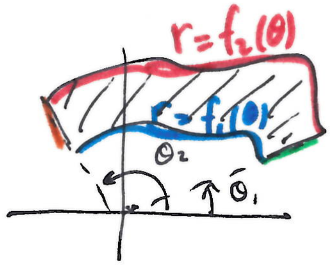
CALCULUS IN POLAR COORDINATES

- BEST EXPLAINED IN CALCULUS III, I let you consult text for reasoning here.

AREA:



If $f_1(\theta) < r < f_2(\theta)$ for $\theta_1 \leq \theta \leq \theta_2$ then the area bounded is given by:

$$\boxed{A = \frac{1}{2}\int_{\theta_1}^{\theta_2} [f_2^2 - f_1^2]\,d\theta}$$

In contrast, the following I can naturally explain here w/o much trouble,

ARCLENGTH: If $C: r = f(\theta)$ for $\theta_1 \leq \theta \leq \theta_2$

then

$$\boxed{S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{where } r = f_1(\theta)}$$

Derivation: if $r = f(\theta)$ then

$$x = f(\theta)\cos\theta \qquad \therefore \qquad \frac{dx}{d\theta} = \frac{df}{d\theta}\cos\theta - f\sin\theta$$

$$y = f(\theta)\sin\theta \qquad\qquad \frac{dy}{d\theta} = \frac{df}{d\theta}\sin\theta + f\cos\theta$$

Thus,

$$dS = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{\left(\frac{df}{d\theta}\cos\theta - f\sin\theta\right)^2 + \left(\frac{df}{d\theta}\sin\theta + f\cos\theta\right)^2}\, d\theta$$

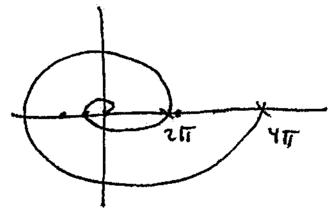
$$= \sqrt{\left(\frac{df}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta) + f^2(\sin^2\theta + \cos^2\theta) + \text{terms which cancel out}}\, d\theta$$

$$= \sqrt{\left(\frac{df}{d\theta}\right)^2 + f^2}\, d\theta$$

and the formula for arclength follows.

E1 If $r = \theta^2$ then for $0 \le \theta \le 4\pi$ we could calculate the arclength of this spiral by

$$S = \int_0^{4\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$



$$= \int_0^{4\pi} \sqrt{\theta^2 + (2\theta)^2}\, d\theta$$

$$= \int_0^{4\pi} \sqrt{5}\, \theta\, d\theta$$

since $\theta \in [0, 4\pi]$ we have $\theta \ge 0$ and $\sqrt{\theta^2} = |\theta| = \theta$.

$$= \sqrt{5}\, \frac{\theta^2}{2} \Big|_0^{4\pi}$$

$$= \boxed{8\pi\sqrt{5}}$$

You can also mix concepts

E2 find slope of tangent line to curve parametrized by $r = t^2$ and $\theta = e^t$ at $t = \ln(\pi)$.

$$x = r\cos\theta = t^2 \cos(e^t)$$
$$y = r\sin\theta = t^2 \sin(e^t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t\cos(e^t) - t^2 e^t \sin(e^t)}{2t\sin(e^t) + t^2 e^t \cos(e^t)}$$

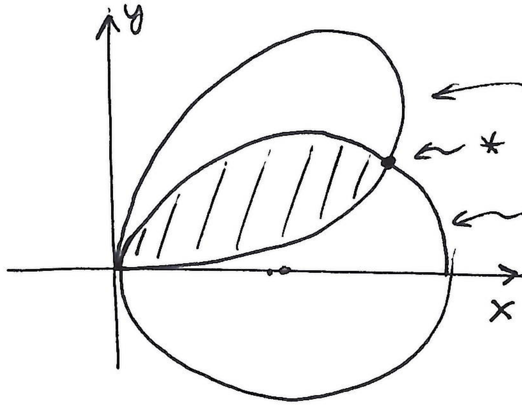
Notice $t = \ln(\pi)$ gives $e^t = e^{\ln\pi} = \pi$ thus $\sin(e^t) = \sin(\pi) = 0$ whereas $\cos(e^t) = \cos(\pi) = -1$

Hence at $t = \ln(\pi)$ we calculate,

$$\frac{dy}{dx} = \frac{2\ln\pi\, \cancel{\cos(\pi)}}{(\ln\pi)^2 e^{\ln\pi}\, \cancel{\cos(\pi)}} = \frac{2}{(\ln\pi)\,\pi}$$ slope of curve at $t = \ln\pi$

$t = \ln\pi$ gives $\left.\begin{array}{l} x = (\ln\pi)^2\cos(\pi) \\ y = (\ln\pi)^2 \sin(\pi) \end{array}\right\}$ $(-(\ln\pi)^2, 0)$ $\leftarrow$ aha

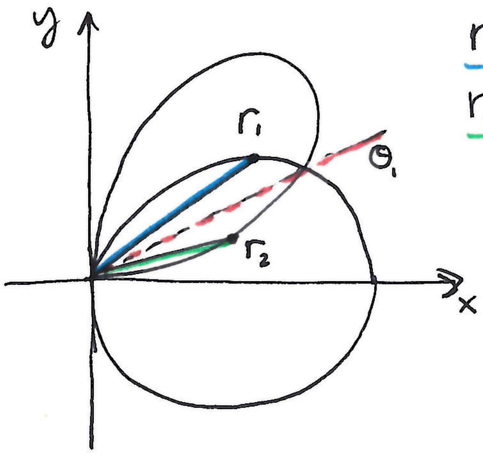
$\boxed{E3}$ Find area bounded by $r = 2\sin(2\theta)$ and $r = 2\cos\theta$ in the 1st quadrant $(x, y > 0)$



← petal of $r = 2\sin(2\theta)$

← $*$

← $r = 2\cos\theta$ is circle centered at $(1,0)$ of radius 1.



$r_1 = 2\cos\theta$

$r_2 = 2\sin 2\theta$

$\Big\}$ outter radius for $\theta > \theta_1$ or $\theta < \theta_1$

to find $*$ at θ_1 we need

$$\underbrace{2\sin(2\theta_1)} = 2\cos\theta_1$$

$$2(2\sin\theta_1\cos\theta_1) = 2\cos\theta_1$$

$$2\sin\theta_1 = 1 \quad \text{provided } \underline{\cos\theta \neq 0}$$

reasonable given plot.

$$\therefore \sin\theta_1 = \frac{1}{2}$$

$$\underline{\theta_1 = \pi/6}.$$

Thus,

$$\text{AREA} = \int_0^{\pi/6} \frac{1}{2}(2\sin(2\theta))^2 d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}(2\cos\theta)^2 d\theta$$

$$= \int_0^{\pi/6} 2\sin^2(2\theta)\,d\theta + \int_{\pi/6}^{\pi/2} 2\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/6} [1 - \cos(4\theta)]\,d\theta + \int_{\pi/6}^{\pi/2} [1 + \cos 2\theta]\,d\theta$$

$$= \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) + \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$= \boxed{\frac{\pi}{2} - \frac{3\sqrt{3}}{8}}$$