8. L'HOPITAL'S RULE

In Chapter 3 we were able to resolve many indeterminant limits with purely algebraic arguments. You might have noticed we have not really tried to use calculus to help us solve limits better. In our viewpoint, limits were just something we needed to do in order to carefully define the derivative and integral. However, we were certainly happy enough once those limits vanished and were replaced by a few essentially algebraic rules. Linearity, product, quotient and chain rules all involve a limiting argument if we consider the technical details. The fact that we can do calculus without dwelling on those details is in my view why calculus is so beautifully simple.

In this chapter we will learn about L'Hopital's Rule which allows us to use calculus to resolve limits which are indeterminant. We need to have limits of type $0/0$ or ∞/∞ in order to apply the rule. Often we will need to rewrite the given expression in order to change it to either type $0/0$ or ∞/∞. We will see that $\infty - \infty, 1^\infty, \infty^0, 0^0$ can all be resolved with the help of L'Hopital's Rule.

L'Hopital's Rule says that the limit of an indeterminant quotient of functions should be the same as the limit of of the quotient of the derivatives of those functions. Essentially the idea is to compare how the numerator changes verses the how the denominator changes. This can be done at a finite limit point or with limits at $\pm\infty$.

I will give a proof of the Theorem, but my proof is only for a relative special case. L'Hopital's Rule holds in a context more general than the assumptions for my proof. You should consult a more serious calculus text if you wish to see the details. Ask me if you are interested. (Thomas' Calculus is one good source)

8.1. L'HOPITAL'S RULE

Just a reminder, (look at Section 3.3 for more)

1.) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ is of type $\left(\frac{0}{0}\right)$ if $\lim_{x\to a}(f(x))=0$ AND $\lim_{x\to a}(g(x))=0$

2.) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ is of type $\left(\frac{\infty}{\infty}\right)$ if $\lim_{x\to a}(f(x))=\infty$ AND $\lim_{x\to a}(g(x))=\infty$

We say the same for one-sided limits and for $a=\pm\infty$. Other indeterminant forms like $\infty-\infty$, $0\cdot\infty$, 0^0, 1^∞, ∞^0 are defined similarly

And now for the rule,

L'Hôpital's Rule

If $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ is type $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ and f and g are differentiable near $x=a$ then we have that

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$$

Notation I like to use to clarify the application of L'Hopital's Rule is

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)\underset{(\frac{0}{0})}{\overset{\mathcal{L}}{=}}\lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right) \quad \text{OR} \quad \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)\underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}}\lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$$

Example 8.1.1:

$\boxed{E1}$ Notice $\lim_{x\to 0}(\sin(x))=\sin(0)=0$ and $\lim_{x\to 0}(x)=0$ thus

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)\underset{(\frac{0}{0})}{\overset{\mathcal{L}}{=}}\lim_{x\to 0}\left(\frac{\cos(x)}{1}\right)=\cos(0)=\boxed{1}.$$

We gave a geometric argument to prove this limit in the discussion leading up to the derivatives of sine and cosine. Given that the derivatives of sine and cosine require knowledge of this limit it is not surprising that this limit is trivially reproduced with the help of the derivative of sine and cosine. I used to think this proved this limit, but it is circular since we cannot know the derivative of sine is cosine unless we have already derived this limit. Chicken, Egg, I say the Chicken is the limit.

Example 8.1.2:

$\boxed{E2}$ This one is type $\frac{\infty}{\infty}$, we'll apply L'Hopital's Rule twice,

$$\lim_{x\to\infty}\left(\frac{x^2+x-2}{x^2+3}\right)\underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}}\lim_{x\to\infty}\left(\frac{2x+1}{2x}\right)\underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}}\lim_{x\to\infty}\left(\frac{2}{2}\right)=\boxed{1}$$

Remark: please notice that the rule says to differentiate the numerator and denominator separately. There is no such rule as $\lim f(x) = \lim f'(x)$. Let's see why the rule holds true. The following is a proof of a weak form of L'Hopital's rule. The rule holds even when $f'(a), g'(a)$ do not exist (they might be holes in the graph of the derivatives). You can find the complete technically correct proof in a good calculus text (ask me if interested)

Let $\lim\limits_{x\to a}\left(\frac{f(x)}{g(x)}\right)$ be of type $\left(\frac{0}{0}\right)$ and let $f'(a)$ and $g'(a)$ exist with $g'(a) \neq 0$. We'll prove the rule in this <u>simple</u> case, the general proof can be found in Appendix 6 of Thomas' 10^{th} Ed. of "Calculus". Recall by the very definition of $f'(a)$ and $g'(a)$ we find,

$$f'(a) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right) = \lim_{x\to a}\left(\frac{f(x)}{x-a}\right) - \lim_{x\to a}\left(\frac{f(a)^{\nearrow 0}}{x-a}\right)$$

$$g'(a) = \lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right) = \lim_{x\to a}\left(\frac{g(x)}{x-a}\right) - \lim_{x\to a}\left(\frac{g(a)^{\nearrow 0}}{x-a}\right)$$

Notice in the above it is crucial that $f(a) = 0$ and $g(a) = 0$, so that

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}}\right) = \frac{\lim\limits_{x\to a}\left(\frac{f(x)}{x-a}\right)}{\lim\limits_{x\to a}\left(\frac{g(x)}{x-a}\right)} = \frac{f'(a)}{g'(a)} \quad //$$

Example 8.1.3 and Example 8.1.4:

$\boxed{E3}$

$$\lim_{x\to\infty}\left(\frac{\ln(x^2)}{\sqrt[3]{x}}\right) \underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x\to\infty}\left(\frac{\frac{2}{x}}{\frac{1}{3}x^{-2/3}}\right) = \lim_{x\to\infty}\left(\frac{6x^{2/3}}{x}\right) = \lim_{x\to\infty}\left(\frac{6}{\sqrt[3]{x}}\right) = \boxed{0}$$

$\boxed{E4}$ These ideas also make sense for one-sided limits. For example,

$$\lim_{x\to 0^+}\left(xe^{\frac{1}{x}}\right) = \lim_{x\to 0^+}\left(\frac{e^{\frac{1}{x}}}{\frac{1}{x}}\right)$$

Notice we had to rewrite the intial expression to make it have the form $\left(\frac{\infty}{\infty}\right)$.

$$\underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x\to 0^+}\left(\frac{e^{\frac{1}{x}}\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}}\right)$$

$$= \lim_{x\to 0^+}\left(e^{\frac{1}{x}}\right)$$

$$= e^{\left(\lim\limits_{x\to 0^+}\left(\frac{1}{x}\right)\right)}$$

$$= e^{\infty} = \boxed{\infty}$$

Pattern worth noticing:

Remark: In $\boxed{E4}$ we had type $(0\cdot\infty)$ to begin. We rewrote it so that 0 became $\frac{1}{1/0} = \frac{1}{\infty}$ meaning $(0\cdot\infty) \to (\frac{\infty}{\infty})$ which is what we need for L'Hopital's to apply. Generally

$$\underset{(\text{type } 0\cdot\infty)}{\lim (fg)} \quad \text{with} \quad \begin{matrix}\lim f = 0 \\ \lim g = \infty\end{matrix} \;\leadsto\; \underset{(\text{type } \frac{\infty}{\infty})}{\lim\left(\frac{g}{1/f}\right)}$$

Example 8.1.5:(exponential growth verses polynomial growth)

$\boxed{E5}$

$$\lim_{x\to\infty}\left(e^{-x}x^2\right) = \lim_{x\to\infty}\left(\frac{x^2}{e^x}\right) \underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{x\to\infty}\left(\frac{2x}{e^x}\right) \underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{x\to\infty}\left(\frac{2}{e^x}\right) = \boxed{0}$$

Notice the exponential grows much quicker that x^2 as $x \to \infty$. In fact its easy to see even if x^2 is replaced with x^n we will find the same result. Exponential growth is faster than polynomial growth.

Example 8.1.6: (logarithmic growth verses polynomial growth)

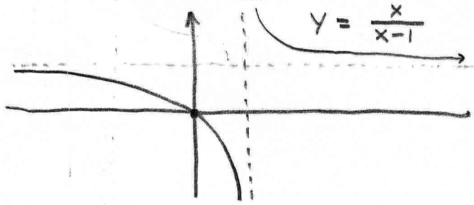
$\boxed{E6}$ Logarithmic growth is slower than polynomial growth,

$$\lim_{x\to\infty}\left(\frac{x^2}{\ln(x)}\right) \underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{x\to\infty}\left(\frac{2x}{\frac{1}{x}}\right) = \lim_{x\to\infty}\left(2x^2\right) = \boxed{\infty}$$

Example 8.1.7: (why is the application of L'Hoptial's Rule not applicable here?)

$$\lim_{x \to 1}\left(\frac{x}{x-1}\right) \stackrel{L}{=} \lim_{x \to 1}\left(\frac{1}{1}\right) = 1. \qquad (\underline{\text{FALSE}}!)$$

<u>WHAT DID I DO WRONG</u> ?



As you can see from graph $\lim\limits_{x \to 1}\left(\frac{x}{x-1}\right)$ d.n.e.

Example 8.1.8: (see Chapter 2 for graphs)

$$\lim_{\theta \to 0^+}\left(\csc(\theta) - \cot(\theta)\right) = \lim_{\theta \to 0^+}\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)$$

$$= \lim_{\theta \to 0^+}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

they had a common denom. it was just hidden by notation.

$$\underset{(\frac{0}{0})}{\stackrel{L}{=}} \lim_{\theta \to 0^+}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$= \lim_{\theta \to 0^+}\left(\tan\theta\right)$$

$$= \tan(0) = \boxed{0}$$

Example 8.1.9:

$$\lim_{x \to 1}\left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) = \lim_{x \to 1}\left(\frac{(x-1) - \ln(x)}{\ln(x)(x-1)}\right)$$

made a common denominator.

$$\underset{(\frac{0}{0})}{\stackrel{L}{=}} \lim_{x \to 1}\left(\frac{1 - \frac{1}{x}}{\frac{1}{x}(x-1) + \ln(x)}\right)$$

: used product rule on denom.

$$= \lim_{x \to 1}\left(\frac{1 - \frac{1}{x}}{1 - \frac{1}{x} + \ln(x)}\right)$$

$$\underset{(\frac{0}{0})}{\stackrel{L}{=}} \lim_{x \to 1}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}}\right)$$

$$= \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

8.2. LIMITS OF INDETERMINANT POWERS

INDETERMINANT POWERS (63)

1.) $\lim f = 0$ and $\lim g = 0$ then $\lim (f)^g$ is type (0^0)

2.) $\lim f = \infty$ and $\lim g = 0$ then $\lim (f)^g$ is type $(\infty)^0$

3.) $\lim f = 1$ and $\lim g = \pm\infty$ then $\lim (f)^g$ is type (1^∞)

To resolve these indeterminancies we'll need to employ a little trick,

$$\boxed{[f(x)]^{g(x)} = e^{\ln\left([f(x)]^{g(x)}\right)} = e^{g(x)\ln(f(x))}}$$

Additionally because the exponential function is continous everywhere we can pass limits inside it,

$$\lim_{x\to a}\left([f(x)]^{g(x)}\right) = \lim_{x\to a}\left(e^{g(x)\ln(f(x))}\right) = e^{\lim_{x\to a}\left(g(x)\ln(f(x))\right)}$$

We use this idea throughout the next few examples, we replace the original problem $\lim_{x\to a}\left([f(x)]^{g(x)}\right)$ with the problem of finding $\lim_{x\to a}\left(g(x)\ln(f(x))\right)$ then what happens is:

1.) $\lim_{x\to a}\left(g(x)\ln(f(x))\right) = b \in \mathbb{R} \Rightarrow \lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a}(g\ln f)} = e^b$

2.) $\lim_{x\to a}\left(g(x)\ln(f(x))\right) = \infty \Rightarrow \lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a}(g\ln f)} = e^\infty = \infty$

3.) $\lim_{x\to a}\left(g(x)\ln(f(x))\right) = -\infty \Rightarrow \lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a}(g\ln f)} = e^{-\infty} = 0$

• Now go work thru the examples, then come back and read this and see if it makes more sense.

(the discussion above and the example below include some commentary by Hannah, circa 2007)

Example 8.2.1:

$\boxed{E10}$ (Type 0^0)

$$\lim_{x\to 0^+}(x^x) = \lim_{x\to 0^+}\left(e^{x\ln(x)}\right) = e^{\lim_{x\to 0^+}(x\ln(x))}$$

$$(*)\ \lim_{x\to 0^+}(x\ln(x)) = \lim_{x\to 0^+}\left(\frac{\ln(x)}{\frac{1}{x}}\right) \underset{(\frac{\infty}{\infty})}{\overset{L}{=}} \lim_{x\to 0^+}\left(\frac{\frac{1}{x}}{\frac{-1}{x^2}}\right) = \lim_{x\to 0^+}(-x) = 0.$$

Therefore using $(*)$ we get $\lim_{x\to 0^+}(x^x) = e^0 = \boxed{1}$

Example 8.2.2:(this is a homework problem)

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = \lim_{n\to\infty}\left(e^{nx\ln(1+\frac{1}{n})}\right)$$
$$= e^{\underbrace{\lim_{n\to\infty}(nx\ln(1+\frac{1}{n}))}_{(*)}}$$

Now we need to calculate $(*)$.

$$\lim_{n\to\infty}\left(nx\ln(1+\tfrac{1}{n})\right) = x\lim_{n\to\infty}\left(\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}\right)$$
$$\underset{(\frac{0}{0})}{\overset{\mathcal{L}}{=}} x\lim_{n\to\infty}\left(\frac{\frac{1}{1+1/n}\left(\frac{-1}{n^2}\right)}{\frac{-1}{n^2}}\right)$$
$$= x\lim_{n\to\infty}\left(\frac{1}{1+\underbrace{\tfrac{1}{n}}_{0}}\right)$$
$$= x$$

Thus we find $\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = e^x}$.

The example above connects the definition of the exponential given in these notes to the other definition which is given in terms of continued multiplications, the formulas in this example appear naturally in certain applications about loans or population growth.

Example 8.2.3:

$\boxed{E12}$ Type (∞^0)

$$\lim_{x\to\infty}\left(x^{\frac{1}{x}}\right) = \lim_{x\to\infty}\left(e^{\frac{1}{x}\ln(x)}\right) = e^{\underbrace{\lim_{x\to\infty}\left(\frac{\ln(x)}{x}\right)}_{*}}$$

Again we must find $*$,

$$\lim_{x\to\infty}\left(\frac{\ln(x)}{x}\right) \underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{1}\right) = 0$$

Therefore,

$$\boxed{\lim_{x\to\infty}\left(x^x\right) = e^0 = 1}$$