Intensive in Complex Analysis

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Complex Numbers

- Complex numbers $z = x + iy$ and $w = a + ib$ multiply subject the usual laws of algebra paired with the unusual identity $i^2 = -1$,

$$zw = (a + ib)(x + iy) = ax - by + i(ay + bx).$$

We define **complex conjugation** by $\overline{x + iy} = x - iy$. Also, the **length** of $x + iy$ is denoted $|x + iy|$ and since $z\overline{z} = x^2 + y^2$ we see that $|z| = \sqrt{z\overline{z}}$. It is a simple exercise to verify that $\overline{zw} = \overline{z}\,\overline{w}$ and hence $|zw| = |z||w|$. In words, the length of the product of complex numbers is simply the product of their lengths.

- Let $\theta \in \mathbb{R}$ and define the **imaginary exponential** denoted $e^{i\theta}$ by:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

For $z \neq 0$, if $z = |z|e^{i\theta}$ then we say $|z|e^{i\theta}$ is a **polar form** of z. Given $z = |z|e^{i\theta}$ and $w = |w|e^{i\beta}$ their product is given by $zw = |z||w|e^{i(\theta+\beta)}$. Complex multiplication can be understood in terms of rotations and dilations.

Complex Functions and Multivalued Functions

The following are single-valued assignments from $\mathbb{C}$ to $\mathbb{C}$, that is **functions**

- **principal argument** of $z \neq 0$ is $Arg(z) \in (-\pi, \pi]$ where $z = |z| exp\left(iArg(z)\right)$.
- **principal n-th root** of $z \neq 0$ is $\sqrt[n]{z} = \sqrt[n]{|z|}e^{iArg(z)/n}$.
- **complex exponential** of $z = x + iy$ is $exp(z) = e^x e^{iy} = e^x \cos y + ie^x \sin y$
- **principal logarithm** of $z \neq 0$ is $Log(z) = \ln|z| + iArg(z)$
- **complex power function** for $z \in \mathbb{C}^-$, $z^c = exp(cLog(z))$
- **trigonometric functions** $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$
- **hyperbolic functions** $\cosh z = \frac{1}{2}(e^z + e^{-z})$ and $\sinh z = \frac{1}{2}(e^z - e^{-z})$

The following are multiply-valued maps,

- **n-th roots** of $z \neq 0$ are $z^{1/n} = \sqrt[n]{|z|}e^{iArg(z)/n}\{1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}\}$ where $\omega_n = exp(2\pi i/n)$ is the **principal n-th root of unity**.
- **arguments** of $z \neq 0$ are $arg(z) = Arg(z) + 2\pi\mathbb{Z}$
- **logarithms** of $z \neq 0$ are $log(z) = Log(z) + 2\pi i\mathbb{Z}$

Polynomials and Complex Algebra

- A **polynomial** of degree n over $\mathbb{C}$ is given by $p(z) = a_n z^n + \cdots + a_1 z + a_0$ where $a_n \neq 0$ and $a_n, \dots, a_0 \in \mathbb{C}$ are **coefficients of** $p(z)$. We say $\mathbb{C}[z]$ is the set of all complex polynomials. The **Fundamental Theorem of Algebra** states that every non-constant polynomial over $\mathbb{C}$ has a zero. It follows that every non-constant polynomial over $\mathbb{C}$ can be factored into a product of linear factors, possibly repeated.
- **Example:** consider $p(z) = z^4 + 16i$. Notice $z^4 + 16i = 0$ yields $z^4 = -16i$. Observe $-16i = 16exp(-i\pi/2)$ gives $\sqrt[4]{-16i} = 2exp(-i\pi/8)$ and as $\omega_4 = exp(2\pi i/4) = \cos\pi/2 + i\sin(\pi/2) = i$ we find $z \in (-16i)^{1/4} = 2exp(-i\pi/8)\{1, i, i^2, i^3\}$. Thus,
$$p(z) = (z - 2e^{-i\pi/8})(z - 2e^{3i\pi/8})(z - 2e^{7i\pi/8})(z - 2e^{11i\pi/8}).$$
- $(-1)^{1/5} = \{e^{i\pi/5}, e^{3\pi i/5}, e^{5\pi i/5}, e^{7\pi i/5}, e^{9\pi i/5}\}$
$(-1)^{1/5} = \{e^{i\pi/5}, e^{3\pi i/5}, -1, e^{-3\pi i/5}, e^{-\pi i/5}\}$.
- Go to the roots of these calculations! Group the operations. Classify them according to their complexities rather than their appearances! This, I believe, is the mission of future mathematicians. This is the road on which I am embarking in this work.

Differential Calculus on $\mathbb{C}$

- If $\lim_{z \to z_o} \dfrac{f(z) - f(z_o)}{z - z_o}$ exists then we say f is **complex differentiable at** $\mathbf{z}_o$ and we denote $f'(z_o) = \lim_{z \to z_o} \dfrac{f(z) - f(z_o)}{z - z_o}$.

- Rules of Calculus over $\mathbb{C}$ are not suprising:

$$\frac{d}{dz}(f+g) = \frac{df}{dz} + \frac{dg}{dz}, \qquad \frac{d}{dz}(cf) = c\frac{df}{dz}, \qquad \frac{d}{dz}\big(f(w)\big) = \frac{df}{dw}\frac{dw}{dz}$$

$$\frac{d}{dz}\left(z^n\right) = nz^{n-1}, \quad \frac{d}{dz}\sin z = \cos z, \quad \frac{d}{dz}e^z = e^z, \quad \frac{d}{dz}Log(z) = \frac{1}{z}$$

- Four viewpoints to analyze differentiability; the difference quotient, Caratheodory criterion, Cauchy Riemann Eqns, and the Wirtinger Calculus.
  - If $f = u + iv$ has $\partial_x u = \partial_y v$ and $\partial_x v = -\partial_y u$ (CR-eqns)
  - Let $\partial_z f = \frac{1}{2}(\partial_x f - i\partial_y f)$ and $\partial_{\bar{z}} f = \frac{1}{2}(\partial_x f + i\partial_y f)$. Then $\partial_{\bar{z}} f = 0$ is CR-eqns and for such a function $f'(z) = \partial_z f$.

Contrasting Holomorphic and Analytic Functions

- We say $f \in \mathcal{O}(U)$ which is to say f is **holomorphic** on U if f is complex differentiable on the domain U.
  - **Example:** $\mathbb{C}^-$ is the slit-complex plane, it is the complex plane with the origin and negative real axis deleted. It is crucial to delete these points from the domain of $Log(z)$ if we wish for the identity $\frac{d}{dz}Log(z) = \frac{1}{z}$ to hold true. Note $Log(z) \in \mathcal{O}(\mathbb{C}^-)$ however $Log(z) \notin \mathcal{O}(\mathbb{C})$ for reasons we will soon appreciate.
- A function $f(z)$ is **analytic** on $D_R(z_o) = \{z \in \mathbb{C} \mid |z - z_o| < R\}$ if there exist coefficients $a_k \in \mathbb{C}$ such that $f(z) = \sum_{k=0}^{\infty} a_k(z - z_o)^k$ for all $z \in D_R(z_o)$. A function is said to be **entire** if it is analytic on $\mathbb{C}$
  - **Example:** sine, cosine, cosh, sinh, the exponential, polynomials and products thereof are all entire functions. The power series you learn in Calculus II equally well apply here. For example,
$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ & $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$
- Suppose $f(z)$ is analytic at z_o, with power series expansion centered at z_o; $f(z) = \sum_{k=0}^{\infty} a_k(z - z_o)^k$. The radius of convergence of the power series is the largest number R such that $f(z)$ extends to be holomorphic on the disk $\{z \in \mathbb{C} \mid |z - z_o| < R\}$

Integral Calculus on $\mathbb{C}$

- Given a path $\gamma : [a,b] \to \mathbb{C}$ and complex function $f = u + iv$ we define

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\frac{d\gamma}{dt}\,dt.$$

  Or, as a complex combination of real line-integrals:

$$\int_\gamma f(z)\,dz = \int_\gamma udx - vdy + i\int_\gamma udy + vdx.$$

- Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the unit-circle $\gamma(t) = e^{it}$. Calculate $\int_\gamma \frac{dz}{z}$. Note, if $z = e^{it}$ then $dz = ie^{it}dt$ hence:

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}dt}{e^{it}} = i\int_0^{2\pi} dt = 2\pi i.$$

- Similarly, we can show $\int_C (z - z_0)^n dz = 2\pi i \delta_{n,-1}$ where C is the CCW-oriented circle $|z - z_0| = R$. (see next slide)

Sample Calculation of Complex Integral

Let $z = z_o + Re^{it}$ for $0 \leq t \leq 2\pi$ parametrize $|z - z_o| = R$. Note $dz = iRe^{it}dt$ hence

$$\begin{aligned}
\int_{|z-z_o|=R} (z - z_o)^m \, dz &= \int_0^{2\pi} (Re^{it})^m iRe^{it} dt \\
&= iR^{m+1} \int_0^{2\pi} e^{i(m+1)t} \, dt \\
&= iR^{m+1} \int_0^{2\pi} \big(\cos[(m+1)t] + i\sin[(m+1)t]\big) dt.
\end{aligned}$$

The integral of any integer multiple of periods of trigonometric functions is trivial. However, in the case $m = -1$ the calculation reduces to $\int_{|z-z_o|=R} (z - z_o)^{-1} \, dz = i \int_0^{2\pi} \cos(0) dt = 2\pi i$

Complex FTC II:

Suppose $F' = f$ and $\gamma : [t_1, t_2] \to \mathbb{C}$ is a path from A to B in a domain D. recall the complex derivative can be cast as a partial derivative with respect to x or y in the following sense: $\frac{dF}{dz} = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$,

$$\begin{aligned}
\int_\gamma f(z)\,dz = \int_\gamma \frac{dF}{dz}\,dz &= \int_\gamma \frac{dF}{dz}\,dx + i\int_\gamma \frac{dF}{dz}\,dy \\
&= \int_\gamma \frac{\partial F}{\partial x}\,dx + i\int_\gamma -i\frac{\partial F}{\partial y}\,dy \\
&= \int_\gamma \left(\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy\right) \\
&= \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}\right)\frac{d\gamma}{dt}dt \\
&= \int_{t_1}^{t_2} \frac{d}{dt}\left[F(\gamma(t))\right]dt \\
&= F(\gamma(t_2)) - F(\gamma(t_1)) \\
&= F(B) - F(A).
\end{aligned}$$

Complex Integral Theorems:

- (FTC I) Let D be star-shaped and let $f(z)$ be holomorphic on D. Then $f(z)$ has a primitive on D and the primitive is unique up to an additive constant. A primitive for $f(z)$ is given by

$$F(z) = \int_A^z f(\zeta)\, d\zeta$$

where A is a star-center of D and the integral is taken along some path in D from A to z.

- The following are equivalent:
  - If $f(z)$ is holomorphic and continuously differentiable on D and C_1 and C_2 are two coterminal paths in D then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.
  - If C is a simple closed curve whose interior is within D then $\int_C f(z)dz = 0$.
  - If C_{in} and C_{out} are two CCW oriented loops which bound an annulus where f is continuously differentiable and holomorphic then $\int_{C_{in}} f(z)dz = \int_{C_{out}} f(z)dz$.

Cauchy's Theorem:

If $f(z)$ is holomorphic and continuously differentiable on D and extends continuously to ∂D then $\int_{\partial D} f(z)\, dz = 0$. Here ∂D is the oriented boundary of D where inner boundaries are oriented CW whereas outer boundaries are oriented CCW[1]

Proof: If $f \in \mathcal{O}(U)$ then the Cauchy Riemann equations give $\partial_y f = i\partial_x f$. Apply Green's Theorem,

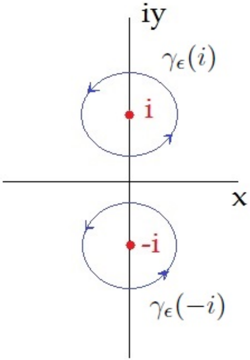
$$\int_{\partial D} f(z) dz = \int_{\partial D} f dx + i f dy = \int_D (\partial_x(if) - \partial_y f) dA = \int_D (i\partial_x f - \partial_y f) dA = 0. \square$$

[1]CW means clockwise and CCW means counterclockwise. If you imagine yourself a tiny person walking the boundary then if you walk in the positively oriented direction then the interior of the space will be on your left. Or you could imagine D as being huge and yourself as a giant, still, the interior is on the left if you walk the positively oriented boundary.

Example illustrating Cauchy's Theorem

$$\int_{\gamma_\epsilon(i)} \left(\frac{dz}{z+i} + \frac{dz}{z-i} \right) = \int_{\gamma_\epsilon(i)} \left(d[\log(z+i)] + \frac{dz}{z-i} \right) = \int_{\gamma_\epsilon(i)} \frac{dz}{z-i} = 2\pi i$$

is a slick notation to indicate the use of an appropriate branch of $\log(z+i)$. In particular, $\text{Log}_{-\pi/2}(z+i)$ is appropriate for $\epsilon < 1$.



Cauchy's (Generalized) Integral Formula

- **Cauchy's Integral Formula** $(m = 0)$**:** let D be a bounded domain with piecewise smooth boundary ∂D. If $f(z)$ is holomorphic with continuous $f'(z)$ on D and $f(z), f'(z)$ extend continuously to ∂D then for each $z \in D$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z}\, dw$$

- We can formally derive the higher-order formulae by differentiation:

$$f'(z) = \frac{1}{2\pi i} \frac{d}{dz} \int_{\partial D} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_{\partial D} \frac{d}{dz} \left[\frac{f(w)}{w-z} \right] dw = \frac{1!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^2}\, dw$$

Differentiate once more,

$$f''(z) = \frac{1}{2\pi i} \frac{d}{dz} \int_{\partial D} \frac{f(w)}{(w-z)^2}\, dw = \frac{1}{2\pi i} \int_{\partial D} \frac{d}{dz} \left[\frac{f(w)}{(w-z)^2} \right] dw = \frac{2!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^3}$$

continuing, we would arrive at:

$$\boxed{f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}\, dw}$$

which is known as **Cauchy's generalized integral formula**.

Example of Cauchy's (Generalized) Integral Formula

- For reference, $\int_{\partial D} \frac{f(z)}{(z-z_o)^{m+1}}\,dz = \frac{2\pi i f^{(m)}(z_o)}{m!}$.

- Let the integral below be taken over the CCW-oriented curve $|z| = 1$:

$$\oint_{|z|=2} \frac{\sin(2z)}{(z-i)^6}\,dz = \frac{2\pi i}{5!}\frac{d^5}{dz^5}\bigg|_{z=i}\sin(2z)$$
$$= \frac{2\pi i}{5\cdot 4\cdot 3\cdot 2}(-32\cos(2i))$$
$$= \frac{-8\pi i\cosh(2)}{15}.$$

Extended Taylor's Theorem

Suppose $f(z)$ is holomorphic for $|z - z_o| < \rho$. Then $f(z)$ is represented by the power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_o)^k, \qquad |z - z_o| < \rho,$$

where for $k \geq 0$,

$$a_k = \frac{f^{(k)}(z_o)}{k!}, \quad \text{(standard result from Calc. II)}$$

and where the power series has radius of convergence $R \geq \rho$. For any fixed r, $0 < r < \rho$, we have

$$a_k = \frac{1}{2\pi i} \oint_{|w - z_o| = r} \frac{f(w)}{(w - z_o)^{k+1}} \, dw, \qquad k \geq 0.$$

(the red part is not available outside of complex analysis)

Proof of Extended Taylor's Theorem

Proof: assume $f(z)$ is as stated in the theorem. Let $z \in \mathbb{C}$ such that $|z| < r < \rho$. Suppose $|w| = r$ then by the geometric series

$$\frac{f(w)}{w-z} = \frac{f(w)}{w}\frac{1}{1-z/w} = \frac{f(w)}{w}\sum_{k=0}^{\infty}\left(\frac{z}{w}\right)^k = \sum_{k=0}^{\infty} f(w)\frac{z^k}{w^{k+1}}.$$

Moreover, we are given the convergence of the above series is uniform for $|w| = r$. This allows us to expand Cauchy's Integral formula into the integral of a series of holomorphic functions which converges uniformly. It follows we are free to exchange the order of the integration and the infinite summation in what follows:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|w|=r}\frac{f(w)}{w-z}\,dw \\ &= \frac{1}{2\pi i}\int_{|w|=r}\left(\sum_{k=0}^{\infty} f(w)\frac{z^k}{w^{k+1}}\right)dw \\ &= \sum_{k=0}^{\infty}\underbrace{\left(\frac{1}{2\pi i}\int_{|w|=r}\frac{f(w)}{w^{k+1}}\,dw\right)}_{a_k} z^k. \end{aligned}$$

Laurent Series

The previous slide concerned a function which was defined at the point of expansion, what is fascinating is there are expansions where despite the function not being defined at the center, we still obtain a Laurent Series representing the function on an annulus about such an isolated singular point.

Laurent Series Decomposition: Suppose $0 \leq \rho < \sigma \leq \infty$, and suppose $f(z)$ is analytic for $\rho < |z - z_o| < \sigma$. Then $f(z)$ can be decomposed as a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

where the coefficients a_n are given by:

$$a_n = \frac{1}{2\pi i} \int_{|z-z_o|=r} \frac{f(z)}{(z-z_o)^{n+1}} \, dz$$

for $r > 0$ with $\rho < r < \sigma$.

Definition of Singularities

- A function f has an **isolated singularity at** z_o if there exists $r > 0$ such that f is analytic on the punctured disk $0 < |z - z_o| < r$.

- Suppose f has an isolated singularity at z_o,
  - If $f(z) = \sum_{k=0}^{\infty} a_k (z - z_o)^k$ then z_o is a **removable singularity**,
  - Let $N \in \mathbb{N}$. If $f(z) = \sum_{k=-N}^{\infty} a_k (z - z_o)^k$ with $a_{-N} \neq 0$ then z_o is a **pole of order** N,
  - If $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_o)^k$ where $a_k \neq 0$ for infinitely many $k < 0$ then z_o is an **essential singularity**.

Behavior of Singularities

- **Riemann's Theorem on Removable Singularities:** let z_o be an isolated singularity of $f(z)$. If $f(z)$ is bounded near z_o then $f(z)$ has a removable singularity.

- Let z_o be an isolated singularity of f. Then z_o is a pole of f of order $N \geq 1$ iff $|f(z)| \to \infty$ as $z \to z_o$.

- **Casorati-Weierstrauss Theorem:** Let z_o be an essential isolated singularity of $f(z)$. Then for every complex number w_o, there is a sequence $z_n \to z_o$ such that $f(z_n) \to w_o$ as $n \to \infty$.

Pole of Order N

- Suppose f has a pole of order N at z_o. If

$$f(z) = \frac{a_{-N}}{(z-z_o)^N} + \cdots + \frac{a_{-1}}{z-z_o} + \sum_{k=0}^{\infty} a_k (z-z_o)^k$$

  then $P(z) = \frac{a_{-N}}{(z-z_o)^N} + \cdots + \frac{a_{-1}}{z-z_o}$ is the **principal part of $f(z)$ about z_o**. When $N=1$ then z_o is called a **simple pole**, when $N=2$ then z_o is called a **double pole**.
- Let z_o be an isolated singularity of f. Then z_o is a pole of f of order N iff $f(z) = g(z)/(z-z_o)^N$ where g is analytic at z_o with $g(z_o) \neq 0$.
- **Example:** Consider $f(z) = \frac{e^z}{(z-1)^5}$. Notice e^z is analytic on $\mathbb{C}$ thus the function f has a pole of order $N=5$ at $z_o = 1$.
- Let z_o be an isolated singularity of f. Then z_o is a pole of f of order N iff $1/f$ is analytic at z_o with a zero of order N.

Residue Definition and Calculation

- Suppose $f(z)$ has an isolated singularity z_o and Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

for $0 < |z - z_o| < \rho$ then we define the **residue of f at z_o** by

$$\text{Res}\left[f(z), z_o\right] = a_{-1}.$$

- **Example Calculation:**

$$f(z) = \frac{1+z}{z^4 - 3z^3 + 3z^2 - z} = -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}$$

By inspection of the above partial fractal decomposition we find:

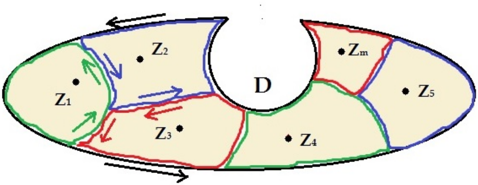
$$\text{Res}\left[f(z), 0\right] = -1 \qquad \& \qquad \text{Res}\left[f(z), 1\right] = 1.$$

Cauchy's Residue Theorem

Let D be a bounded domain in the complex plane with a piecewise smooth boundary ∂D. Suppose that f is analytic on $D \cup \partial D$, except for a finite number of isolated singularities $z_1, \dots, z_m$ in D. Then

$$\int_{\partial D} f(z)\, dz = 2\pi i \sum_{j=1}^{m} \text{Res}\left[f(z), z_j\right].$$

Proof: We simply partition D into m simply connected regions such that each one contains just one singular point. The net-integration only gives the boundary as the cross-cuts cancel. The picture below easily generalizes for $m > 3$.



Rules for Residues

- **Rule 1:** if $f(z)$ has a simple pole at z_o, then

$$\text{Res}\left[f(z), z_o\right] = \lim_{z \to z_o} (z - z_o) f(z).$$

- **Rule 2:** if $f(z)$ has a double pole at z_o, then

$$\text{Res}\left[f(z), z_o\right] = \lim_{z \to z_o} \frac{d}{dz}\left[(z - z_o)^2 f(z)\right].$$

- **Rule 3:** If $f, g \in \mathcal{O}(z_o)$, and if g has a simple zero at z_o, then

$$\text{Res}\left[\frac{f(z)}{g(z)}, z_o\right] = \frac{f(z_o)}{g'(z_o)}.$$

- **Rule 4:** if $g(z)$ has a simple pole at z_o, then

$$\text{Res}\left[\frac{1}{g(z)}, z_o\right] = \frac{1}{g'(z_o)}.$$

Contour Integral Example

Consider $f(z) = \frac{1}{z^4+1}$ has singularities $\{e^{i\pi/4}, ie^{i\pi/4}, -e^{i\pi/4}, -ie^{i\pi/4}\}$. Only two of these fall in the upper-half plane. Thus,

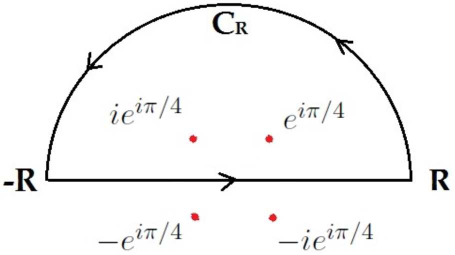
$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^4+1} = 2\pi i\text{Res}\left[\frac{1}{z^4+1}, e^{i\pi/4}\right] + 2\pi i\text{Res}\left[\frac{1}{z^4+1}, ie^{i\pi/4}\right].$$

$$= \frac{2\pi i}{4z^3}\bigg|_{e^{i\pi/4}} + \frac{2\pi i}{4z^3}\bigg|_{ie^{i\pi/4}}$$

$$= \frac{2\pi i}{4e^{i3\pi/4}} + \frac{2\pi i}{4i^3e^{3i\pi/4}}$$

$$= \frac{\pi}{2e^{i3\pi/4}}\left[i + \frac{i}{i^3}\right] = \frac{-\pi}{2e^{i3\pi/4}}[1-i] = \frac{-\pi}{2e^{i3\pi/4}}\sqrt{2}e^{-i\pi/4} = \frac{\pi}{\sqrt{2}}.$$

Hence $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$.



Trigonometric Integral Example

If $z = e^{i\theta} = \cos\theta + i\sin\theta$ then $\bar{z} = e^{-i\theta} = \cos\theta - i\sin\theta$ hence $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ and $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. Of course, we've known these from earlier in the course. But, we also can see these as:

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \& \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

moreover, $dz = ie^{i\theta}d\theta$ hence $d\theta = dz/iz$ for the unit-circle.
Notice $2z^2 + 5iz - 2 = (2z+i)(z+2i) = 2(z+i/2)(z+2i)$ is zero for $z_o = -i/2$ or $z_1 = -2i$. Only z_o falls inside $|z| = 1$ therefore, by Cauchy's Residue Theorem,

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} &= \int_{|z|=1} \frac{dz}{2z^2 + 5iz - 2} \\
&= 2\pi i \,\text{Res}\left[\frac{1}{2z^2 + 5iz - 2}, -i/2\right] \\
&= (2\pi i)\frac{1}{4z + 5i}\bigg|_{z=-i/2} \\
&= \frac{2\pi i}{-2i + 5i} = \frac{2\pi}{3}.
\end{aligned}$$

Story Time

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