15. PARAMETRIZED CURVES AND GEOMETRY

Parametric or parametrized curves are based on introducing a "parameter" which increases as we imagine travelling along the curve. Any graph can be recast as a parametrized curve however the converse is not true. We can find a single set of parametric equations to describe a circle but no single graph can cover the whole circle.

Often students find parametric curves bizarre. Why should we need to introduce a parameter? Where does the parameter come from? Is there just one parameter? What is a parameter anyway? I'll try to answer those questions, but be patient it may take the whole chapter to really convince you.

Parametric curves describe the motion of physical objects. Also, most questions in calculus find their most general answer in terms of parametric equations. The world of graphs and Cartesian coordinates is actually the special case. If you take calculus III with me you'll notice I focus on the parametric viewpoint. Many fundamental definitions are given in terms of parametric curves, and later surfaces. In contrast, many basic calculus texts tend to focus on graphs in calculus III.

15.1. WHEN GRAPHS FAIL

A circle of radius R centered at the origin is the set of all points which satisfy $x^2+y^2=R^2$. We cannot write the circle as a single graph $y=f(x)$ because the circle fails the vertical line test. We would like to assign both $y=\sqrt{R^2-x^2}$ and $y=-\sqrt{R^2-x^2}$ for each x with $-R \leq x \leq R$. Let me show you the parametric equations for the circle based on polar coordinates. Observe that a circle is simply $r=R$ in polar coordinates. Therefore, if we choose the standard angle as a parameter, then the polar coordinate formulas yield:

$$\boxed{x = R\cos(\theta), \qquad y = R\sin(\theta) \qquad \text{where } 0 \leq \theta \leq 2\pi}$$

These are the parametric equations for the circle. The parameter is θ. How can we check that these work? We substitute the parametric equations into the circle equation and see if they are consistent. Notice that

$$x^2+y^2 = (R\cos(\theta))^2 + (R\sin(\theta))^2 = R^2\big(\cos^2(\theta)+\sin^2(\theta)\big) = R^2$$

Thus we verify that $x^2+y^2=R^2$ when $x=R\cos(\theta) \text{ and } y=R\sin(\theta)$.

What good is this? Notice that we more flexibility in our description now. For example, the boxed equations say the circle is oriented in the counterclockwise direction. As the parameter increases we trace out the circle in that direction. Also, we go just once around the circle. It is quite easy to give parametric equations which go twice around the circle in the clockwise direction:

$$\boxed{x = R\cos(-\theta), \qquad y = R\sin(-\theta) \qquad \text{where } 0 \leq \theta \leq 4\pi}$$

Parametric equations for a curve give the curve a direction. They also allow us to select just a small subset of the curve by simply restricting the allowed values for the parameter. This restriction could be much more complicated in a purely Cartesian coordinate based discussion.

We have much freedom in choosing a parameter. I knew that the standard angle would work before we even checked the equations. I knew that because of what I already know about polar coordinates. There is no reason that the parameter has to be chosen in such a natural way. Almost all of the following are parameterizations of the circle of radius R centered at the origin,

$$\begin{aligned}
&1.)\ x = R\cos(2u+1),\ y = R\sin(2u+1), \quad 0 \leq u \leq \pi \\
&2.)\ x = \frac{R}{\sqrt{1+w^2}},\ y = \frac{Rw}{\sqrt{1+w^2}}, \quad -R \leq w \leq R \\
&3.)\ x = R\cos(t),\ y = R\cos(2t), \quad 0 \leq t \leq 2\pi
\end{aligned}$$

In-class Exercise 15.1.1: show that two of the parametric curves given above are indeed the circle of radius R centered at the origin. Explain why the remaining parametric curve is not a parametrization of the circle.

15.2. PARAMETRIC CURVES

The first section focused on just one example. I should probably stop and give a formal definition before we go further.

Definition 15.2.1: Let $C \subset \mathbb{R}^2$ be some curve in the plane. A parametrization of the curve C is a pair of functions $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ such that $C = \{(f(t), g(t)) \mid t \in I\ \}$. In other words, a parametric curve is a mapping from $I \subseteq \mathbb{R} \to \mathbb{R}^2$ given by the rule $t \mapsto (f(t), g(t))$ for each $t \in I$. We say that t is the parameter and that the parametric equations for the curve are $x = f(t)$ and $y = g(t)$. In the case $I = [a, b]$ we say that $(f(a), g(a))$ is the initial point and $(f(b), g(b))$ is the terminal point.

Finding the parametric equations for a curve does require a certain amount of creativity. However, it's almost always some slight twist on the examples I give in this section.

Example 15.2.1: What curve has parametric equations $x = t,\ y = t^2$ for $t \in \mathbb{R}$? To find Cartesian equation we eliminate the parameter (when possible)

$$y = t^2 = x^2 \implies \boxed{y = x^2.}$$

Example 15.2.2: Find parametric equations to describe the graph $y = \sqrt{x+3}$. We can use $x = t$ and $y = \sqrt{t+3}$ for $t \geq -3$.

Example 15.2.3: Find parametric equations to describe the graph $y = f(x)$. We can use $x = t$ and $y = f(t)$ for $t \in dom(f)$. Equivalently we could say the parameter is x and the parametric equations are $x = x$ and $y = f(x)$ for $x \in dom(f)$. Notice this simple construction show us that every graph can be rewritten as a parametric curve.

Example 15.2.4: What curve has parametric equations $x = t,\ y = t^2$ for $t \in [0,1]$? To find Cartesian equation we eliminate the parameter,

$$y = t^2 = x^2 \implies \boxed{y = x^2.}$$

In contrast to Ex. 15.2.1 we only get a finite segment of the parabola.

Example 15.2.5: What curve has parametric equations $x = \tan^{-1}(t),\ y = [\tan^{-1}(t)]^2$ for $t \in \mathbb{R}$? To find Cartesian equation we eliminate the parameter,

$$y = [\tan^{-1}(t)]^2 \text{ and } x^2 = [\tan^{-1}(t)]^2 \implies \boxed{y = x^2.}$$

You might think we'd get the whole parabola since the parameter ranges over all real numbers, however $\tan^{-1}(t) \to \pm\frac{\pi}{2}$ as $t \to \pm\infty$. This means we only get the portion of the parabola with $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Example 15.2.6 (how to parametrize a line segment): Find the parametric equations of the directed line segment from $(1,2)$ to $(3,5)$. I'll write the solution in vector notation, we want $t = 0$ to give $(1,2)$ while $t = 1$ ought to output $(3,5)$. Notice that $(x,y) = (1-t)(1,2) + t(3,5)$ will satisfy these requirements. Simplifying yields $(x,y) = (1-t+3t, 2(1-t)+5t) = (1+2t, 2+3t)$. I propose that

$$\boxed{x = 1+2t, \quad y = 2+3t, \quad \text{for } t \in [0,1]}$$

Are the parametric equations of the line segment. I hope you could repeat this example for other pairs of points.

Example 15.2.7 (how to parametrize a circle): Consider the circle of radius R centered at the point (h,k). The following are parametric equations of this circle:

$$\boxed{x = h + R\cos(t), \qquad y = k + R\sin(t), \qquad 0 \leq t \leq 2\pi.}$$

You can check that $(x-h)^2 + (y-k)^2 = R^2$.

Example 15.2.8 (how to parametrize part of a horizontally opening hyperbola):
Consider the hyperbola $x^2 - y^2 = R^2$. The parametric equations for the right branch are conveniently written via the hyperbolic sine and cosine functions,

$$\boxed{x = R\cosh(t), \qquad y = R\sinh(t), \qquad t \in \mathbb{R}}$$

Notice that this only covers the right branch because $\cosh(t) = \frac{1}{2}\left(e^t + e^{-t}\right) \geq 1$. To cover the other branch were $x = -\sqrt{R^2 + y^2}$ we can use

$$\boxed{x = -R\cosh(s), \qquad y = R\sinh(s), \qquad s \in \mathbb{R}}$$

Both of the parametrizations above are built from the identity $\cosh^2(t) - \sinh^2(t) = 1$.

Example 15.2.9 (how to parametrize an ellipse):
Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for $a, b > 0$. The parametric equations for the ellipse are as follows, you can think of it as a circle with different radii in the x and y directions,

$$\boxed{x = a\cos(t), \qquad y = b\sin(t), \qquad t \in [0, 2\pi]}$$

This parametrization is built from the knowledge that $\cos^2(t) + \sin^2(t) = 1$.

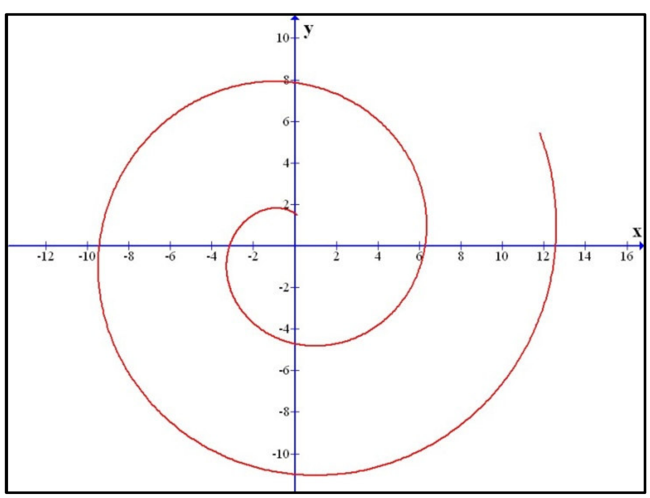
In-class Exercise 15.2.10: Find parametric equations for the right part of the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1.$$

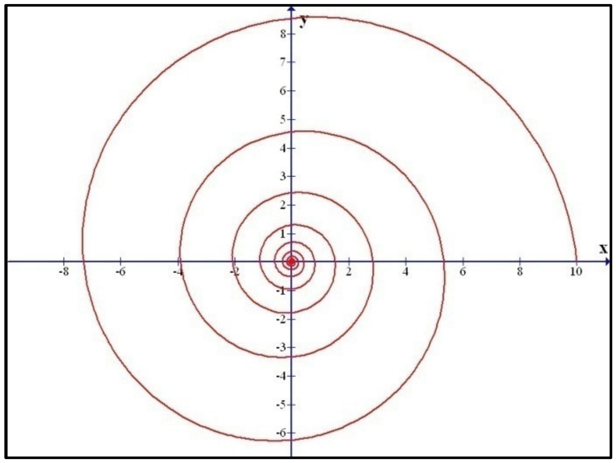
Parametric Curves in Polar Coordinates

Same idea as we have discussed thus far for Cartesian coordinates, except now we need a parametric equation for r, θ.

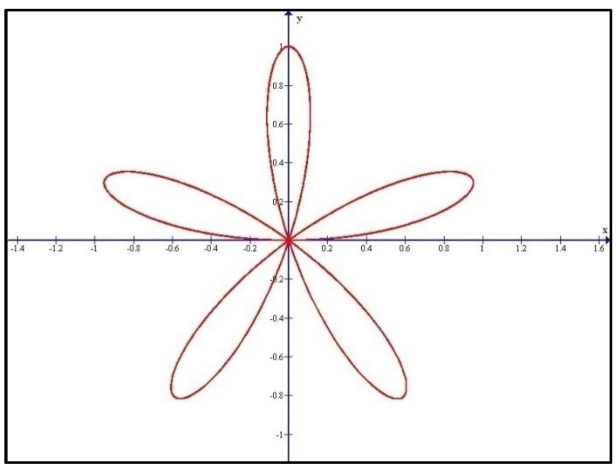
Example 15.2.11: Describe the curve given by the parametric equations $r = t,\ \theta = t$ for $1.5 \leq t \leq 13$. We can make a table of values to helps graph the curve,



Example 15.2.12: Describe the curve given by the parametric equations $r = 10e^{-t/10}, \ \theta = t$ for $t \geq 0$. This is spiral goes inward,



Example 15.2.13: Describe the curve given by the parametric equations $r = \sin(5t), \ \theta = t$ for $t \in \mathbb{R}$. The parametrization covers the pictured regions infinitely many times since it orbits the shape again and again and again…



Remark: it is interesting to study the curves $r = \sin(pt)$, $\theta = t$ for various natural numbers p. You'll find it always gives a flower. When p is odd the number of petals is just p. However, when p is even the number of petals is $2p$. You saw this in the case $p = 2$ we got four petals for the graph of $r = \sin(2\theta), \ \theta = \theta$.

I have drawn a few graphs in these notes. It is likely I'll add pictures in lecture for some of the examples without pictures. Pictures help me organize my thoughts for many examples.

15.3. TANGENTS TO PARAMETRIC CURVES

Consider a graph $y = F(x)$. Suppose we can parametrize this curve via $x = f(t)$ and $y = g(t)$. What can we say about the derivatives of the parametric curves with respect to the parameter verses the derivative of the graph with respect to x ? Consider,

$$\begin{aligned} y = F(x),\ x = f(t),\ y = g(t) \quad &\Longrightarrow \quad g(t) = F(f(t)) \\ &\Longrightarrow \quad \frac{dg}{dt} = \frac{dF}{dx}(f(t))\frac{df}{dt} \\ &\Longrightarrow \quad \frac{dy}{dt} = \frac{dy}{dx}(f(t))\frac{dy}{dt} \\ &\Longrightarrow \quad \frac{dy}{dx}(f(t)) = \frac{\frac{dy}{dt}}{\frac{dy}{dt}} \end{aligned}$$

This means we can calculate the slope of the tangent line to a parametric curve by taking the quotient of the derivatives of the parametrizing functions. In the case that both of the derivatives are zero at the point in question we should understand this calculation in the limit as the parameter approaches the point. This means L'Hopital's rule will be useful in the case $\lim_{t\to a} f'(t) = 0 = \lim_{t\to a} g'(t)$. We should consider the limit as $t \to a$can determine what $\frac{dy}{dx}(f(t))$ tends to as $t \to a$. If this limit tends to infinity then we find the point has a vertical tangent. If $\frac{dy}{dt} \to 0$ and $\frac{dx}{dt}$ does not tend to zero then that point has a horizontal tangent.

Points where both the derivatives tend to zero are somewhat pathological. Generally, if we avoid such points we can say something much nicer. The tangent line to the parametric curve $\vec{r}(t) = (x(t), y(t))$ at the point $\vec{r}(a)$ points in the direction $\frac{d\vec{r}}{dt}(a)$. Moreover, the parametric equations of the tangent line to the curve at $t = a$ are simply,

$$\vec{L}(h) = \vec{r}(a) + h\frac{d\vec{r}}{du}(a)$$

Which is analogous to the tangent line to a curve $L(a + h) = f(a) + hf'(a)$.

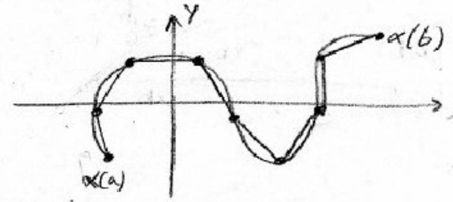
<u>Example 15.3.1:</u>
Suppose that $x = 3\cos(t),\ y = 3\sin(t)$ then $\frac{dx}{dt} = -3\sin(t),\ \frac{dy}{dt} = 3\cos(t)$. Thus the tangent line will point in the $\vec{v} = 3 < -\sin(a), \cos(a) >$ direction for $t = a$. For example, the tangent line when $t = \pi$ points in the $\vec{v} =< 0, -3 >$ direction. In contrast, if we just described the circle as $x^2 + y^2 = 9$ then we would find dy/dx is undefined at the point (-3,0). Parametric equations do not care which direction is x and which is y. The parametric viewpoint puts x and y on an equal footing to start with.

<u>Remark:</u> if you would like for me to cover more of section 11.2 please ask. As it stands we have ignored certain aspects of the discussion in Stewart.

15.4. ARCLENGTH

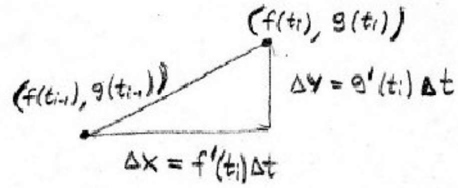
We are given some parametrized curve $\alpha : I \longrightarrow \mathbb{R}^2$ the xy-plane. That is $\alpha(t) = (f(t), g(t)) \quad \forall t \in I \subset \mathbb{R}$.

Lets picture it : Let $I = [a, b]$



if we calculated the length of each line segment and added-em up that would give some estimation of the "arclength"

Divide $[a,b]$ into n - subintervals $[t_{i-1}, t_i]$ with $t_i = a + i\Delta t$ & $\Delta t = \frac{b-a}{n}$ then the length of a particular segment



$$\Delta S_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$
$$= \sqrt{(f'(t_i))^2(\Delta t)^2 + (g'(t_i))^2 \Delta t^2}$$
$$= \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\,\Delta t$$

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta S_i$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i)]^2 + [g'(t_i)]^2}\,\Delta t$$
$$\boxed{S = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\,dt}$$

- Derivation I like $\alpha(t) = (x(t), y(t)) \leftarrow$ a parametrized curve

$$ds^2 = dx^2 + dy^2 = \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)dt^2$$

$$\therefore \quad \boxed{S = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$

Remark when we parametrize by x then $\frac{dx}{dx} = 1$ and

$$\boxed{S = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx}$$

Technically, the last two boxed equation are incorrect, or at least I should say they are incomplete. We must give bounds. The bounds should be given so that they yield the initial and terminal points for the curve.

There is also an arclength function which gives the arclength as a function of the parameter, same idea but varying endpoint"

$$\boxed{s(t) = \int_a^t \sqrt{\frac{dx}{du}^2 + \frac{dy}{du}^2}\, du.}$$

We could use arclength as a parameter for the curve if we wished. That choice is particularly nice, many formulas concerning the geometry of curves find their simplest form for arclength parametrized curves.

Examples 15.4.1 and 15.4.2

$\boxed{E1}$ Let $\alpha(t) = (r\cos t, r\sin t)$ $\quad 0 \le t \le 2\pi$

$x = r\cos t \qquad \frac{dx}{dt} = -r\sin t \qquad \therefore \left(\frac{dx}{dt}\right)^2 = r^2\sin^2 t$

$y = r\sin t \qquad \frac{dy}{dt} = r\cos t \qquad \therefore \left(\frac{dy}{dt}\right)^2 = r^2\cos^2 t$

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt = \int_0^{2\pi} r\, dt = \boxed{2\pi r}$$

Which is good since $x^2 + y^2 = r^2$, is a circle of radius r.

$\boxed{E2}$ Let $\alpha(x) = (x, \ln(\cos(x)))$ $\quad 0 \le x \le \pi/4$ $\quad x = x$, $y = \ln(\cos(x))$

$y = \ln(\cos(x))$ for $0 \le x \le \pi/4$

$$S = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \left(-\frac{\sin(x)}{\cos(x)}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2(x)}\, dx \quad ; \quad \tan^2(x) + 1 = \sec^2(x)$$

$$= \int_0^{\pi/4} \sec(x)\, dx$$

$$= \int_1^{1+\sqrt{2}} \frac{du}{u}$$

$u = \sec(x) + \tan(x)$

$\frac{du}{u} = \sec(x)dx$

$u(0) = \sec(0) + \tan(0) = 1$

$u(\pi/4) = \sec(\pi/4) + \tan(\pi/4) = \sqrt{2} + 1$

$$= \left(\ln|u|\right)\Big|_1^{1+\sqrt{2}}$$

$$= \ln(1+\sqrt{2}) - \ln(1)$$

$$= \boxed{\ln(1+\sqrt{2})}$$

Example 15.4.3: Arclength integrals cannot be solved with elementary functions in many cases. We must resort to a power series approximation or Simpson's rule in order to tackle the problem numerically. The examples which work out nice are somewhat contrived, with the notable exception of the circle.

(149)

E3 Find circumference of an ellipse: $x = a\cos t$, $y = b\sin t$; $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \sin^2 t + \cos^2 t = 1$

$\frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = b\cos t$ where we are given $0 \leq t \leq 2\pi$.

$$S = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$$

$$= \int_0^{2\pi} a\sqrt{\sin^2 t + \left(\frac{b\cos t}{a}\right)^2}\, dt$$

$$= \int_0^{2\pi} a\sqrt{1 - \cos^2 t + \left(\frac{b\cos t}{a}\right)^2}\, dt$$

$$= \int_0^{2\pi} a\sqrt{1 - (1 - (b/a)^2)\cos^2 t}\, dt \quad : \text{define } \beta = \sqrt{|1 - (b/a)^2|}$$

$$= \int_0^{2\pi} a\sqrt{1 - \beta^2\cos^2 t}\, dt$$

This integral is not elementary. We'll need a specific a and b in order to proceed. Note that when $a = b$ then $\beta = 0$ so $S = 2\pi a$ happily. Consider $a = 1$, $b = \sqrt{2}$ then $\beta = 1$

$$\int_0^{2\pi} \sqrt{1 - \cos^2 t}\, dt = \quad (\text{numerically})$$

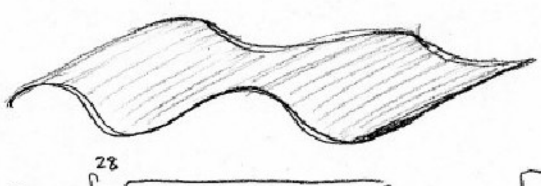
$$= \int_0^{2\pi} |\sin t|\, dt = \int_0^{\pi} \sin t\, dt - \int_{\pi}^{2\pi} \sin t\, dt = 4$$

Example 15.4.4 (find length of aluminum sheet that is picture below)

If a sheet of aluminum is pressed into a curved mold such that its finished form is 28" long with a 2" sinusoidal wave then how long a piece of flat aluminum is needed to construct the wrinkled sheet of aluminum?

E4 Produce panels 28" wide & 2" thick. From the picture the panels have 2 full-waves so $2\lambda = 28''$ and amplitude 2"

$y = \sin(\pi x/7)$ should have $4\pi = \frac{\pi}{7} \cdot 28$ ✓



$$\frac{dy}{dx} = \frac{\pi}{7}\cos\left(\frac{\pi x}{7}\right)$$

$$S = \int_0^{28} \sqrt{1 + \left(\frac{\pi}{7}\cos(\pi x/7)\right)^2}\, dx = \boxed{29.36''}$$

In-class Exercise 15.4.5: Let the curve C be parametrized by $t \in [0, 3]$ where

$$x = e^t + e^{-t}, \qquad y = 5 - 2t.$$

Find the arclength of C. (yes you can do the integral here)

Example 15.4.6:

$\boxed{E6}$ $y = \frac{x^3}{6} + \frac{1}{2x}$ with $\frac{1}{2} \le x \le 1$ find arclength.

$$\frac{dy}{dx} = \frac{x^2}{2} - \frac{1}{2x^2} \quad \therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)$$

$$S = \int_{1/2}^{1} \sqrt{1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)}\,dx = \int_{1/2}^{1} \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\,dx = \int_{1/2}^{1} \sqrt{\frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2}\,dx$$

$$= \int_{1/2}^{1} \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_{1/2}^{1} = \frac{31}{48} = \boxed{0.6458}$$

15.5. SURFACE AREA

In calculus I we calculated the volumes of solids of revolution. We now address the calculation of the surface area of such solids.

Actually, I'll let you read Stewart on this question. His discussions on page 568-573 and 671-672 are a good starting point. There should be a few solved homeworks on the webpage as well.