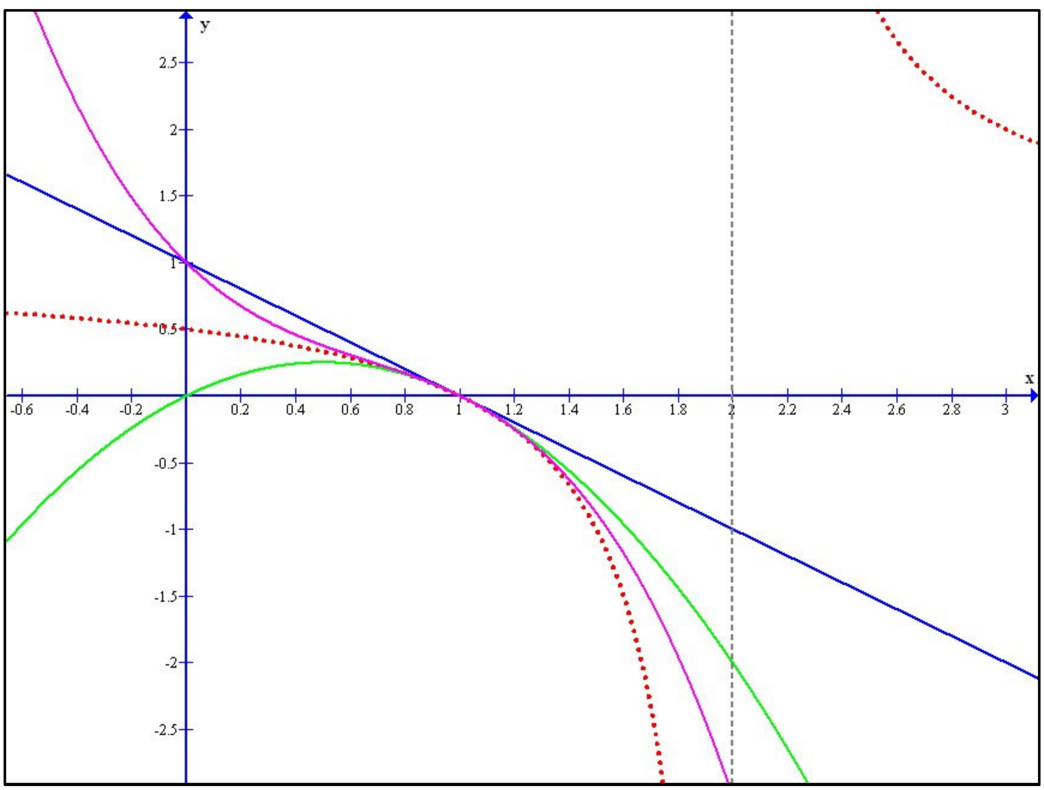
Consider $f(x) = \frac{1}{x-2} + 1$. Let's calculate several Taylor polynomials centered at $x = 1$ and $x = 3$. Graph and compare.

$$
\begin{aligned}
f(x) &= \frac{1}{x-2} + 1 \quad \Longrightarrow \quad f(1) = 0 \\
f'(x) &= \frac{-1}{(x-2)^2} \quad \Longrightarrow \quad f'(1) = -1 \\
f''(x) &= \frac{2}{(x-2)^3} \quad \Longrightarrow \quad f''(1) = -2 \\
f'''(x) &= \frac{-6}{(x-2)^4} \quad \Longrightarrow \quad f'''(1) = -6
\end{aligned}
$$

We can assemble the first few interesting Taylor polynomials centered at one,

$$
\begin{aligned}
T_1(x) &= -(x-1) \\
T_2(x) &= -(x-1) + (x-1)^2 \\
T_3(x) &= -(x-1) + (x-1)^2 - (x-1)^3
\end{aligned}
$$

Let's see how these graphically compare against $y = f(x)$:



$y = f(x)$ is the dotted red graph, $y = T_1(x)$ is the blue line, $y = T_2(x)$ is the green quadratic and $y = T_3(x)$ is the purple graph of a cubic. The vertical asymptote is gray. Notice the Taylor polynomials are defined at $x = 2$ even though the function is not.

Remark: We could have seen this coming, after all this function is a geometric series,

$$f(x) = 1 + \frac{1}{x-2} = 1 + \frac{-1}{1-(x-1)} = 1 - 1 - (x-1) - (x-1)^2 - (x-1)^3 + \cdots$$

The IOC for $r = x - 1$ is $|x-1| < 1$. It is clear that the approximation cannot extend to the asymptote. We can't approximate something that is not even defined. On the other hand perhaps is a bit surprising that we cannot extend the approximation beyond one unit to the left of $x = 1$. Remember the IOC is symmetric about the center.

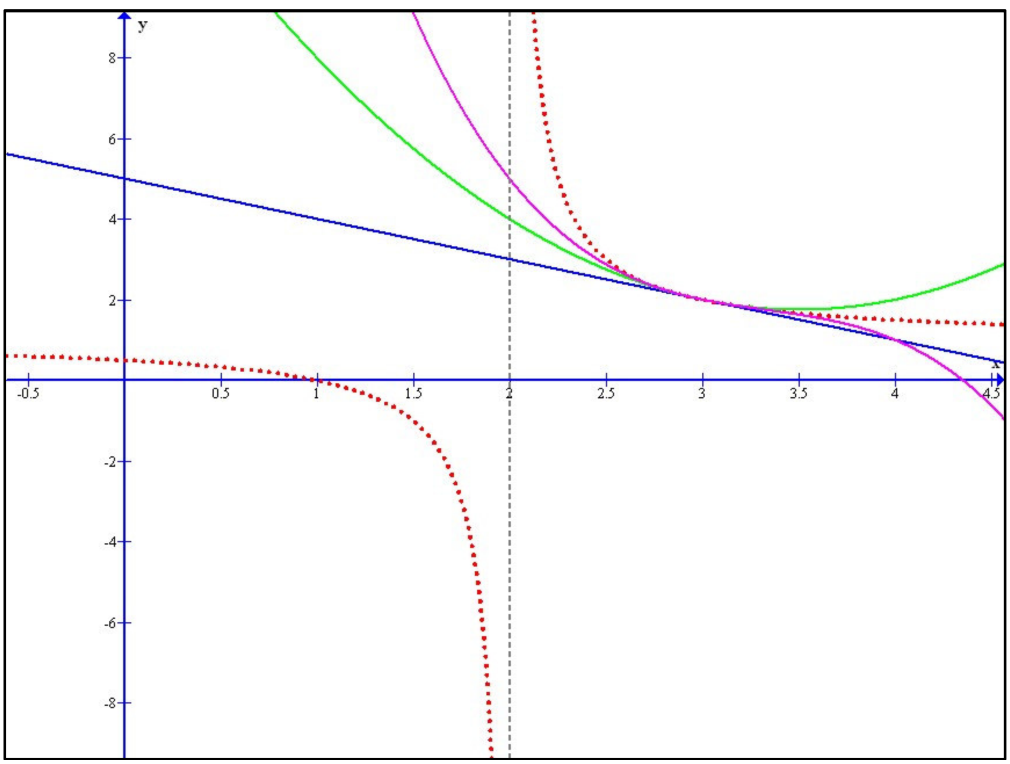
Given the remark we probably can see the Taylor polynomials centered about $x = 3$ from the following geometric series,

$$f(x) = 1 + \frac{1}{x-2} = 1 + \frac{1}{1-(3-x)} = 1 + 1 + (3-x) + (3-x)^2 + \cdots$$

We can calculate (relative to $a = 3$):

$$T_1(x) = 2 + (3-x)$$
$$T_2(x) = 2 + (3-x) + (3-x)^2$$
$$T_3(x) = 2 + (3-x) + (3-x)^2 + (3-x)^3$$

Let's graph these and see how they compare to the actual graph. I used the same color-code as last time,



Again we only get agreement close to the center point. As we go further away the approximation fails. Any agreement for x outside $2 < x < 4$ is coincidental.

Example 13.1.3

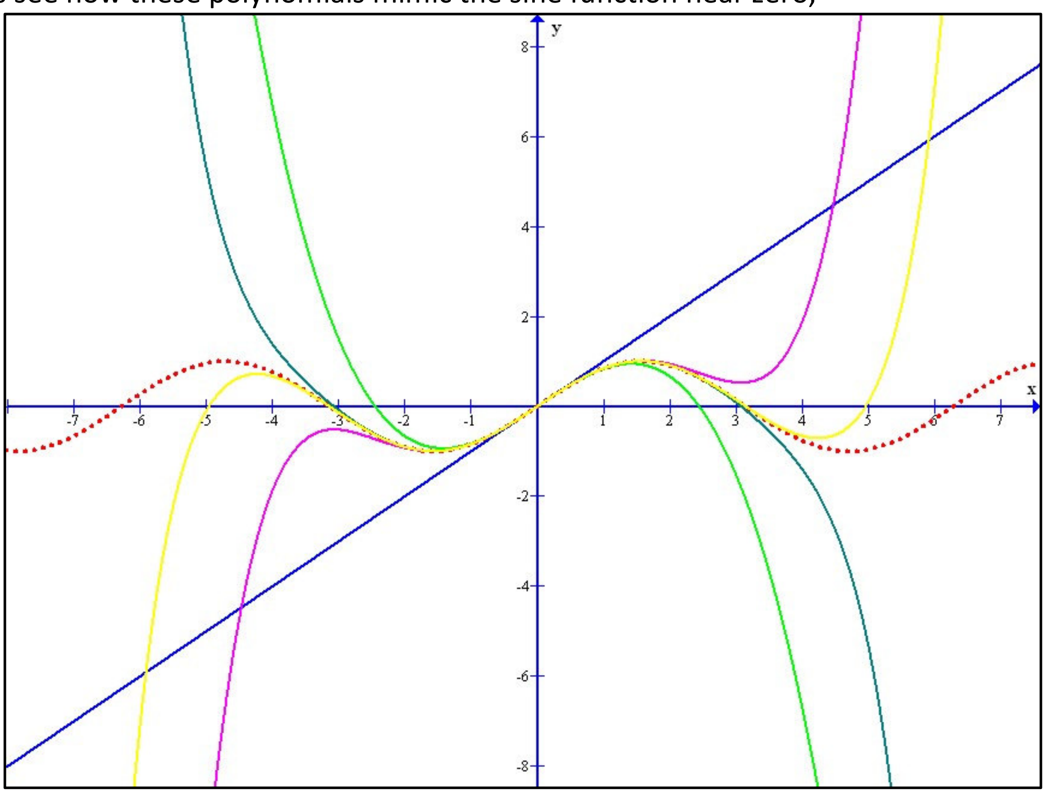
Let $f(x)=\sin(x)$. Find several Taylor polynomials centered about zero.

$$\begin{gathered}
f(x)=\sin(x) \implies f(0)=0 \\
f'(x)=\cos(x) \implies f'(0)=1 \\
f''(x)=-\sin(x) \implies f''(0)=0 \\
f'''(x)=-\cos(x) \implies f'''(0)=-1 \\
f^{(4)}(x)=\sin(x) \implies f^{(4)}(0)=0 \\
f^{(5)}(x)=\cos(x) \implies f^{(5)}(0)=1
\end{gathered}$$

It is clear this pattern continues. Given the above we find:

$$\begin{aligned}
T_1(x) &= x && \text{blue graph} \\
T_3(x) &= x-\frac{1}{6}x^3 && \text{green graph} \\
T_5(x) &= x-\frac{1}{6}x^3+\frac{1}{120}x^5 && \text{purple graph}
\end{aligned}$$

Let's see how these polynomials mimic the sine function near zero,



The grey-blue graph is $y=T_7(x)$. The yellow graph is of $y=T_9(x)$. As we add more terms we will pick up further cycles of the sine function. We have covered three zeros of the sine function fairly well via the ninth Taylor polynomial. I'm curious, how many more terms do we need to add to get within 0.1 of the zeros for since at $\pm 2\pi$? From basic algebra we know we need at least a 5-th order polynomial to get 5 zeros. Of course, we can see from what we've done so far that it takes more than that. I have made a homework problem that let's you explore this question via Mathematica.

13.2. TAYLOR'S THEOREM

Geometric series tricks allowed us to find power series expansions for a few of the known functions but there are still many elementary functions which we have no series representation for as of now. Taylor's Theorem will allow us to generate the power series representation for many functions through a relatively simple rule. Before we get to that we need to do a few motivating comments.

Suppose a function f has the following power series representation

$$f(x) = \sum_{n=0} c_n (x-a)^n$$

We call the constants c_n the coefficients of the series. We call $x = a$ the center of the series. In other words, the series above is centered at a.

In-class Exercise 13.2.1: find the eqn. relating the derivatives of the function evaluated at $x = a$ and the coefficients of the series. **[*the answer is* $c_n = \frac{1}{n!} f^{(n)}(a)$ *for all* $n \geq 0$]**

Definition of Taylor Series

We say that $T(x)$ is the Taylor series for $f(x)$ centered at $x = a$,

$$\begin{aligned} T(x) &= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \end{aligned}$$

You should recognize that $T(x) = \lim_{n\to\infty} T_n(x)$ where $T_n(x)$ is the n-th order Taylor polynomial we defined in the last section.

Comment: Exercise 13.2.1 shows that **if** a given function has a power series representation then it has to be the Taylor series for the function.

Remark: One might question, do all functions have a power series representation? It turns out that in general that need not be the case. It is possible to calculate the Taylor Series at some point and find that it does not match the actual function near the point. The good news is that such examples are fairly hard to come by. If a function has a power series expansion on an interval $I \subset \mathbb{R}$ then the function is said to be **analytic on** I**.** I should remind you that if we can take arbitrarily many continuous derivatives on $I \subset \mathbb{R}$ then the function is said to be **smooth or infinitely differentiable.** It is always the case that an analytic function is smooth, however the converse is not true. There are smooth functions which fail to be analytic at a point. The following is probably the most famous example of a smooth yet non-analytic function:

Example 13.2.1(example of smooth function which is not analytic)

$$f(x) = \begin{cases} exp(\frac{-1}{x^2}), & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \Longrightarrow \; f'(0) = 0, \; f''(0) = 0, \ldots$$

Notice this yields a vanishing Taylor series at $x = 0$; $T(x) = f(0) + f'(0)x + \cdots = 0$. However, you can easily see that the function is nonzero in any open interval about zero. This example shows there are functions for which the Taylor series fails to match the function. In other words, the Taylor series does not converge to the function.

Definition of analytic: A function f is analytic on $I \subset dom(f)$ iff $f(x) = T(x)$ for all x in an open interval I. In particular, a function is analytic on I if

$$f(x) = \lim_{n\to\infty} T_n(x).$$

Question: "How do we test if $f(x) = T(x)$?"

Definition of n-th remainder of Taylor series: The n-th partial sum in the Taylor series is denoted T_n (this is the n-th order Taylor polynomial for f). We define R_n as follows:

$$R_n(x) = T_n(x) - f(x)$$

Taylor's Theorem:

If f is a smooth function with Taylor polynomials $T_n(x)$ such that $f(x) = T_n(x) + R_n(x)$ where the remainders $R_n(x)$ have $\lim_{n\to\infty} R_n(x) = 0$ for all x such that $|x-a| < R$ then the function f is analytic on $I = \{x \mid |x-a| < R\}$. To reiterate, if the remainder goes to zero on I then the Taylor Series converges to f for all $x \in I$,

$$\boxed{f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

We are still faced with a difficult task, how do we show that the remainder $R_n(x)$ goes to zero for particular examples? Fortunately, the following inequality helps.

Th^{m} (TAYLOR'S INEQUALITY). If $|f^{(n+1)}(x)| \leq M$ for $|x-a| < R$ then the remainder $R_n(x)$ of the TAYLOR SERIES is bounded by

$$0 \leq |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| < R$$

This inequality is easy to apply in the case of sine or cosine.

Example 13.2.3

E3 Prove $f(x)=\sin(x)$ is rep. by it's Maclaurin series $\forall x$,

$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$f'''(x) = -\cos(x)$$
$$f''''(x) = \sin(x)$$

Thus $f^{(n)}(x) = \pm\sin(x)$ OR $\pm\cos(x)$ $\therefore$ $|f^{n}(x)| \le 1 = M$.

By TAYLOR'S INEQ. Th^{m},

$$0 \le |R_n(x)| \le \frac{x^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as } n\to\infty$$

Thus by squeeze th^{m} $R_n(x) \longrightarrow 0$ as $n\to\infty$ (for all x).

Moreover, the Maclaurin series is easily calculated.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$
$$= \sin(0) + \cos(0)\cdot x - \frac{\sin(0)}{2!}x^2 - \frac{\cos(0)}{3!}x^3 + \cdots$$
$$= \boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = \sin(x)}$$

In-class Exercise 13.2.2

Do E3 in the case $f(x) = \cos(x)$.

Example 13.2.4 (assuming that the exponential has a power series representation)

E4 Let $f(x) = e^x$ then $f'(x) = f''(x) = \dots = f^{(n)}(x) = e^x$.

Thus the taylor series about zero for e^x is simple,

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$
$$= \sum_{n=0}^{\infty}\frac{e^0}{n!}x^n$$
$$= \boxed{\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x}$$

Maclaurin series for the exponential fnct.

Remark: We would like to show that the power the exponential function is analytic. To do that we should discuss the other version of Taylor's Theorem (which is a generalization of the mean value theorem). Once that is settled, a half-page of inequalities and the squeeze theorem will show that the remainder for the exponential function goes to zero independent of the argument. You can earn 3 bonus points if you work out these things in reasonable detail. Ask me if you are interested, I'll get you started.

Examples 13.2.5 and 13.2.6

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$\boxed{E5}$ Geometric Series: Given $f(x) = \frac{1}{1-x}$ what do we find,

$$f(x) = \frac{1}{1-x} \qquad f(0) = 1$$
$$f'(x) = \frac{1}{(1-x)^2} \qquad f'(0) = 1$$
$$f''(x) = \frac{2}{(1-x)^3} \qquad f''(0) = 2!$$
$$f'''(x) = \frac{3 \cdot 2 \cdot 1}{(1-x)^4} \qquad f'''(0) = 3!$$
$$\vdots \qquad \vdots$$
$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \qquad f^{(n)}(0) = n!$$

Checking consistency. we already knew the result here w/o work

Hence $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=0}^{\infty} x^n$

That is $\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots}$

$\boxed{E6}$ $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \underset{d/dx}{\overset{d/dx}{\rightleftarrows}} \sinh(x) = \frac{1}{2}(e^x - e^{-x})$

Let us find Taylor expansion of $\cosh(x)$ about $x = 0$,

$$f(x) = \cosh(x) \qquad f(0) = \tfrac{1}{2}(e^0 + e^{-0}) = 1$$
$$f'(x) = \sinh(x) \qquad f'(0) = \tfrac{1}{2}(e^0 - e^{-0}) = 0$$
$$f''(x) = \cosh(x) \qquad f''(0) = 1$$
$$f'''(x) = \sinh(x) \qquad f'''(0) = 0$$
$$f''''(x) = \cosh(x) \qquad f''''(0) = 1$$

Hence $\boxed{\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$

Very similarly $\boxed{\sinh(x) = 1 + x + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$

Remark: just like $\cos(x)$ and $\sin(x)$ just no alternating signs.

Example 13.2.7 (multiplying series verses direct-Taylor expanding)

$\boxed{E7}$ $f(x) = \sin^2(x)$. One way is $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$

Using $\boxed{E4}$ with $2x$ in place of x we find

$$\cos(2x) = 1 - \frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6$$

Thus subst. this into identity above gives

$$\sin^2(x) = \frac{1}{2}\left(\cancel{1} - [\cancel{1} - 2x^2 + \frac{16}{24}x^4 - \frac{64}{720}x^6 +]\right)$$
$$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \dots$$

A second method is to multiply the series for $\sin(x)$

$$\sin^2(x) = \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right)\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right)$$
$$= x^2 - \frac{1}{3!}x^4 + \frac{1}{5!}x^6 - \dots - \frac{1}{3!}x^4 + \frac{1}{(3!)^2}x^6 + \dots + \frac{1}{5!}x^6 + \dots$$
$$= x^2 - \frac{2}{3!}x^4 + \left(\frac{2}{5!} + \frac{1}{(3!)^2}\right)x^6 + \dots$$
$$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \dots$$

A third method is to simply taylor expand,

$$f(x) = \sin^2(x) \qquad f(0) = 0$$
$$f'(x) = 2\sin(x)\cos(x) \qquad f'(0) = 0$$
$$f''(x) = 2(\cos^2(x) - \sin^2(x)) \qquad f''(0) = 2$$
$$f'''(x) = -4\sin(x)\cos(x) - 4\sin(x)\cos(x) \qquad f'''(0) = 0$$
$$f''''(x) = -8(\cos^2(x) - \sin^2(x)) \qquad f''''(0) = -8$$

Thus $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \frac{2x^2}{2!} - \frac{8x^4}{4!} + \dots = x^2 - \frac{1}{3}x^4 \dots$

Which method do you think is best?

Obviously it depends on the example and what we're asked

Example 13.2.8 and 13.2.9

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E8 Expand $f(x) = \sqrt{x}$ around $a = 4$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2}\frac{1}{\sqrt{x}}$ $\qquad f(4) = 2$, $\quad f'(4) = \frac{1}{4}$

$f''(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)x^{-3/2} = -\left(\frac{1}{2}\right)^2\frac{1}{(\sqrt{x})^3}$ $\qquad f''(4) = \frac{-1}{32}$

$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^{-5/2} = 3\left(\frac{1}{2}\right)^3\frac{1}{(\sqrt{x})^5}$ $\qquad f'''(4) = \frac{3}{2^8}$

$f''''(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)x^{-7/2} = -3\cdot 5\left(\frac{1}{2}\right)^4\frac{1}{(\sqrt{x})^7}$ $\qquad f''''(4) = \frac{-15}{2^{11}}$

Thus we find

$$f(x) = \sum \frac{f^{(n)}(4)}{n!}(x-4)^n$$

$$= \boxed{2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{5}{16,384}(x-4)^4 + \cdots = \sqrt{x}}$$

E9 Find taylor exp. of $f(x) = x^3 + 3x^2 + 3x + 1$ about $x = 0$ and $x = -1$. Notice $f(0) = 1$ and $f(-1) = 0$ and

$f'(x) = 3x^2 + 6x + 3$ $\qquad f'(0) = 3$ $\qquad f'(-1) = 0$

$f''(x) = 6x + 6$ $\qquad f''(0) = 6$ $\qquad f''(-1) = 0$

$f'''(x) = 6$ $\qquad f'''(0) = 6$ $\qquad f'''(-1) = 6$

$f''''(x) = 0$

about: zero $\rightarrow$ $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = 1 + 3x + \frac{6}{2!}x^2 + \frac{6}{3!}x^3 = x^3 + 3x^2 + 3x + 1$

about $x = -1$ $\rightarrow$ $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n$

$$= 0 + 0\cdot(x+1) + \frac{0}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3$$

$$= \boxed{(x+1)^3 = x^3 + 3x^2 + 3x + 1 = f(x)}$$

Polynomials provide Taylor series which *truncate*. There is still something to learn from E9, we can use derivatives to center the polynomial about any point we wish. Notice the Taylor series revealed $f(x) = (x+1)^3$. Algebraically that is clear anyway, but it's always nice to find a new angle on algebra.

Example 13.2.10

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$\boxed{E10}$ Find Maclaurin series for $\tan(x) = f(x)$. Find 1st 2 non-zero terms.

$$f(x) = \tan(x) \qquad f(0) = 0$$
$$f'(x) = \sec^2(x) \qquad f'(0) = 1$$
$$f''(x) = 2\sec^2(x)\tan(x) \qquad f''(0) = 0$$
$$f'''(x) = 4\sec^2(x)\tan^2(x) + 2\sec^4(x) \qquad f'''(0) = 2$$

Hence $\tan(x) = x + \frac{2}{3!}x^3 + \cdots = \boxed{x + \frac{1}{3}x^3 + \cdots = \tan(x)}$

We could differentiate further if we want to generate higher order terms.

Summary of known Maclaurin Series

	I.O.C
$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + u^3 + \cdots$	$(-1, 1)$
$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3$	$(-\infty, \infty)$
$\sin(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 \cdots$	$(-\infty, \infty)$
$\cos(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{1}{2}u^2 + \frac{1}{4!}u^4 - \cdots$	$(-\infty, \infty)$

Example 13.2.11(note this calculation uses what we already calculated)

$\boxed{E11}$ $x\sin(x/2) = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{[\frac{1}{2}x]^{2n+1}}{(2n+1)!}$

$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \frac{x^{2n+2}}{(2n+1)!}} = x\left(\frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 + \cdots\right)$ "sigma notation"

$= \boxed{\frac{1}{2}x^2 - \frac{1}{48}x^4 + \frac{1}{3840}x^6 - \cdots}$ 1st three non-zero terms.

Once we have a few of the basic Maclaurin series established the examples built from them via substitution are much easier than direct application of Taylor's Theorem.

13.3 BINOMIAL SERIES

There is a neat trick for calculating $(a+b)^k$. It's called Pascal's Triangle, I use it occasionally. Below I write the triangle and what the line $\Rightarrow$ for $(a+b)^k$.

$$\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & \longrightarrow & (a+b)^0 = 1 \\
 & & & & 1 & & 1 & & & & & \longrightarrow & (a+b)^1 = a+b \\
 & & & 1 & & 2 & & 1 & & & & \longrightarrow & (a+b)^2 = a^2+2ab+b^2 \\
 & & 1 & & 3 & & 3 & & 1 & & & \longrightarrow & (a+b)^3 = a^3+3a^2b+3ab^2+b^3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & \longrightarrow & (a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & \longrightarrow & (a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{array}$$

And so on, hopefully the pattern is clear. Well we can write this more compactly

$$\boxed{\begin{aligned}
&(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n \qquad \text{Binomial Th}^{\underline{m}} \\
&\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} = \frac{k!}{n!(k-n)!} \qquad (k \in \mathbb{N}) \\
&\text{“}k \text{ choose } n\text{”}, \text{ where } \binom{k}{0} = 1
\end{aligned}}$$

This has been known for some time, however once k is allowed to be any real number we need an infinite series, to keep it simple we'll study $(1+x)^k$, once we know that we can easily find $(a+b)^k$ since $(a+b)^k = a^k(1+b/a)^k$.

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}$$

Hence

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Therefore, assuming $R_n \to 0$ as $n \to \infty$,

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots \qquad \text{Binomial Series}}$$

Examples 13.3.1 through 13.3.3 and the IOC for various k:

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Remark: $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ converges for $|x| < 1$

While the I.O.C. depends on k, the result is

$$\text{I.O.C} = (-1, 1] \quad \text{if } -1 < k \le 0$$
$$\text{I.O.C} = [-1, 1] \quad \text{if } k \ge 0$$
$$\text{I.O.C} = (-1, 1) \quad \text{if } -1 > k$$

Proof left to reader. (I won't ask you to prove these)

$\boxed{E1}$ Expand $\frac{1}{(1+x)^2}$ using binomial series (this was $\boxed{E6}$ of §8.5)

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + \frac{(-2)(-2-1)}{2} x^2 + \frac{(-2)(-2-1)(-2-2)}{3!} x^3 + \cdots$$

$$= \boxed{1 - 2x - 3x^2 + 4x^3 - \cdots = \frac{1}{(1+x)^2}}$$

$\boxed{\text{for } x \in (-1, 1)}$

Same as $\boxed{E6}$ ☺

$\boxed{E2}$ $\frac{1}{\sqrt{1 - v^2/c^2}} = (1 - v^2/c^2)^{-1/2}$ let $u = -v^2/c^2$

$$= (1+u)^{-1/2}$$
$$= 1 - \tfrac{1}{2}u + \frac{(-1/2)(-1/2-1)}{2} u^2 + \cdots$$
$$= 1 - \tfrac{1}{2}u + \tfrac{3}{8}u^2 + \cdots$$
$$= 1 + \tfrac{1}{2}\left(\tfrac{v}{c}\right)^2 + \tfrac{3}{8}\left(\tfrac{v}{c}\right)^4 + \cdots$$

for $\left|\frac{v}{c}\right| < 1$ aka $\underline{-c < v < c}$

$\boxed{E3}$ $\frac{3}{1-x^2} = 3(1-x^2)^{-1}$ $\quad u = -x^2$

$$= 3(1+u)^{-1}$$
$$= 3\left[1 - u + \frac{(-1)(-1-1)}{2} u^2 + \cdots\right]$$
$$= \boxed{3\left[1 + x^2 + x^4 + \cdots\right] = \frac{3}{1-x^2}}$$

(with radius of convergence 1 $\quad |x^2| < 1$)

- Alternatively you could have identified this to be a geometric series with $a = 3$ and $r = x^2$

13.4 NUMERICAL APPLICATIONS OF TAYLOR SERIES

Numerical Methods: Given $f(x)$ which is analytic we can approximate $f(x)$ near $x=a$ by

$$f(x) \cong T_n(x)$$

Where $T_n(x) \longrightarrow f(x)$ as $n \to \infty$. The natural questions that arise are.

1. for a particular n how good an approx is $T_n(x)$ to $f(x)$?
2. if we desire a certain accuracy for $f(x)$ what is the minimum n for which $f(x) \cong T_n(x)$

Both of these questions can be answered if we know $|R_n(x)| = |f(x) - T_n(x)|$. Which is possible to estimate

1. graphically (a bit cheesy assumes we can graph $f(x)$ right?)
2. Alt. Series Th^{m} (much better, don't need the answer to get the answer!)
3. Taylor's INEQ $\quad 0 \le |R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ when $|f^{(n+1)}(x)| \le M$ for $|x-a| < R$.

Example 13.4.1

$\boxed{E1}$ For what interval about zero can we approx $\sin(x)$ by x to two decimals?

$\sin(x) = x - \frac{1}{3!}x^3 + \cdots \leftarrow$ alternating series.

If we keep upto x then $\sin(x) \cong x$ upto an error $\le \left|\frac{x^3}{3!}\right|$

the max error is

$$\text{error} = \frac{|x|^3}{3!} = 0.01 \quad \Rightarrow \quad |x| = \sqrt[3]{0.06} = 0.39$$

That means $\sin\theta \cong \theta$ for $-0.39 \le \theta \le 0.39$

In degrees $|\theta| \le 22.3°$.

Example 13.4.2

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$\boxed{E2}$ So we're faced with the task of accurately calculating the $\sqrt{4.03}$ to seven decimals. For the purposes of this example assume all calculators are evil, its after the robot holocaust so they can't be trusted. What to do? We'll use $\boxed{E8}$ on 225 plus the Alternating series estimation theorem,

$$\sqrt{x} = \underbrace{2 + \tfrac{1}{4}(x-4)}_{\text{nothing new see pg. 54}} \underbrace{- \tfrac{1}{64}(x-4)^2 + \tfrac{1}{512}(x-4)^3 + \cdots}_{\text{thank you Mr. Taylor this makes it even better than a linear approximation.}}$$

Comments aside lets calculate, try using 1st 3 terms,

$$\begin{aligned}\sqrt{4.03} &= 2 + \tfrac{1}{4}(4.03-4) - \tfrac{1}{64}(4.03-4)^2 \\ &= 2 + \tfrac{1}{4}(0.03) - \tfrac{1}{64}(0.03)^2 \\ &= 2 + \tfrac{3}{4}\tfrac{1}{100} - \tfrac{9}{64}\left(\tfrac{1}{100}\right)^2 \\ &= 2.00 + 0.0075 - 0.000014062 \\ &= 2.007485938\end{aligned}$$

arithmatic.

$$\begin{array}{r} 0.14062 \\ 64\,\overline{)9.0} \\ \underline{64} \\ 260 \\ \underline{256} \\ 400 \\ \underline{384} \\ 160. \end{array} \qquad \begin{array}{r} 0.007500000 \\ \underline{0.000014062} \\ 0.007485938 \end{array}$$

How many of these digits are certain? Well the series is alternating thus we know the error is smaller than the next term in the series,

$$\text{Error} \leq \frac{1}{512}(0.03)^3 = \frac{27}{512}\left(\frac{1}{100}\right)^3 \cong 0.000000058$$

$$\begin{array}{r} 0.058 \\ 512\,\overline{)27.00} \\ \underline{25.60} \\ 4400 \\ \underline{4112} \end{array}$$

Thus $\sqrt{4.03} = 2.007485938 \pm 0.0000000058$

For sure $\boxed{\sqrt{4.03} \cong 2.0074859}$ the next digit 3 is uncertain.

Compare this to $\sqrt{4.03} = 2.00748598999$ from my TI-89

13.5 CALCULUS APPLICATIONS OF TAYLOR SERIES.

Example 13.5.1(using power series to integrate)

E1 Find the complete power series sol^{n} to $\int \frac{e^x}{x}\,dx$.

$$\frac{e^x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{x^{n-1}}{n!}\right)$$

Now we can integrate term by term, (not at zero!)

$$\int \frac{e^x}{x}dx = \int\left(\sum_{n=0}^{\infty}\frac{x^{n-1}}{n!}\right)dx$$

$$= \sum_{n=0}^{\infty}\int\frac{x^{n-1}}{n!}dx + C$$

$$= \int\frac{1}{x}dx + \sum_{n=1}^{\infty}\int\frac{x^{n-1}}{n!}dx + C$$

$$= \boxed{\ln|x| + \sum_{n=1}^{\infty}\frac{x^n}{n\,n!} + C}$$

Lets differentiate and see if it works, is this really the antiderivati

$$\frac{d}{dx}\left(\ln|x| + \sum_{n=1}^{\infty}\frac{x^n}{n\,n!}\right) = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{d}{dx}\left(\frac{x^n}{n\,n!}\right) = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}$$

Example 13.5.2(power series solution to integral)

$$\sin(x^2) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^2)^{2n+1}}{(2n+1)!}$$: using the Maclaurin series for sine which we know.

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!} = \sin(x^2)$$

Now use the series to represent the integrand (just as we did in E

$$\int\sin(x^2)dx = \int\sum_{n=0}^{\infty}(-1)^n\frac{x^{4n+2}}{(2n+1)!}dx$$

$$= \sum_{n=0}^{\infty}\int(-1)^n\frac{x^{4n+2}}{(2n+1)!}dx + C$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int x^{4n+2}dx + C$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{4n+3}x^{4n+3} + C}$$ ← complete sol^{n}

$$= \boxed{C + \frac{1}{3}x^3 - \frac{1}{3!}\frac{1}{7}x^7 + \frac{1}{5!}\frac{1}{11}x^{11} + \cdots}$$ ← 1st 3 non trivial terms.

Example 13.5.3(what's not right here?)

E3 $\int \cos(e^x)\,dx$, hmm. this one is more interesting!

$$\cos(e^x) = \sum_{n=0}^{\infty} (-1)^n \frac{(e^x)^{2n}}{(2n)!}$$ is this ok? Why or why not?

$$\int \cos(e^x)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{(e^x)^{2n}}{(2n)!}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \int e^{2nx}\,dx + C$$

$$= \int e^0\,dx + \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!} \int e^{2nx}\,dx + C$$

$$= \boxed{C + x + \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!}\frac{1}{2n} e^{2nx}}$$ ← complete sol^n

$$= \boxed{C + x - \frac{1}{4}e^{2x} + \frac{1}{4!}\frac{1}{4}e^{4x} + \dots}$$ ← 1st 3 non-trivial terms.

What is wrong with this example? Bonus Point if you can clearly tell me

Example 13.5.4

$$\int \frac{\sin(x)}{x}\,dx = \int \frac{1}{x}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)dx$$

$$= \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}\right)dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int x^{2n}\,dx + C$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n+1}}{(2n+1)} + C}$$

Example 13.5.5(power series solution to integral)

Approximately calculate $\int_0^{0.2} \frac{1}{1+x^5}\,dx$ to at least 6 correct decimal places. Notice this integral is not elementary, however we can find a power series solution:

$$\int \frac{1}{1+x^5}\,dx = \int (1 - x^5 + x^{10} - x^{15} + \cdots)\,dx$$

$$= x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \cdots + C$$

Now the definite integral will reveal the answer is an alternating series!

$$\int_0^{0.2} \frac{dx}{1+x^5} = 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \frac{(0.2)^{16}}{16} + \cdots$$

It's quite clear $(0.2)^{11}/11$ is several decimals beyond the 6^{th} decimal and by the Alt. Series Estimation Th^{m} we know that

$$\int_0^{0.2} \frac{dx}{1+x^5} = 0.2 - \frac{(0.2)^6}{6} \quad \underline{\text{within}} \quad \frac{(0.2)^{11}}{11}$$

$$\cong \boxed{0.199989}$$

(could do even better given $\frac{(0.2)^{11}}{11}$ is actually beyond the 6^{th} decimal.)

(the alternating series error estimation Theorem is quite useful for questions like this one, sadly not all series alternate.)

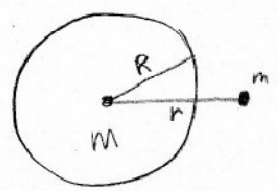
13.6 PHYSICAL APPLICATIONS OF TAYLOR SERIES

I sometimes cover E1 in calculus I but it needs repeating here. E2 is a discussion of the electric dipole.

Applications to Physics (233)

E1 Why does $U = mgy$ for gravity? Isn't it the case that $U = \frac{-GmM}{r}$

$$U(R) = \frac{-GmM}{R}$$
$$U'(R) = \frac{GmM}{R^2}$$
$$U''(R) = \frac{-2GmM}{R^3}$$



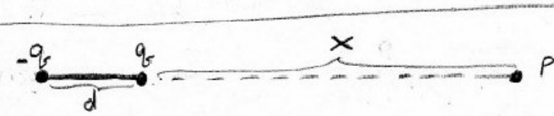
Well taylor expand $U(r)$ about $r = R$

$$U(r) \cong -\frac{GmM}{R} + \frac{GmM}{R^2}(r-R) - \frac{GmM}{R^3}(r-R)^2 + \cdots$$

Think about it, $r - R$ is height above ground so noting $\frac{GM}{R^2} = g = 9.81$.

$$\boxed{U(y) \cong -\frac{GmM}{R} + mgy, \quad \text{upto error of } \frac{GmM}{R^3}y^2}$$

That is $P.E. = mgy$ near surface of earth

E2 

find $\vec{E}$ field at the point P, assuming $x \gg d$.

$$E = \frac{q}{x^2} - \frac{q}{(x+d)^2}$$

Notice, $\frac{1}{(x+d)^2} = \frac{1}{x^2(1+d/x)^2} = \frac{1}{x^2}\left(1+\frac{d}{x}\right)^{-2}$ ← Binomial Series for $\frac{d}{x} < 1$ as is the case.

$$\therefore \quad \frac{1}{(x+d)^2} \cong \frac{1}{x^2}\left(1 - 2\frac{d}{x} + \cdots\right)$$

Thus $E = \cancel{\frac{q}{x^2}} - q\left[\cancel{\frac{1}{x^2}}\left(1 - 2\frac{d}{x} + \cdots\right)\right] = \boxed{\frac{2qd}{x^3} \cong E}$

this gives the "far-field" approximation, it shows how the dominant contribution of E scales with distance.

Example 13.6.3 (Special Relativity: Einstein's famous equation)

Special Relativity: for a particle with mass m_o

$E = \gamma m_o c^2$ (Total Energy of free particle)

$K = \gamma m_o c^2 - m_o c^2$ (Relativistic Kinetic Energy)

Where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ with v = velocity of particle, c = speed of light

Let's Calculate the power series expansion of $E(v)$ and see what it means physically, recall we found

$$\frac{1}{\sqrt{1-v^2/c^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4$$

Then we see if $v \approx 0$ we have

$$E = \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right) m_o c^2$$

$$= \underbrace{m_o c^2}_{\text{Rest Energy of Particle}} + \underbrace{\frac{1}{2} m_o v^2}_{\text{Usual Kinetic Energy.}} + \cdots$$

It's easy to see that $K \cong \frac{1}{2} m_o v^2$ for $|v| << c$, which is good, the Relativistic Kinetic Energy should become the ordinary Newtonian Kin. Energy when $|v| << c$, that

Is called the "non-relativistic" case. Much of special relativity amounts to adding a γ-factor to the classical equations. For example,

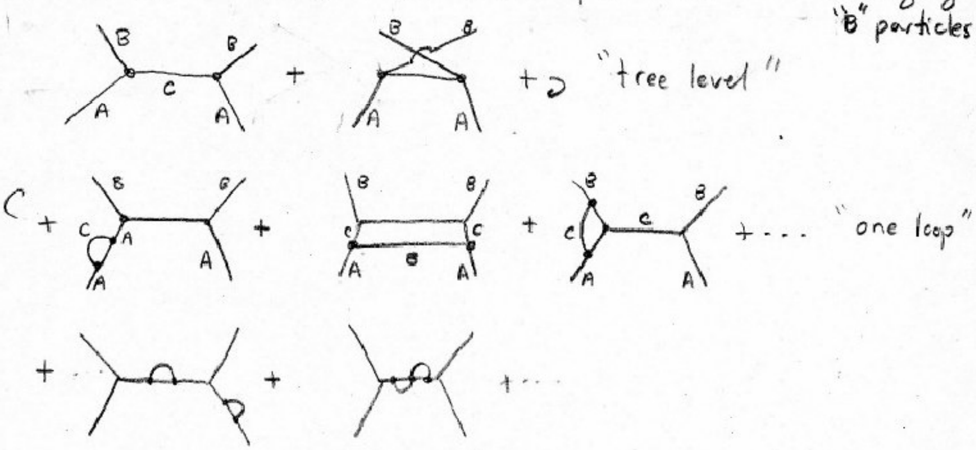
$$\vec{p} = \gamma m \vec{v} \qquad \text{Relativistic 3-momentum}$$

$$F = \frac{\gamma m v^2}{R^2} \qquad \text{Force for circular motion of radius } R$$

Generally, the proper stage to discuss special relativity is Minkowski space. In Minkowski space, time is treated as the fourth dimension. Anyhow, there is much more to say about special relativity, lot's of interesting and relatively easy mathematics. You can peruse my ma430 course notes or ask me for things to read. I hope we will be able to offer a course which covers special relativity in the physics minor sometime soon.

On the use of series in pertubative modern physics:

TAYLOR SERIES are nice for <u>known</u> functions. In modern physical theories the equations are so difficult to solve that we often have only a "perturbative" description of the physics. What this means is we have to find a series to describe physical things (like how big a particle is, or its mass, ...). You might wonder how can we find the series, afterall $f(x)$ is not known, well we use what are called "FEYNMAN DIAGRAMS" let me illustrate for $A + A \longrightarrow B + B$ (two "A" particles collide and become two outgoing "B" particles)



From such pictures one can make predictions about the physical characteristics of elementary particles. The man who invented these diagrams had a van with them painted on long before anybody knew what they were, needless to say he lived in California.

The type of physics I am sketching above generally falls under what it known as *field theory.* There are many open problems in field theory, yet we know that the most precise equations follow from field theoretic models. It's not crazy to start thinking about field theory as an undergraduate. I have some good books if you would like to do an independent study. I'd wager you could teach me a few things.