Calculus of Parametric Curves:

Given $x = f_1(t)$ and $y = f_2(t)$ parametrize curve C then at point $P_0 = (f_1(t_0), f_2(t_0))$ on C we may (in the case the quotient below exists) calculate the slope of the tangent line to C at $P_0 = \vec{r}(t_0)$ via

$$\boxed{\frac{dy}{dx}(P_0) = \frac{\frac{df_2}{dt}(t_0)}{\frac{df_1}{dt}(t_0)}} \quad (*)$$

$\frac{df_2}{dt} = 0 \therefore$ Horizontal Tangent

$\frac{df_1}{dt} = 0 \therefore$ Vertical Tangent

If we use notation, $x = x(t)$, $y = y(t)$ then $(*)$ reads,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

However, if both are zero then cannot say w/o further analysis

Furthermore, continuing to abuse notation,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

$\boxed{E1}$ $x = t^3 + 3t^2$
$y = t\sin(t)$

$$\frac{dy}{dx} = \frac{\frac{d}{dt}(t\sin t)}{\frac{d}{dt}(t^3+3t^2)} = \frac{\sin t + t\cos t}{3t^2+6t}$$

: $\frac{dy}{dx}$ at the point $(t^3+3t^2, t\sin t)$ of the given path

$$\frac{d^2y}{dx^2} = \frac{1}{\frac{dx}{dt}}\frac{d}{dt}\left[\frac{\sin t + t\cos t}{3t^2+6t}\right]$$

quotient rule!

$$= \underbrace{\frac{1}{3t^2+6t}\left[\frac{(\cos t + \cos t - t\sin t)(3t^2+6t) - (\sin t + t\cos t)[6t+6]}{(3t^2+6t)^2}\right]}_{\frac{d^2y}{dx^2} \text{ at } (3t^2+t^3,\ t\sin t) \text{ on given path}}$$

Could use this to test <u>concavity</u> of path etc.

Comment: can find tangent line as graph or parametrized curve. Can match-up parameter to give "colliding" tangent line ($\underline{\vec{l}(t) = \vec{r}(t_o) + (t-t_o)\vec{r}\,'(t_o)}$ does nicely)
**

E2 Consider $x = \sqrt{2}\cos t$, $y = \sqrt{2}\sin t$ at $t = \pi/4$ find parametric eq^{n}s for colliding tangent line

$$\frac{dy}{dx} = \frac{\frac{d}{dt}(\sqrt{2}\sin t)}{\frac{d}{dt}(\sqrt{2}\cos t)} = \frac{\sqrt{2}\cos t}{-\sqrt{2}\sin t} = -\tan(t)$$

At $t = \pi/4$ find $\frac{dy}{dx} = -\tan(\pi/4) = -1$. The point of tangency is $(\sqrt{2}\cos\frac{\pi}{4}, \sqrt{2}\sin\frac{\pi}{4}) = (1,1)$ hence $\underline{y = 1 - 1(x-1)}$ gives eq$^{\underline{n}}$ of tangent line

$$\vec{l}(t) = \vec{r}(t_o) + (t-t_o)\frac{d\vec{r}}{dt}(t_o) \qquad (** \text{ again})$$

Where $\underbrace{\vec{r} = (x,y)}_{\text{position}}$ and $\underbrace{\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)}_{\text{velocity}}$ btw $\underbrace{\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)}_{\text{acceleration}}$

$$\vec{r} = (\sqrt{2}\cos t, \sqrt{2}\sin t) \qquad \vec{r}(\pi/4) = (1,1)$$

$$\frac{d\vec{r}}{dt} = (-\sqrt{2}\sin t, \sqrt{2}\cos t) \qquad \vec{r}\,'(\pi/4) = (-1,1)$$

Hence using ** as guide, here $t_o = \pi/4$

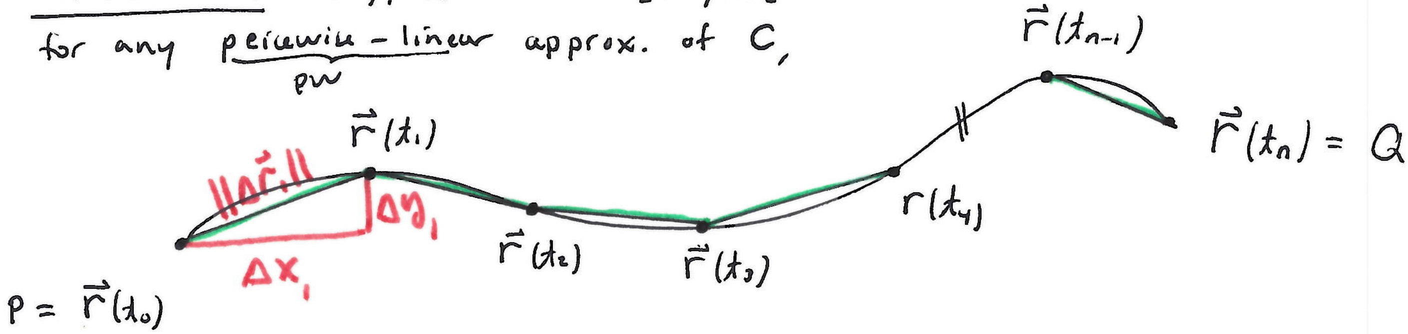
$$\vec{l}(t) = (1,1) + (t-\pi/4)(-1,1)$$

$$\boxed{\vec{l}(t) = (1-(t-\pi/4),\, 1+t-\pi/4)} \longleftrightarrow \boxed{\begin{cases} x = 1 + \frac{\pi}{4} - t \\ y = t - \frac{\pi}{4} + 1 \end{cases}}$$

eq^{n}s of the colliding tangent line.

Remark: the example in the text on cycloids is nice.

ARCLENGTH: suppose $C = \vec{r}\,[t_0, t_n]$ where $a = t_0$, $b = t_n$
for any $\underbrace{\text{piecewise}}_{pw}$-linear approx. of C,



We approximate the length of C with the length of the pw-linear approximation or interpolation of C. Let $\Delta\vec{r}_i = \vec{r}(t_i) - \vec{r}(t_{i-1})$ and notice the length of $\Delta\vec{r} = \langle \Delta x, \Delta y\rangle$ given by $\underbrace{\|\Delta\vec{r}\| = \sqrt{(\Delta x)^2 + (\Delta y)^2}}_{*}$ and

$$\text{pw-line length} = \sum_{i=1}^{n} \|\Delta\vec{r}_i\|$$

However, by Mean Value Th$^{\underline{m}}$ (MVT) for $x(t)$ on $t_{i-1} \le t \le t_i$ we find $\exists\, t_i^* \in [t_{i-1}, t_i]$ such that $\frac{dx}{dt}(t_i^*) = \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} = \frac{\Delta x_i}{\Delta t}$ assuming $t_i - t_{i-1} = \Delta t$ $\forall\, i = 1, 2, \dots, n$. We likewise select $t_i^{**} \in [t_{i-1}, t_i]$ such that $\frac{dy}{dt}(t_i^{**}) = \frac{y(t_i) - y(t_{i-1})}{t_i - t_{i-1}} = \frac{\Delta y_i}{\Delta t}$

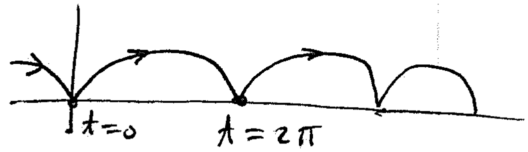
Thus $\Delta\vec{r}_i = \left(\frac{dx}{dt}(t_i^*)\Delta t, \frac{dy}{dt}(t_i^{**})\Delta t \right)$. As $n \to \infty$ we have $\Delta t = \frac{b-a}{n} \to 0$ and our approx. pw-line length yields (factoring Δt out using $*$ to calculate $\|\Delta\vec{r}_i\|$)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{dx}{dt}(t_i^*)\right)^2 + \left(\frac{dy}{dt}(t_i^{**})\right)^2}\,\Delta t$$

But $t_i^*, t_i^{**} \to t_i$ as $n \to \infty$ and $[t_i - \Delta t, t_i] \to \{t_i\}$ and we identify the above is the arclength,

$$\text{arclength of } C = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

[E3] $\left.\begin{array}{l} x = r(\theta - \sin\theta) \\ y = r(1-\cos\theta) \end{array}\right\}$ parametrizes cycloid



The length on one arch given by,

$$\text{Arc Length} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{[r(1-\cos\theta)]^2 + [r(\sin\theta)]^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2 - 2r\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2r^2 - 2r^2\cos\theta}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

: Recall $\sin^2\beta = \frac{1}{2}(1-\cos(2\beta))$
thus $4\sin^2\beta = 2(1-\cos 2\beta)$
or $4\sin^2(\theta/2) = 2(1-\cos\theta)$
Setting $\theta = 2\beta$ so $\beta = \theta/2$

$$= r\int_0^{2\pi} \sqrt{4\sin^2(\theta/2)}\, d\theta$$

$$= 2r\int_0^{2\pi} \sin(\theta/2)\, d\theta$$

$$= 2r\left(\frac{-1}{\frac{1}{2}}\cos\left(\frac{\theta}{2}\right)\Big|_0^{2\pi}\right)$$

$$= 2r\left[2\cos(0) - 2\cos(\pi)\right]$$

$$= \boxed{8r}$$

<u>Remark</u>: we've shown $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ in the argument on the last page. Essentially to find net S simply add up (integrate) ds. This can be used in conjunction with other geometrically motivated applications of ds such as $dS = 2\pi y\, ds$ for surface area of surface of revolution... (see your text)

E4 $x = R\cos\theta$

$y = R\sin\theta$

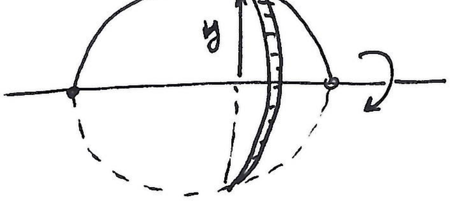
Thus $\dfrac{dx}{d\theta} = -R\sin\theta$ and $\dfrac{dy}{d\theta} = R\cos\theta$ hence (assume $R > 0$)

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \sqrt{(-R\sin\theta)^2 + (R\cos\theta)^2}\, d\theta = R\, d\theta$$

and $$s = \int_0^{2\pi} R\, d\theta = \boxed{2\pi R}$$ ← arclength of circle of radius R.

E5 Find surface area of sphere radius R. Rotate about x-axis

$$\left.\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \end{aligned}\right\} \quad 0 \le \theta \le \pi$$



$ds = R\,d\theta$ (just like E4)

$$dS = 2\pi y\, ds = 2\pi R\sin\theta\, R\, d\theta$$

$$S = \int_0^{\pi} 2\pi R^2 \sin\theta\, d\theta = 2\pi R^2 (-\cos\theta)\Big|_0^{\pi}$$

$$= 2\pi R^2(-\cos\pi + \cos 0)$$

$$= \boxed{4\pi R^2}$$ surface area of sphere.

PHYSICS CORNER:

$\vec{r}(t) = (x(t), y(t))$: position

$\vec{V}(t) = \dfrac{d\vec{r}}{dt} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right)$: velocity

$\vec{a}(t) = \dfrac{d\vec{V}}{dt} = \left(\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}\right)$: acceleration

} ALL VECTORS

$$\vec{V} = \underbrace{V}_{\text{magnitude}}\ \underbrace{\hat{V}}_{\text{unit-vector}}$$

However, the magnitude of velocity $\|\vec{V}\| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2} = V$

So, we see speed $\boxed{V = \dfrac{ds}{dt}}$ to find arclength we integrate ds which means integrating $V\,dt$.