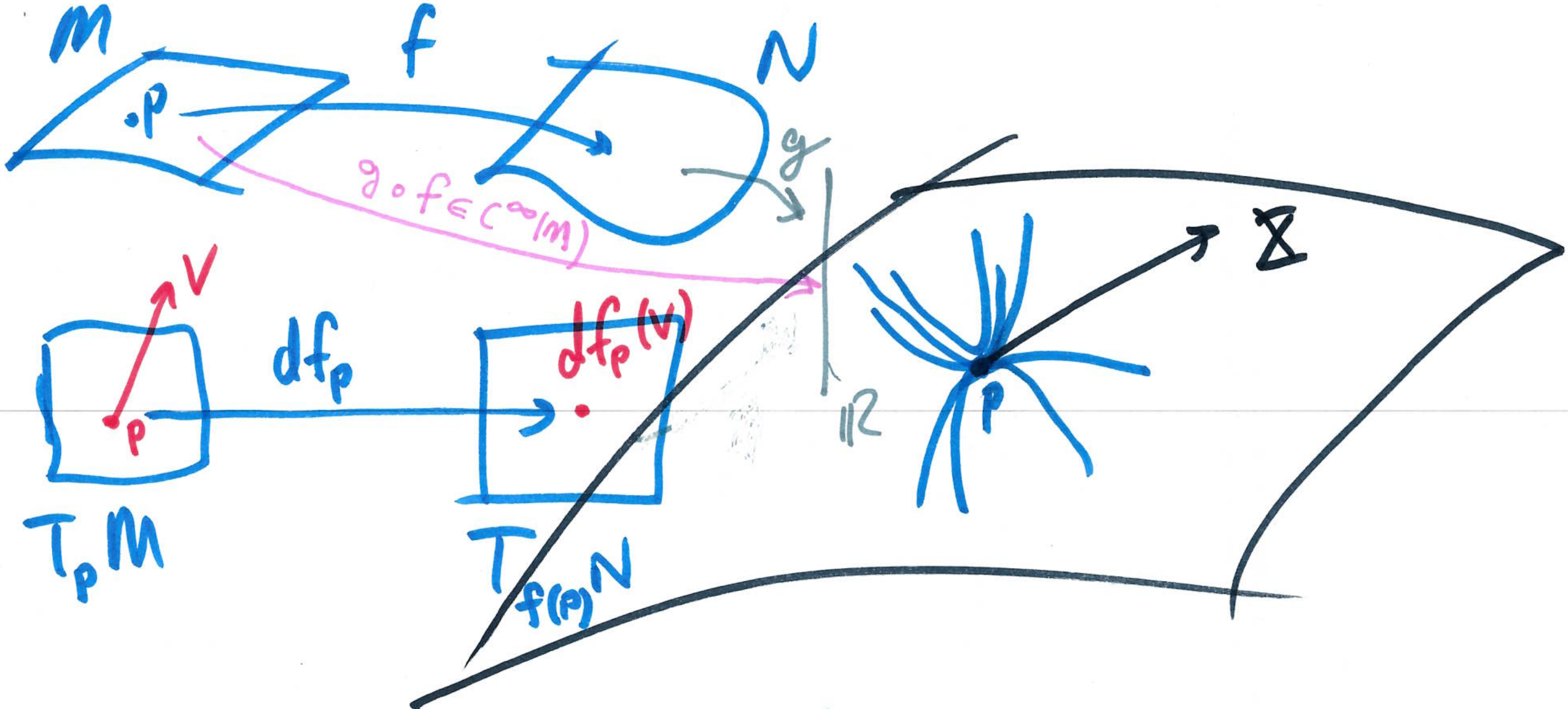


$[\gamma] \longleftrightarrow X$

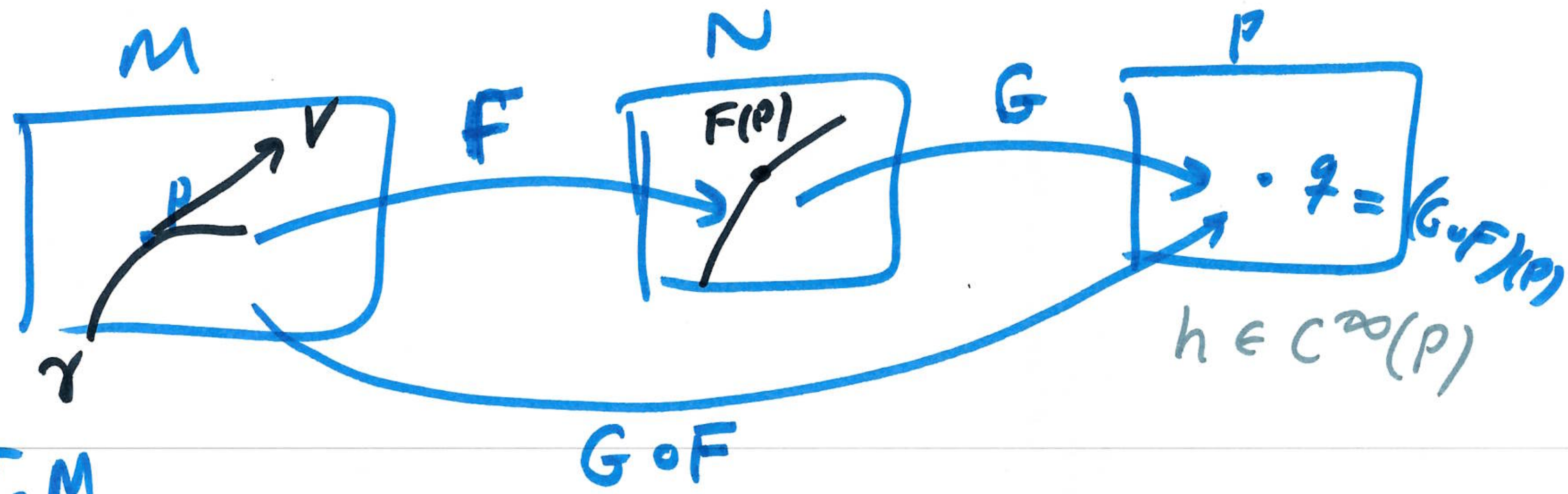
$df_p(v) \in \underbrace{T_{f(p)}N}_{\text{derivation on } C^\infty N}$

$v \in T_pM$

$v : C^\infty M \to \mathbb{R}$

$(df_p(v))(g) = v(g \circ f)$

$\gamma'(0) = v$



$h \in C^\infty(P)$

$\underline{v \in T_p M}$

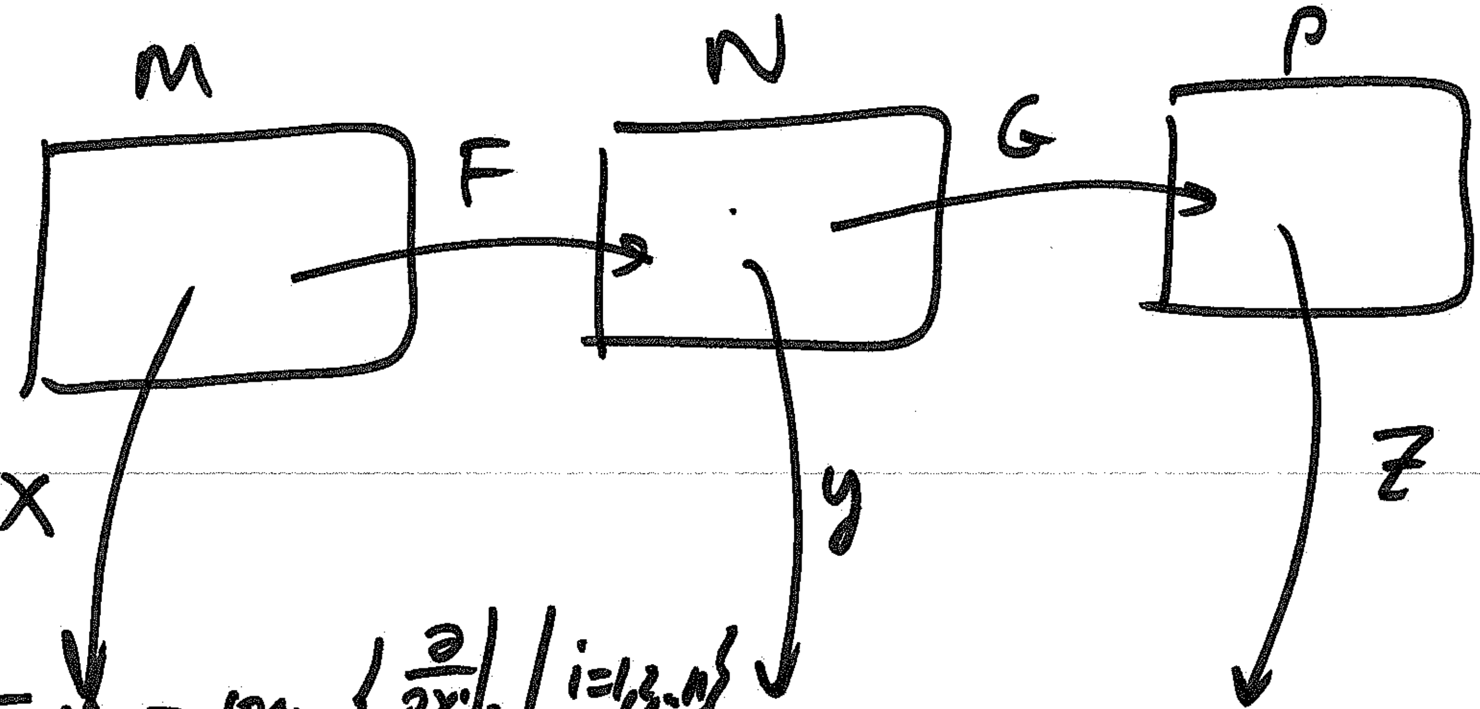
$\big(d(G \circ F)_p(v)\big)(h) = v\big(h \circ (G \circ F)\big)$

$\underbrace{\big(dG_{F(p)} \circ dF_p\big)(v)(h) = \big(dG_{F(p)}\big(dF_p(v)\big)\big)(h)}$

$dF_p([\gamma]) = [F \circ \gamma]$

$d(G \circ F)_p([\gamma]) = [G \circ F \circ \gamma]$

$dG_{F(p)}[F \circ \gamma] = [G \circ (F \circ \gamma)]$



$$\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM = \mathrm{span}\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \,\middle|\, i=1,2,\dots,n \right\}$$

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \sum_{j=1}^{n} \frac{\partial (y^j \circ F)}{\partial x^i} \left.\frac{\partial}{\partial y^j}\right|_{F(p)}$$

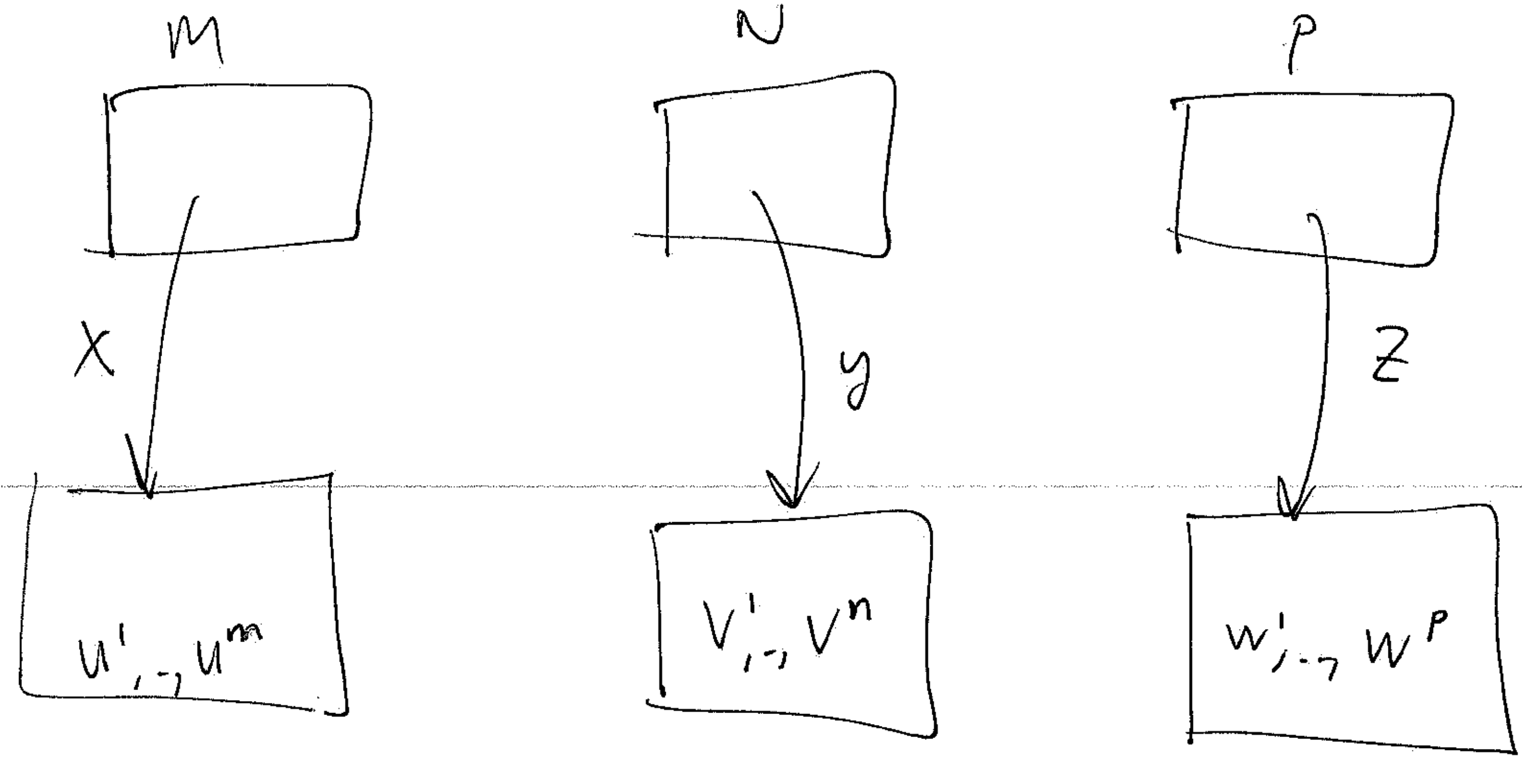
$$dG_q\left(\left.\frac{\partial}{\partial y^j}\right|_q\right) = \sum_{k=1}^{p} \frac{\partial (z^k \circ G)}{\partial y^j} \left.\frac{\partial}{\partial z^k}\right|_{G(q)}$$

$G \circ F : M \longrightarrow N \longrightarrow P$

$$d(G \circ F)_p \left(\frac{\partial}{\partial x^i}\Big|_p \right) = \sum_{k=1}^{p} \frac{\partial (z^k \circ (G \circ F))}{\partial x^i} \frac{\partial}{\partial z^k}\Big|_{(G \circ F)(p)} \quad \mathrm{I}$$

$$\begin{aligned}
(dG \circ dF)\left(\frac{\partial}{\partial x^i}\Big|_p \right) &= dG\left(dF \left(\frac{\partial}{\partial x^i}\Big|_p \right)\right) \\
&= dG \left(\sum_{j=1}^{n} \frac{\partial (y^j \circ F)}{\partial x^i} \frac{\partial}{\partial y^j}\Big|_{F(p)} \right) \\
&= \sum_{j=1}^{n} \frac{\partial (y^j \circ F)}{\partial x^i} \, dG \left(\frac{\partial}{\partial y^j}\Big|_{F(p)} \right) \\
&= \sum_{j=1}^{n} \frac{\partial (y^j \circ F)}{\partial x^i} \sum_{k=1}^{p} \frac{\partial (z^k \circ G)}{\partial y^j} \frac{\partial}{\partial z^k}\Big|_{G(F(p))} \quad \mathrm{II}
\end{aligned}$$

Suffices to show $\displaystyle \frac{\partial (z^k \circ (G \circ F))}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial (y^j \circ F)}{\partial x^i} \frac{\partial (z^k \circ G)}{\partial y^j}$



m
n
p
x
y
z
u', .., u^m
v', .., v^n
w', .., w^p

$$\left.\frac{\partial f}{\partial x^i}\right|_p = \frac{\partial}{\partial u^i}\left[\left(f \circ x^{-1}\right)(u)\right]\Big|_{u=x(p)} \qquad f = z^k \circ (G \circ F)$$

$$= \frac{\partial}{\partial u^i}\left[\left(z^k \circ G \circ F \circ x^{-1}\right)(u)\right]\Big|_{u=x(p)}$$

$$= \left.\frac{\partial}{\partial u^i}\right|_{u=x(p)}\left[z^k \circ G(v)\right]\Big|_{v=(F\circ x^{-1})(u)} \qquad z^k \circ G$$

$$= \left.\frac{\partial}{\partial u^i}\right|_{u=x(p)}\left[\left(z^k \circ G \circ y^{-1}\right)(v)\right]\Big|_{v=(y\circ F\circ x^{-1})(u)}$$

$$= \sum_{j=1}^{n} \frac{\partial\left(z^k \circ G \circ y^{-1}\right)}{\partial v^j}\,\frac{\partial v^j}{\partial u^i} \qquad z^k \circ G \circ y^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$= \sum_{j=1}^{n} \frac{\partial\left(z^k \circ G \circ y^{-1}\right)(v)}{\partial v^j}\,\frac{\partial}{\partial u^i}\left[\left(y^j \circ F \circ x^{-1}\right)(u)\right]$$

$$= \sum_{j=1}^{n} \frac{\partial\left(z^k \circ G\right)}{\partial y^j}\,\frac{\partial\left(y^j \circ F\right)}{\partial x^i} = \frac{\partial\left(z^k \circ (G \circ F)\right)}{\partial x^i}$$