Examples of Comparison Tests

Th$^{\underline{m}}$/ Direct Comparison Test (DCT)

Suppose $\sum a_n$ & $\sum b_n$ are series with $a_n, b_n > 0$

(i.) if $\sum b_n$ is convergent <u>and</u> $a_n \leq b_n$ then $\sum a_n$ is convergent

(ii.) if $\sum b_n$ is divergent <u>and</u> $a_n \geq b_n$ then $\sum a_n$ is divergent.

1.) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{10+3^n}$ is a series with positive $a_n = \dfrac{1}{10+3^n}$

and $\dfrac{1}{10+3^n} < \dfrac{1}{3^n}$. Note $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-1/3}$ by

geometric series. Thus by D.C.T. we find

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{10+3^n}$ converges.

<u>Remark</u>: in fact, in case (i.) $\sum a_n \leq \sum b_n$

so we also obtain some estimate of $\sum a_n$. In

1.) notice $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-1/3} = \frac{3}{2}$ so $\displaystyle\sum_{n=0}^{\infty} \frac{1}{10+3^n} < \frac{3}{2}$.

2.) $\displaystyle\sum_{n=3}^{\infty} \frac{n}{n^2-4}$. Observe $\dfrac{n}{n^2-4} > 0$ for $n \geq 3$.

Also, $\dfrac{n}{n^2-4} > \dfrac{n}{n^2} = \dfrac{1}{n}$ and since $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ diverges as

it is ~~the~~ a tail of the $p=1$ series we find

$\displaystyle\sum_{n=3}^{\infty} \frac{n}{n^2-4}$ diverges by the D.C.T.

Th$^{\underline{m}}$/ LIMIT COMPARISON TEST (L.C.T.)

Suppose $\sum_1 a_n$ & $\sum_1 b_n$ are series with positive terms

If $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = c$ where c is finite # with $c > 0$

then either both series converge or both diverge.

3.) $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ since $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$ could use D.C.T.

But, I'll use L.C.T against $p = 1/2$ series

$$\lim_{n\to\infty}\left(\frac{\frac{1}{\sqrt{n}-1}}{\frac{1}{\sqrt{n}}}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{n}}{\sqrt{n}-1}\right) = \lim_{n\to\infty}\left(\frac{1}{1-\underbrace{1/\sqrt{n}}_{0}}\right) = 1.$$

Thus $\sum\limits_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$ diverges by L.C.T. with $c=1$

to the $p = 1/2$ divergent series.

4.) $S = \sum\limits_{n=1}^{\infty} \frac{1}{n} e^{\frac{1}{n}}$ is nicely compared to divergent $p=1$ series,

$$\lim_{n\to\infty}\left(\frac{\frac{1}{n}e^{\frac{1}{n}}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(e^{\frac{1}{n}}\right) = e^0 = 1 > 0$$

Thus S diverges by L.C.T. against $p=1$ series.

5.) $S = \sum\limits_{n=1}^{\infty}\frac{2}{\sqrt{n}+2}$ observe $\lim\limits_{n\to\infty}\left(\frac{1/\sqrt{n}}{2/(\sqrt{n}+2)}\right) = \lim\limits_{n\to\infty}\left(\frac{\sqrt{n}+2}{2\sqrt{n}}\right) = \frac{1}{2}$

thus S diverges by L.C.T. with $p = 1/2$ series compared to S.

6.) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n} = \underbrace{1 + \frac{1}{4} + \frac{1}{27} + \frac{1}{4^4} + \frac{1}{5^5} + \cdots}_{\text{when I compare against } p=2 \text{ it appears this converges.}} < 1 + \frac{1}{4} + \frac{1}{9} + \cdots$

For $n \geq 2$ notice $\dfrac{1}{n^n} < \dfrac{1}{n^2}$ since $n^n > n^2$

thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges by D.C.T. with $p=2$ series.

Alternatively:

$$\lim_{n\to\infty}\left(\frac{1/n^n}{1/n^2}\right) = \lim_{n\to\infty}\left(n^{2-n}\right)$$

$$= \exp\left[\lim_{n\to\infty}\left(\ln\left(n^{2-n}\right)\right)\right]$$

$$= \exp\left(\lim_{n\to\infty}\left[\frac{\ln(n)}{\frac{1}{2-n}}\right]\right)$$

extend. n continuously, L-Hop. rule

$$= \exp\left(\lim_{n\to\infty}\left[\frac{\frac{1}{n}}{\frac{1}{(2-n)^2}}\right]\right)$$

$$= \exp\left(\underbrace{\lim_{n\to\infty}\left[\frac{(2-n)^2}{n}\right]}_{\infty}\right) = \infty$$

oh well, looks like L.C.T. not working for $p=2$ choice!

Instead, try L.C.T. with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\frac{\frac{1}{2^n}}{\frac{1}{n^n}} = \frac{n^n}{2^n} = \left(\frac{n}{2}\right)^n = \exp\left(n\ln\left(n/2\right)\right)$$

$$= \exp\left(\frac{\ln\left(n/2\right)}{1/n}\right) \longrightarrow \exp\left(\frac{1/n}{-1/n^2}\right) \to 0$$

still no success, sometimes D.C.T. is better choice ☺

7.) $$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \underbrace{\frac{1}{24} + \frac{1}{120} + \cdots}_{*}$$

compare to

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \underbrace{\frac{1}{25} + \frac{1}{36} + \cdots}_{**}$$

Observe $n^2 < n!$ for $n \geq 4$ (easy to see, harder to prove?)

(this could be proved via induction perhaps,

Note $n = 4$, $16 < 4! = 4 \cdot 3 \cdot 2 = 24$. Assume

$n^2 < n!$ for some $n \geq 4$.

$$(n+1)! = (n+1)n! > (n+1)\underline{\underline{n^2}} > (n+1)\underline{\underline{(n+1)}} = (n+1)^2$$

Notice, for $n \geq 4$, $n^2 > n+1$ is clear since

$n^2 > n+1 \iff n^2 - n - 1 > 0 \iff \underbrace{\left(n - \frac{1}{2}\right)^2 - \frac{3}{4} > 0}_{\text{true for } n \geq 4.}$

Thus $n^2 < n! \Rightarrow (n+1)^2 < (n+1)!$

and we conclude $n^2 < n!$ for all $n \geq 4$.

Observe $\sum_{n=4}^{\infty} \frac{1}{n^2}$ converges by $p = 2$ series

and thus $\sum_{n=4}^{\infty} \frac{1}{n!}$ converges by D.C.T. since $\frac{1}{n!} < \frac{1}{n^2}$

Thus it follows $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.