14. GEOMETRY AND COORDINATES

We define $\mathbb{R}^2 \equiv \{(x,y) \mid x,y \in \mathbb{R}\ \}$. Given $(a,b) \in \mathbb{R}^2$ we say that the x-coordinate is a while the y-coordinate is b. We can view the coordinates as mappings from $\mathbb{R}^2$ to $\mathbb{R}$:

$$x(a,b) = a$$
$$y(a,b) = b$$

Coordinates take in a point in the plane and output a real number.

Our goal in this chapter go beyond Cartesian coordinates for the plane. Are there other useful descriptions of $\mathbb{R}^2$? How do those other coordinate systems relate to the standard Cartesian coordinate system?

Geometrically, the Cartesian coordinates of a point (a,b) correspond to a horizontal displacement of a-units and a vertical displacement of b-units that reach from the origin to the point (a,b). Basically, the Cartesian coordinates are built from travelling lines from the origin. Moreover, these lines are perpendicular.

We'll learn that we can construct other coordinate systems which are not based just on perpendicular lines. We can use circles, hyperbolas, parabolas, ellipses and many other curves as well as lines. The coordinates will be prescribed according to how the point is reached by travelling from the (new) origin along those curves. We define the origin for a particular coordinate system to be the point at which the coordinates are both zero. This may or may not line up with the Cartesian coordinate origin. For example,

$$\bar{x} = x + 3$$
$$\bar{y} = y + 2$$

Puts the barred-origin at the point $(-3,-2)$. In other words, $(-3,-2) \mapsto (0,0)$. Obviously there is danger of confusion here, what is meant by $(0,0)$? Is the origin in the xy-coordinates or is this the origin in the $\bar{x}\bar{y}$-coordinate system? The solution is context, when dealing with several coordinate systems at once it is important to say which coordinates are used for a particular statement. We will discuss a variety of coordinate systems. We'll go into the most depth on polar coordinates since those are used the most in undergraduate coursework.

To begin we define conic sections. This will help us remember those graphs other than lines which we have less recent experience with. However, it should be a review, your highschool mathematics ought to have covered conic sections in some depth.

By the way, the concept of a coordinate system in physics is a bit more general. I would say that a coordinate system in physics is an "observer". An "observer" is a mapping from $\mathbb{R}$ to the space of all coordinates on $\mathbb{R}^2$. At each time we get a different coordinate system, the origin for an observer can move. I describe these ideas carefully in my ma430 notes. Let me just say for conceptual clarity that our coordinate systems are fixed and immovable.

14.1. CONIC SECTIONS

Take a look in Stewart, he has a nice picture of how a cone can be sliced by a plane to give a parabola, hyperbola or an ellipse. Let me remind you the basic definitions and canonical (standard) formulae for the conic sections. I have saved the task of deriving the formula from the geometric definition for homework. Each derivation is a good algebra exercise, and they can be found in many text books.

Definition 14.1.1(Parabola): A parabola is the collection of all points in some plane that are equidistant from a given line and point in that plane. The given line is called the *directerix* and the point called the *focus.* It is assumed that the focus is not a point on the directerix. The perpendicular to the directerix which intersects the focus is called the *axis* of the parabola. The *vertex* is at the intersection of the axis and the parabola.

Theorem 14.1.1: The parabola in the xy-plane with focus $(0, p)$ and directerix $y = -p$ is the set of all points satisfying the equation $x^2 = 4py$.

Proof: see homework.

There are similar equations for parabolas built from a vertical directerix. If you understand this equation it is a simple matter to twist it to get the sideways parabola equation.

Definition 14.1.2(Ellipse): An ellipse is the collection of all points in some plane for which the sum of the distances to a pair of fixed points is constant. Each of the points (which must be in the plane) is called a focus of the ellipse. The line through the focus points intersects the ellipse at its *vertices.* The *major axis* is the line segment which connects the vertices. The *minor axis* is a line segment which is a perpendicular bisector of the major axis with endpoints on the ellipse.

Theorem 14.1.2: Let $a, c > 0$. Define an ellipse in the xy-plane with foci $(-c, 0), (c, 0)$ by taking the collection of all points which have the sum of distance from $(-c, 0)$ and the distance from $(c, 0)$ equal to $2a$. Given these assumptions, there exists $b > 0$ such that $b < a$ and $b^2 = a^2 - c^2$. Moreover, the equation of such an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Proof: see homework.

Again it is not hard to mimick the proof for the case that the foci lie at $(0, \pm c)$ and the sum of the distances is $2b > 0$. That yields $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a^2 = b^2 - c^2$ and $a < b$which is an ellipse with vertices $(0, \pm b)$. (compare my notation with Stewart, I keep a, b in the same location) I usually just think of an ellipse at the locus of points satisfying

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Different values for a, b give different types of ellipses. For instance, $a = b = R$ yields a special ellipse which we call a circle. It is nice that we broke it down into the horizontal and vertical ellipse cases so that the proof of Theorem 14.1.2 doesn't involve cases.

Definition 14.1.3(Hyperbola): A hyperbola is the collection of all points in some plane for which the difference between the distances to a pair of fixed points is constant. Each of the points (which must be in the plane) is called a focus of the hyperbola. The line connecting the foci intersects the hyperbola at the *vertices* of the hyperbola.

Theorem 14.1.3: If a hyperbola with foci $(-c, 0)$ and $(c, 0)$ is formed from the collection of points for which the difference between the distance to $(-c, 0)$ and the distance to $(c, 0)$ is the constant value $2a > 0$ then there exists a $b > 0$ such that $c^2 = a^2 + b^2$ and the hyperbola consists of points satisfying the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

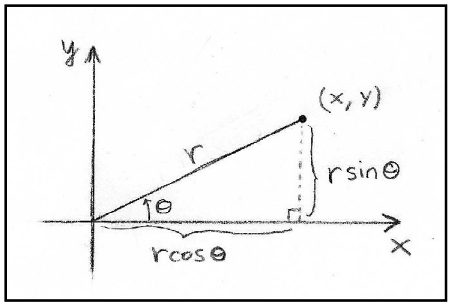
Proof: see homework.

Remark: the Theorem's given in this section provide standard equations which we identify with the ellipse, parabola or hyperbola. These basic graphs can be shifted and/or rotated and the resulting graph will still be the same shape.

I have provided several examples in the solved homework problems. However, this should really just be a review. I don't expect that all of you have gone through the steps that connect the geometric definition of the conic sections with the equations that more usefully define them. That's why it's in your homework.

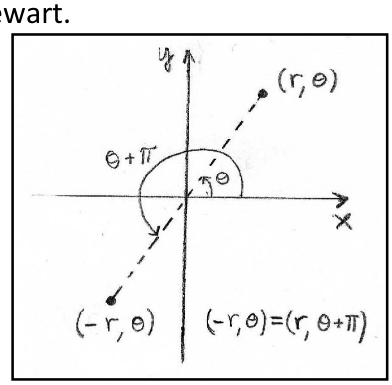
14.2. POLAR COORDINATES

Polar coordinates are denoted (r, θ). Strictly speaking we should insist that $r \geq 0$ and $0 \leq \theta \leq 2\pi$. The radius $r = \sqrt{x^2 + y^2}$ is the distance from the origin. The standard angle θ is the angle measured in radians in the counterclockwise direction from the x-axis. This means that $\tan(\theta) = \frac{y}{x}$ and $\theta = \tan^{-1}(\frac{y}{x})$. We should draw a picture to better understand these equations:



Let us spend a little time discussing a somewhat subtle point. Observe that the $range(\tan^{-1}) = (\frac{-\pi}{2}, \frac{\pi}{2})$ since the domain of the the tangent function about zero is that very interval. Thus the formula $\theta = \tan^{-1}(\frac{y}{x})$ does not yield angles in the standard interval for θ. Fortunately we can make certain natural geometric identifications. In particular if we replace $\theta \mapsto \theta + 2\pi$ we describe the same direction. So while I would like to make polar coordinates *single valued* I will allow for the natural geometric identifications. We can trade $0 \leq \theta \leq 2\pi$ for $-\pi \leq \theta \leq \pi$ or $\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{2}$. This means that for a single direction in the plane we have many values of θ which give that direction. In this sense polar coordinates are *multiply valued.*

We can also trade the point $(-r, \theta) \mapsto (r, \theta + \pi)$ if we extend the values of r to include negative values. This identification is not as prevalent in applications. I would tend to avoid it unless it was a real convenience. There are a few homework problems devoted to this correspondence in Stewart.



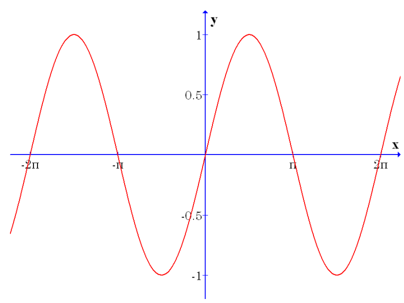
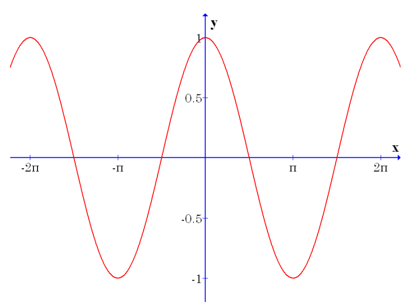
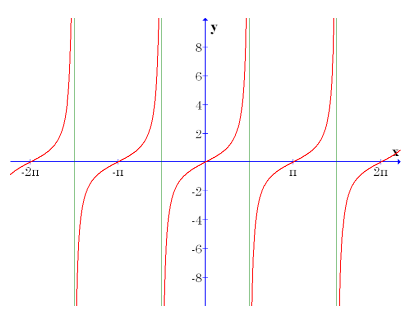
I will assume we are working with the strict version of polar coordinates unless otherwise stated. I mention the other versions because there is no universal agreement in so far as applications are concerned.

Fundamental Equations for Polar Coordinates

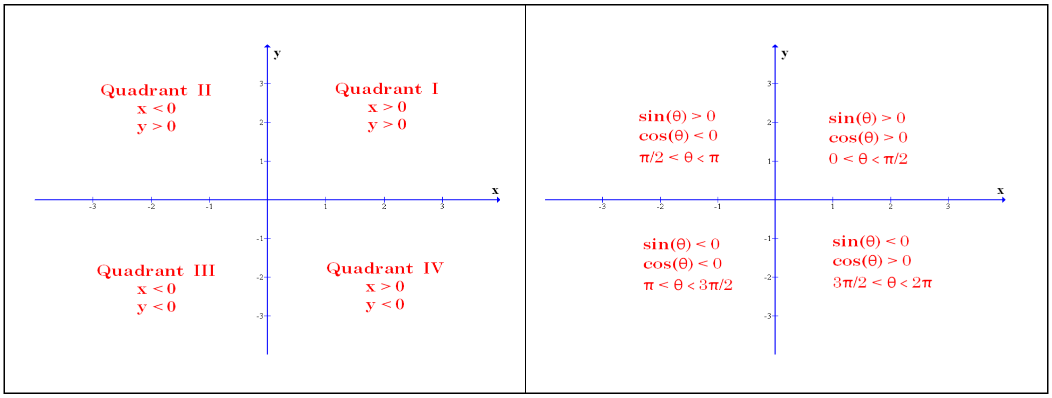
$x = r\cos(\theta)$	$r^2 = x^2 + y^2$	$r = \sqrt{x^2+y^2}$
$y = r\sin(\theta)$	$\tan(\theta) = \frac{y}{x}$	$\theta = \tan^{-1}(\frac{y}{x})$

The third column requires the most thought. When we convert from Cartesian coordinates to polar coordinates we must think about the angle. We must make sure it corresponds to the correct quadrant.

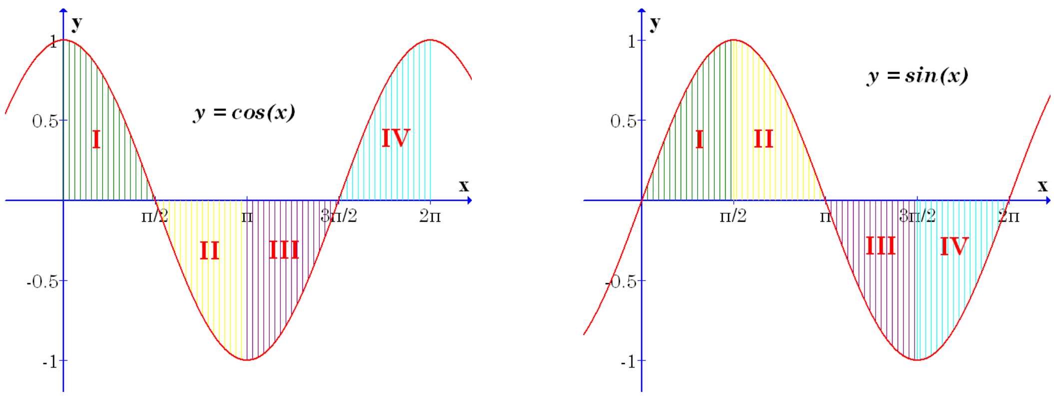
Let me remind you how sine and cosine behave in various quadrants.

notation	name	Zeros	Graph
$\sin(x)$	Sine	$x = 0, \pm\pi, \pm 2\pi, \ldots$ Equivalently, $x = n\pi,\ n \in \mathbb{Z}$	
$\cos(x)$	cosine	$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ Equivalently, $x = n\pi + \frac{\pi}{2},\ n \in \mathbb{Z}$	
$\tan(x)$	tangent	Same as sine. The green lines are the vertical asymptotes which happen where cosine is zero.	

See below how the sine and cosine of the standard angle yield the needed signs.



Or perhaps the following diagrams make more sense to you,



Since $r = \sqrt{x^2 + y^2} \geq 0$ we see that the formulas $x = r\cos(\theta)$ and $y = \sin(\theta)$ reproduce the correct signs for the Cartesian coordinates x and y.

Calculator Warning: Given the Cartesian coordinates of a point it is a common task to find the standard angle θ, we can solve $\tan(\theta) = y/x$ for θ by taking the inverse tangent to obtain $\theta = \tan^{-1}(y/x)$. Let me explain some of the dangers of this formula. Notice that $\tan(\theta) = \sin(\theta)/\cos(\theta)$ is positive in quadrants I and III and is negative in quadrants II and IV. If you try to solve for θ with a calculator it cannot detect the difference between I and III or II and IV. Let's see how the formula is ambiguous if you are not careful,

i.) Suppose $x = 1, y = 1$ then $tan(\theta) = 1/1 = 1$. We can solve for θ by taking the inverse tangent of both sides, $\tan^{-1}(\tan(\theta)) = \theta = \tan^{-1}(1)$ now most scientific calculators will calculate the inverse tangent, it gives $\tan^{-1}(1) = \pi/4$. In this case the calculator has not misled, the standard angle is $\pi/4$.

ii.) Suppose $x = -1, y = -1$ then $tan(\theta) = (-1)/(-1) = 1$. We can solve for θ by taking the inverse tangent of both sides, $\tan^{-1}(\tan(\theta)) = \theta = \tan^{-1}(1)$. Now the scientific calculator will again calculate that $\tan^{-1}(1) = \pi/4$. But in this case the calculator might mislead us, the standard angle is not $\pi/4$. In fact the standard angle here lies in quadrant III and so we have to add π to the angle the calculator found to get the correct angle; $\theta = 5\pi/4$.

Remark: the last couple pages is partly recycled from the section 2.4.5 of my calculus I notes. I figure it needed repeating here.

Example 14.2.1

Find the polar coordinates for the point (1,1).

$$r = \sqrt{1^2+1^2} = \sqrt{2}, \qquad \tan(\theta) = \frac{1}{1} = 1$$

If tangent is one that means $\sin(\theta) = \cos(\theta)$, in quadrant I we find solution $\theta = \frac{\pi}{4}$. Hence the polar coordinates of this point are $\boxed{(\sqrt{2}, \frac{\pi}{4}).}$

Example 14.2.2

Find the polar coordinates for the point $(-\sqrt{3}, 1)$.

$$r = \sqrt{3+1} = 2,$$

$$x = r\cos(\theta) \implies -\sqrt{3} = 2\cos(\theta) \implies \cos(\theta) = \frac{-\sqrt{3}}{2}$$

The given point is in quadrant II thus we find $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. Hence the polar coordinates of this point are $\boxed{(2, \frac{5\pi}{6}).}$

Graphing with polar coordinates:

The goal for rest of this section is to get a feel for how to graph in polar coordinates. We work a few purely polar problems. We also try translating a few standard Cartesian graphs. The general story is that both polar and Cartesian coordinates have their own respective virtues. Broadly speaking, polar coordinates help simplify equations of circles while Cartesian coordinates make equations of lines particularly nice.

Example 14.2.3

Find the equation of the line $y = x + 1$ in polar coordinates. We simply substitute the transformation equations $x = r\cos(\theta), \;\; y = r\sin(\theta)$ to obtain:

$$r\sin(\theta) = r\cos(\theta) + 1 \implies \boxed{r = \frac{1}{\sin(\theta) - \cos(\theta)}}$$

We don't need to solve for the radius necessarily. I like to solve for it when I can since it's easy to think about constructing the graph. Logically the angle is just as primary. In the same sense it is essentially a matter of habit and convenience that we almost always solve for y in the Cartesian coordinates.

Example 14.2.4

Consider the polar equation $r = 2$. Find the equivalent Cartesian equation:

$$r = 2 \implies r^2 = 4 \implies \boxed{x^2 + y^2 = 4.}$$

This is a circle of radius two centered at the origin.

Question: what are the coordinates of the origin in polar coordinates? I'll take the easy part, clearly $r = 0$. The ambiguity enters when we try to ascertain the value of θ at the origin. The standard angle is undefined at the origin. This is not due to a genuine divergence. Rather, we call this sort of problem a *coordinate defect*. These are not generally avoidable. In manifold theory there is a precise and rather exacting definition of a "coordinate map". At a minimum a coordinate map needs to be one-one everywhere. In order to cover most spaces it is necessary to use several coordinate maps such that they paste together in a nice way. Fortunately all of that fussy manifold theory is not needed for calculations over $\mathbb{R}^2$ or $\mathbb{R}^3$. We can calculate without worrying too much about these coordinate defects. In contrast, when the integrand has a vertical asymptote we must approach the asymptote via a limit and many times the integral will diverge as a result. Anyhow, this issue is more or less ignored in much of Stewart and my notes for that matter. I do think coordinate defects can lead to the wrong answer for a calculation, but I don't have a convincing example to make us worry. If this paragraph doesn't make any sense to you don't sweat it, I'm thinking out loud here at the moment.

Example 14.2.5

If $x^2 + (y-1)^2 = 1$ then what is the corresponding polar equation? Substitute as usual,

$$\begin{aligned} & r^2\cos^2(\theta) + (r\sin(\theta) - 1)^2 = 1 \\ \implies \quad & r^2\cos^2(\theta) + r^2\sin^2(\theta) - 2r\sin(\theta) + 1 = 1 \\ \implies \quad & r^2(\cos^2(\theta) + \sin^2(\theta)) = 2r\sin(\theta) \\ \implies \quad & \boxed{r = 2\sin(\theta).} \end{aligned}$$

When the circle is not centered at the origin the equation for the circle in polar coordinates will involve the standard angle in a non-trivial manner.

Example 14.2.6

What is the Cartesian equation that is equivalent to $\theta = \frac{\pi}{6}$? We have two equations:

$$x = r\cos\left(\frac{\pi}{6}\right) = \frac{r\sqrt{3}}{2}$$
$$y = r\sin\left(\frac{\pi}{6}\right) = \frac{r}{2}$$

We eliminate the radius by dividing the equations above:

$$\frac{y}{x} = \frac{1}{\sqrt{3}} \quad \implies \quad \boxed{y = \frac{x}{\sqrt{3}}.}$$

To be careful I should emphasize this is only valid in quadrant I. The other half of the line goes with $\theta = \frac{7\pi}{6}$. In other words, the graph of $\theta = \theta_o$ is a ray based at the origin. It would be the whole line if we allowed for negative radius.

Remark: if we fix the standard angle to be some constant we have seen it gives us a ray from the origin. If we set the radius to be some constant we obtain a circle. Contrast this to the Cartesian case; $x = 3$ is a vertical line, $y = 1$ is a horizontal line.

In-class Exercise 14.2.7
Graph $r = 3\cos(\theta)$. Make a table of values to begin then use an algebraic argument to verify your suspicion about the identity of this curve.

Not all polar curves correspond to known Cartesian curves. Sometimes we just have to make a table of values and graph by connecting the dots, or Mathematica.

In-class Exercise 14.2.8
Graph $r = \sin(2\theta)$. Consider using a graph in the $r\theta$-plane to generate the graph of the given curve in the xy-plane. *(Stewart calls the $r\theta$-plane "Cartesian coordinates" , see Fig. 10 or 12 on page 679)*

I suppose it took all of us a number of examples before we really understood Cartesian coordinates. The same is true for other coordinate systems. Your homework explores a few more examples that will hopefully help you better understand polar coordinates.

Polar Form of Complex Numbers

Recall that we learned a complex number $z = x + iy$ corresponds to the point (x, y). What are the polar coordinates of that point and how does the imaginary exponential come into play here? Observe,

$$z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

The calculation above justifies the assertion that a complex number may be put into its *polar form.* In particular we define,

> ***Definition 14.2.1:*** (Polar Form of Complex Number) Let $z = x + iy$ the polar form of z is $re^{i\theta}$ where $r = \sqrt{x^2 + y^2}$ and θ is the standard angle of (x, y).

Much can be said here, but we leave that discussion for the complex variables course. If this interests you I have a book which is quite readable on the subject. Just ask.
In-class Exercise 14.2.9
Find the polar form of the complex number $z_1 = 3 + 4i$. Find the polar form of the complex number $z_2 = i$. Find the polar form of the complex number $z_1 z_2$. Graph both z_1 and $z_1 z_2$ in the complex plane. How are the points related?

Remark: In electrical engineering complex numbers are used to model the impedance Z. The neat thing is that AC-circuits can be treated as DC-circuits if the voltage source is sinusoidal. This is called the Phasor method. In short, resistances give real impedance while capacitors and inductors give purely imaginary impedance. The net impedance for a circuit is generally complex.

14.3. ROTATED COORDINATES

If we want new coordinates $\bar{x}, \bar{y}$ which are rotated an angle β in the counter clockwise direction relative to the standard x, y coordinates then we'll want

$$\boxed{\begin{aligned} \bar{x} &= x\cos(\beta) + y\sin(\beta) \\ \bar{y} &= -x\sin(\beta) + y\cos(\beta) \end{aligned}}$$

Let's check to see if the positive part of the $\bar{x}$-axis is at angle β in the xy-coordinates. The equation of the $\bar{x}$-axis is simply $\bar{y} = 0$ so by the definition of the rotated coordinate boxed above we have,

$$0 = -x\sin(\beta) + y\cos(\beta) \implies \tan(\beta) = \frac{y}{x}$$

Thus the point (x, y) on the $\bar{x}$-axis has $\theta = \beta$.

Example 14.3.1

Let's rotate by $\beta = \frac{\pi}{2}$, since $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$ we have the rather simple transformation laws:

$$\begin{aligned} \bar{x} &= y \\ \bar{y} &= -x \end{aligned}$$

Now let's see what the parabola $y = x^2$ looks like to the rotated coordinates. To answer such a question we simply substitute as follows:

$$y = x^2 \implies \bar{x} = \bar{y}^2$$

Thus in the rotated coordinate system the parabola is a side-ways parabola.

In-class Exercise 14.3.2

Find the equation of the circle $x^2 + y^2 = R^2$ in rotated coordinates. You will get the same equation for any value of β. Please calculate it for an arbitrary β.

Inverse Transformations

We can relate the rotated coordinates to the xy-coordinates as follows:

$$\boxed{\begin{aligned} x &= \bar{x}\cos(\beta) - \bar{y}\sin(\beta) \\ y &= \bar{x}\sin(\beta) + \bar{y}\cos(\beta) \end{aligned}}$$

Example 14.3.3

Let's explore the case $\beta = \frac{\pi}{6}$. We find,

$$x = \frac{\sqrt{3}}{2}\bar{x} - \frac{1}{2}\bar{y}, \qquad y = \frac{1}{2}\bar{x} + \frac{\sqrt{3}}{2}\bar{y}$$

Or, in other words,

$$\bar{x} = \frac{\sqrt{3}}{2}x + \frac{1}{2}y, \qquad \bar{y} = -\frac{1}{2}x + \frac{\sqrt{3}}{2}y$$

Now, we might wonder what does a standard parabola in the rotated coordinates look like in the xy-coordinates? Let's transform $\bar{y} = \bar{x}^2$ into xy-coordinates:

$$\begin{aligned} \bar{y} = \bar{x}^2 \quad &\Longrightarrow \quad -\frac{1}{2}x + \frac{\sqrt{3}}{2}y = \left(\frac{\sqrt{3}}{2}x + \frac{1}{2}y\right)^2 \\ &\Longrightarrow \quad -2x + 2\sqrt{3}y = 4\left(\frac{3}{4}x^2 + \frac{2\sqrt{3}}{4}xy + \frac{1}{4}y^2\right) \\ &\Longrightarrow \quad -2x + 2\sqrt{3}y = 3x^2 + 2\sqrt{3}xy + y^2 \\ &\Longrightarrow \quad \boxed{3x^2 - 2x + 2\sqrt{3}xy + y^2 + 2\sqrt{3}y = 0.} \end{aligned}$$

Remark: Does the boxed equation look like a parabola to you? It is. This is the beauty of rotated coordinates, reverse this example, start with the boxed equation and ask the question what system of coordinates makes the equation simple? We would need to spend some time to get good at answering that sort of question. My goal here is just to alert you to the fact that rotated coordinates are worth entertaining for difficult questions. Essentially the idea is this: if the given problem looks like a standard problem just rotated a bit then use rotated coordinates to simplify the problem. Use rotated coordinates to reveal the true nature of the problem. Sometimes our initial choice of coordinates is poor, we create problems for ourselves simply through a wrong choice of coordinates. If we use coordinates which respect the symmetry of a given problem then the mathematics tends to fall into place much easier. Just contrast the boxed equation with $\bar{y} = \bar{x}^2$. This principle becomes very important in calculus III, we must choose coordinates which are best for the problem. Otherwise we can make simple problems needlessly difficult.

14.4. OTHER COORDINATE SYSTEMS

This section looks at a variety of non-standard coordinate systems.

HYPERBOLICALLY ROTATED COORDINATES

We can define hyperbolic coordinates as follows: for some $\gamma \in \mathbb{R}$,

$$\boxed{\begin{aligned}\bar{x} &= x\cosh(\gamma) + y\sinh(\gamma)\\ \bar{y} &= x\sinh(\gamma) + y\cosh(\gamma)\end{aligned}}.$$

Notice that the hyperbolas $x^2 \pm y^2 = \mp 1$ have the same form in the hyperbolically rotated coordinates,

$$\begin{aligned}\bar{x}^2 - \bar{y}^2 = 1 \quad &\implies \Big(x\cosh(\gamma) + y\sinh(\gamma)\Big)^2 - \Big(x\sinh(\gamma) + y\cosh(\gamma)\Big)^2 = 1\\ &\implies \Big(x^2\cosh^2(\gamma) + 2xy\cosh(\gamma)\sinh(\gamma) + y^2\sinh^2(\gamma)\Big)\\ &\qquad - \Big(x^2\sinh^2(\gamma) + 2xy\sinh(\gamma)\cosh(\gamma) + y^2\cosh^2(\gamma)\Big) = 1\\ &\implies x^2\big(\cosh^2(\gamma) - \sinh^2(\gamma)\big) - y^2\big(\cosh^2(\gamma) - \sinh^2(\gamma)\big) = 1\\ &\implies x^2 - y^2 = 1\end{aligned}$$

Circles have the same form of equation in the xy and rotated $\bar{x}\bar{y}$-coordinates. With hyperbolic coordinates the form of the equation of a hyperbola is maintained for the hyperbolically rotated coordinates.

TRANSLATED COORDINATES

Translated coordinates are defined as follows: let $a, b \in \mathbb{R}$

$$\boxed{\begin{aligned}\bar{x} &= x + a\\ \bar{y} &= y + b\end{aligned}}.$$

The origin $(\bar{x}, \bar{y}) = (0,0)$ is at the point $(-a, -b)$. Sometimes it's nice to combine these with other coordinate transformations. For example,

$$\boxed{\begin{aligned}\bar{x} &= x\cos(\beta) + y\sin(\beta) + a\\ \bar{y} &= -x\sin(\beta) + y\cos(\beta) + b\end{aligned}}$$

These coordinates allow us to take something like a shifted and rotated parabola in the xy-coordinate system and morph it into a standard parabola at the origin in the transformed coordinate picture.

SKEW-LINEAR COORDINATES

Translated coordinates are defined as follows: let $a, b, c, d \in \mathbb{R}$

$$\boxed{\begin{aligned} \bar{x} &= ax + by \\ \bar{y} &= cx + dy \end{aligned}}.$$

We require that $ad - bc \neq 0$. That requirement is needed to insure the coordinates cover the whole plane.

In-class Exercise 14.4.1: Find conditions on $a, b, c, d \in \mathbb{R}$ such that:

$$\boxed{x^2 + y^2 = \bar{x}^2 + \bar{y}^2, \qquad \forall\ x, y \in \mathbb{R}}.$$

What system of coordinates are a special case of skew-linear coordinates?

HYPERBOLIC COORDINATES

***Warning:* the equations given here are perhaps nonstandard, these might not be natural coordinates, I think they may be double-valued.**

Hyperbolic coordinates will have hyperbolas playing the same role that circles played for polar coordinates.

$$\begin{aligned} r^2 &= y^2 - x^2, \qquad \text{for } y^2 \geq x^2 \\ r^2 &= x^2 - y^2, \qquad \text{for } y^2 \leq x^2 \\ x\sinh(\gamma) &= y\cosh(\gamma) \end{aligned}$$

Which can be inverted to reveal that:

$$\begin{aligned} x &= r\cosh(\gamma) \\ y &= r\sinh(\gamma) \end{aligned}$$

for $r, \gamma \in \mathbb{R}$. In these coordinates we find that $r = k$ corresponds to the hyperbolas $\boxed{y^2 - x^2 = \pm k}$. On the other hand, if we look at $\gamma = k$ then we find

$$x = r\cosh(k),\ \ y = r\sinh(k) \quad \Longrightarrow \quad \frac{y}{x} = \frac{\sinh(k)}{\cosh(k)} \quad \Longrightarrow \quad \boxed{y = \tanh(k)x}$$

The points along this curve are at constant hyperbolic angle k. You can verify that the boxed equation is an orthogonal trajectory of the hyperbolas just as the rays from the origin are orthogonal trajectories to the circles centered about the origin.

Remark: *I don't expect you understand each and every type of coordinate system I've introduced in this Chapter. Certainly, you are expected to understand the polar coordinates in some depth, but these other examples were by in large an attempt on my part to expand your concept of what a coordinate system can be.There is a geometric supposition that flows throughout. The plane exists independent of the coordinates that describe it. We have seen there are many ways to put coordinates on a plane.*

Coordinate Maps on a Surface (a window on higher math)

In abstract manifold theory we find it necessary to refine our concept of a coordinate mapping to fit the following rather technical prescription. The coordinates discussed in this chapter don't quite make the grade. We realize certain coordinates are not one-one. There are multiple values of the coordinate which map to the same point on the surface. Also, we have not even begun to worry about smoothness. We need tools from calculus III to tackle that question and even then this topic is a bit beyond calculus III.

Let S be a surface. In fact, let S be a two-dimensional manifold. What this means is that there is a family of open sets U_i for $i = 1, 2, \dots n$ which cover $S = \cup_{i=1}^{n} U_i$. For each one of these open sets U_i there exists a one-one mapping $\phi_i : U_i \to \phi_i(U_i) \subseteq \mathbb{R}^2$ which is called a **coordinate map.** These coordinate maps must be *compatible.* This means

$$\phi_i \circ \phi_j^{-1} : V \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$

is a smooth mapping on $\mathbb{R}^2$.

The following picture illustrates the concept: (notation not consistent, in the picture below the "x" and "y" are coordinate maps and ϕ_{xy} is the smooth transition function)