Constant Acceleration:

If $\frac{d\vec{a}}{dt} = 0$ for $t_1 \le t \le t_2$ then $\vec{a}(t) = \langle a_x, a_y, a_z \rangle$ is a <u>constant</u> vector on the time interval $[t_1, t_2]$.

Th^m/ If $\frac{d\vec{a}}{dt} = 0$ and $\vec{a}_o = \langle a_x, a_y, a_z \rangle$ for all $t \in [t_o, t_1]$ then the position $\vec{r}(t)$ satisfies the <u>vector equation</u>

$$\vec{r}(t) = \vec{r}_o + (t - t_o)\vec{V}_o + \frac{1}{2}(t - t_o)^2 \vec{a}_o \qquad (\text{for } t_o \le t \le t_1)$$

where $\vec{r}_o = \langle x_o, y_o, z_o \rangle = \vec{r}(t_o)$ and $\vec{V}_o = \langle V_{ox}, V_{oy}, V_{oz} \rangle = \vec{V}(t_o)$. Moreover, we have <u>scalar equations</u> for $t_o \le t \le t_1$,

$$x(t) = x_o + (t - t_o)V_{ox} + \frac{1}{2}(t - t_o)^2 a_x$$

$$y(t) = y_o + (t - t_o)V_{oy} + \frac{1}{2}(t - t_o)^2 a_y$$

$$z(t) = z_o + (t - t_o)V_{oz} + \frac{1}{2}(t - t_o)^2 a_z$$

<u>Proof</u>: we need to use the FTC a few times

$$\frac{d\vec{V}}{dt} = \vec{a} \Rightarrow \int_{t_o}^{t} \frac{d\vec{V}}{d\bar{t}}\, d\bar{t} = \int_{t_o}^{t} \vec{a}(\bar{t})\, d\bar{t} = \vec{a}_o \int_{t_o}^{t} d\bar{t}$$

see pg. 9 for Calculus justification

$$\Rightarrow \vec{V}(t) - \vec{V}(t_o) = (t - t_o)\vec{a}_o$$

$$\Rightarrow \vec{V}(t) = \vec{V}(t_o) + (t - t_o)\vec{a}_o$$

$$\Rightarrow \underline{\vec{V}(t) = \vec{V}_o + (t - t_o)\vec{a}_o}.$$

Likewise,

$$\frac{d\vec{r}}{dt} = \vec{V}(t) \Rightarrow \int_{t_o}^{t} \frac{d\vec{r}}{d\bar{t}}\, d\bar{t} = \int_{t_o}^{t} \left(\vec{V}_o + (\bar{t} - t_o)\vec{a}_o\right) d\bar{t}$$

$$\Rightarrow \vec{r}(t) - \vec{r}(t_o) = \vec{V}_o \int_{t_o}^{t} d\bar{t} + \vec{a}_o \int_{t_o}^{t} (\bar{t} - t_o)\, d\bar{t}$$

$$\Rightarrow \underline{\vec{r}(t) = \vec{r}_o + (t - t_o)\vec{V}_o + \frac{1}{2}(t - t_o)^2 \vec{a}_o}.$$

The scalar eq^n's follow quickly from the formula above. Also, notice we have <u>derived</u> eq^n's for velocity

$\boxed{\begin{array}{l} \text{Th}^{m}/ \text{ Given the hypotheses of the preceding Th}^{m}, \\ \quad \vec{V}(t) = \vec{V}_0 + (t - t_0)\vec{a}_0 \\ \text{Moreover,} \\ \quad V_x(t) = V_{0x} + (t-t_0)a_x \\ \quad V_y(t) = V_{0y} + (t-t_0)a_y \\ \quad V_z(t) = V_{0z} + (t-t_0)a_z \end{array}}$

Example: GRAVITY studying the motion of a mass m close to the surface of the earth and if we neglect friction, rotation of the earth etc... then Newton's 2^{nd} Law reads $\vec{F}_g = \langle 0, -mg \rangle = m\vec{a} = m\langle a_x, a_y \rangle$

or $F_g = \langle 0, 0, -mg \rangle = m\vec{a} = \langle a_x, a_y, a_z \rangle$.

Usually our problem considers motion confined to a particular plane so we just use the two dimensional notation.

different choices of coordinates.

$$\vec{F}_g = \langle 0, -mg \rangle = m\langle a_x, a_y \rangle$$

$$\Rightarrow \vec{a} = \langle 0, -g \rangle \quad \underline{\text{constant acceleration}}.$$

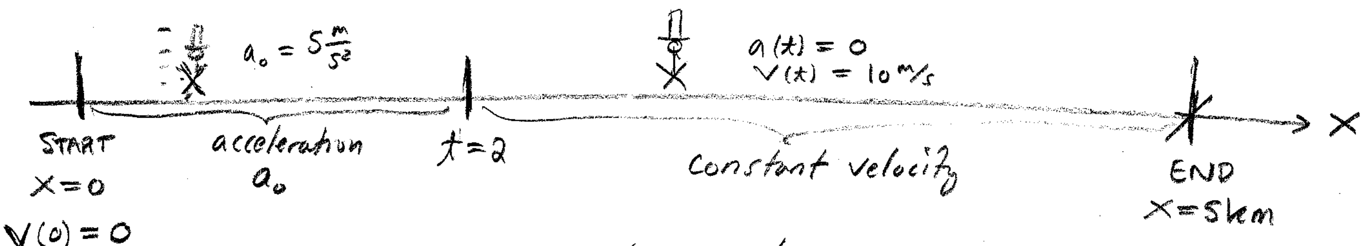
Thus,

$$\boxed{\begin{aligned} V_x(t) &= V_{0x} \\ V_y(t) &= V_{0y} - g(t-t_0) \\ x(t) &= x_0 + V_{0x}(t-t_0) \\ y(t) &= y_0 + V_{0y}(t-t_0) - \tfrac{1}{2}g(t-t_0)^2 \end{aligned}}$$

where $x_0 = x(t_0)$ and $y_0 = y(t_0)$.
$V_{0x} = V_x(t_0)$ and $V_{0y} = V_y(t_0)$.

<u>Remark</u>: in lecture I often take $t_0 = 0$ for convenience, but if you're working a problem with several parts linked together it's important to basically reset the time on each sub-problem. These eq^{n}'s help.

Example: Suppose a runner sprints for 2 seconds at a constant acceleration of $a_0 = 5 m/s^2$. Then the runner finishes the race at constant velocity. If the race is 5km then what was the time of the runner?



For the first 2 seconds we have,

$$V(t) = a_0 t \quad \text{and} \quad x(t) = \tfrac{1}{2} a_0 t^2 \quad \text{for} \quad 0 \leq t \leq 2s$$

Notice $V(2s) = \left(5\frac{m}{s^2}\right)(2s) = 10\frac{m}{s}$ and $x(2) = \frac{1}{2}\left(5\frac{m}{s^2}\right)(4s^2) = 10m$.

Thus,

$$\begin{cases} V(t) = 10\frac{m}{s} & \text{for} \quad 2s \leq t \leq t_{RACE} = t_R \\ x(t) = 10m + (t - 2s)10\frac{m}{s} & \text{for} \quad 2s \leq t \leq t_R \end{cases}$$

Notice that the position and velocity at $t = 2$ give the initial conditions for the motion during times $2s \leq t \leq t_R$. We wish to determine t_R, recall we're given $x(t_R) = 5km$.

$$x(t_R) = 10m + (t_R - 2s)10\frac{m}{s} = 5000m$$

$$\Rightarrow 4990s = 10(t_R - 2s)$$

$$\Rightarrow 499s = t_R - 2s$$

$$\therefore \boxed{t_R = 497s}$$

Remark: if you wish you may state from the outset that you are omitting units in the calculations. However, for laboratory calculations <u>and</u> the final answer units must be made explicit (for full credit)

SPECIAL FORMULAS FOR CONSTANT ACCERATION

Thm/ For one-dimensional motion with position s, velocity $v = \frac{ds}{dt}$ and acceleration $a = \frac{dv}{dt}$ if $a = a_o$ is constant then

$$V_f^2 = V_i^2 + 2a_o(S_f - S_i)$$

where $V(t_f) = V_f$ and $V(t_i) = V_i$ and $S(t_f) = S_f$ and $S(t_i) = S_i$

__Proof:__ for one dimensional motion we can write either $V = V(t)$ or we could write $V = V(s)$ (velocity as function of position) The chain-rule from calculus yields,

$$\frac{dV}{dt} = \frac{ds}{dt}\frac{dV}{ds} = V\frac{dV}{ds}$$

Hence $a_o = V\frac{dV}{ds} \implies a_o\, ds = V dV$ then __integrate__

$$\int_{S_i}^{S_f} a_o\, ds = \int_{V_i}^{V_f} V dV$$

(let $\Delta S = S_f - S_i$)

$$a_o(S_f - S_i) = \frac{1}{2}(V_f^2 - V_i^2) \implies \boxed{V_f^2 = V_i^2 + 2a_o\Delta S}$$

__Remark:__ this formula is useful for problems where we either don't know or don't care about the time.

__Example:__ A coin thrown directly downward with a velocity of $V = -5\,m/s$. What is it's velocity as it hits the ground 20 m below? (on __EARTH__)

$$V_f^2 = V_i^2 + 2a_o(y_f - y_i)$$

← helps me follow your work if you write w/o #'s to start. (partial credit more likely)

$$= (-5\tfrac{m}{s})^2 - 2(9.8\tfrac{m}{s^2})(-20m)$$

$$= (25 + 39.2)\frac{m^2}{s^2}$$

$$= 64.2\, m^2/s^2$$

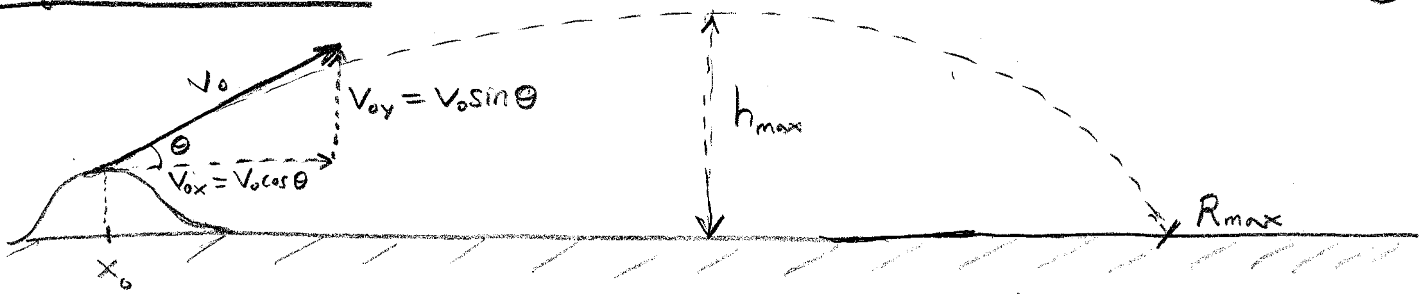
$$\implies V_f = \pm\sqrt{64.2\frac{m^2}{s^2}}$$

$$\implies \boxed{V_f = -8.01\,\frac{m}{s}}$$

Given our __choice__ of coordinates, ($V_f < 0$ since motion downward)

__Remark:__ $V_f^2 = V_o^2 - 2g\Delta y$, $g = 9.8\, m/s^2$

Projectile Motion



The equations below model the trajectory of a projectile under the influence of $\vec{F}_g = -mg\hat{j}$. This is an approximation which ignores friction, rotation of earth and the higher order contributions in the non constant force of gravity. I take $t_o = 0$ in what follows,

$$\boxed{\begin{aligned} x(t) &= x_o + (V_o \cos\theta)t \\ y(t) &= y_o + (V_o \sin\theta)t - \tfrac{1}{2}gt^2 \end{aligned}}$$

Th^m/ The eq^s above give parabola with vertex $\left(x_o + \dfrac{V_o^2 \sin(2\theta)}{2g},\ y_o + \dfrac{V_o^2 \sin^2\theta}{2g}\right)$

Proof: eliminate t. Note $t = \dfrac{x - x_o}{V_o \cos\theta}$ and substitute into y formula,

$$y - y_o = (V_o \sin\theta)\left(\frac{x - x_o}{V_o \cos\theta}\right) - \frac{1}{2}g\left(\frac{x - x_o}{V_o \cos\theta}\right)^2$$

$$\Rightarrow\ y - y_o = (\tan\theta)(x - x_o) - \frac{g}{2V_o^2\cos^2\theta}(x - x_o)^2$$

$$\Rightarrow\ \underline{y = y_o + \tan\theta\,(x - x_o) - \frac{g}{2V_o^2 \cos^2\theta}(x - x_o)^2}$$

Now I'll attempt to write it in vertex form. Need to complete the square to write $y = A(x-h)^2 + k$. Substitute, $u = x - x_o$ then $y = y_o + \tan\theta\, u - \dfrac{g}{2V_o^2\cos^2\theta}u^2$

$$\underbrace{-\frac{2V_o^2\cos^2\theta}{g}(y - y_o)}_{☺} = u^2 - \frac{2V_o^2\cos^2\theta\tan\theta}{g}u$$

$$☺ = \left(u - \frac{V_o^2\cos\theta\sin\theta}{g}\right)^2 - \left[\frac{V_o^2\cos\theta\sin\theta}{g}\right]^2$$

Hence, undoing the $u = x - x_o$ substitution we find,

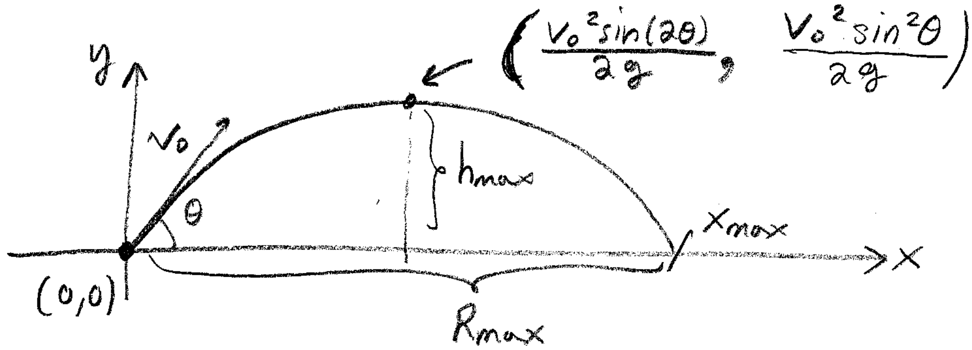
$$\boxed{y = \frac{-g}{2V_o^2\cos^2\theta}\left[x - \left(x_o + \frac{V_o^2\cos\theta\sin\theta}{g}\right)\right]^2 + y_o + \frac{V_o^2\sin^2\theta}{2g}}$$

(see pg. 9 for calculus justification of vertex.)

<u>Remark</u>: when $x_0 = y_0 = 0$ then the vertex formula I just derived yields

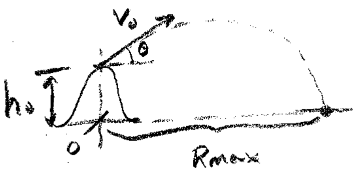
$$R_{max} = \frac{V_0^2 \sin(2\theta)}{g} \qquad (R_{max} = x_{max} - x_0)$$

$$h_{max} = \frac{V_0^2 \sin^2\theta}{2g} \qquad (h_{max} = y_{max} - y_0)$$



These formulas are for the special case pictured above.

<u>Example</u>: find R_{max} as function of θ, V_0, h_0 assuming that the projectiles flight ends upon hitting the ground $y = 0$.



Let $x_0 = 0$. (choose coordinates!)

$$x = (V_0 \cos\theta)t$$
$$y = h_0 + (V_0 \sin\theta)t - \tfrac{1}{2}gt^2$$

Then $y = 0 \Rightarrow 0 = h_0 + (V_0 \sin\theta)t - \frac{1}{2}gt^2$

$$\Rightarrow t = \frac{-V_0\sin\theta - \sqrt{V_0^2\sin^2\theta + 2gh_0}}{-g}$$: choose (−) sol$^{\underline{n}}$ to get $t > 0$ sol$^{\underline{n}}$.

Then substitute,

$$R_{max} = V_0\cos\theta\left[\frac{V_0\sin\theta + \sqrt{V_0^2\sin^2\theta + 2gh_0}}{g}\right]$$

$$\boxed{R_{max} = \frac{V_0^2\sin(2\theta)}{2g}\left[1 + \sqrt{1 + \frac{2gh_0}{V_0^2\sin^2\theta}}\right]}$$

Note $h_0 = 0$ returns the usual formula.

Motion due to Several Forces on Constant Mass

Concept: Newton's 2nd Law states $\sum \vec{F} = m\vec{a}$. Thus if three forces are given, say $\vec{F}_1, \vec{F}_2, \vec{F}_3$, and act on some mass m then we must solve

$$m\vec{a} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{F}_{net} \leftarrow \text{"net force on } m\text{"}$$

This breaks into two tasks,

1.) add the given forces using vector algebra. (may need to break into components etc...)

2.) solve $m\vec{a} = \vec{F}_{net} \longrightarrow \vec{a} = \frac{1}{m}\vec{F}_{net}$. If $\vec{F}_{net}$ is constant then I already told you the resulting equations on pg. 1 and 2. However, if $\vec{F}_{net}$ is nonconstant then we can try to solve

$$\frac{d\vec{V}}{dt} = \frac{1}{m}\vec{F}_{net} \quad \text{and} \quad \frac{d\vec{r}}{dt} = \vec{V}$$

by direct integration. Generally that calculation is difficult because $\vec{F}_{net}$ could depend on velocity or position. In the special case $\vec{F}_{net} = \vec{F}_{net}(t)$ we can solve w/o much trouble. We'll stick to that case for now.

Example: $\vec{F}_1 = \langle F_o, 0, 0\rangle$, $\vec{F}_2 = \langle 0, -F_o, F_o\rangle$, $\vec{F}_3 = \langle 0, 0, 3F_o\rangle$ or we could write $\vec{F}_1 = F_o\hat{i}$, $\vec{F}_2 = -F_o\hat{j} + F_o\hat{k}$, $\vec{F}_3 = 3F_o\hat{k}$. We are given that F_o is a constant. Find motion of mass m which has initial velocity $\vec{V}(0) = \vec{V}_o$ and position $\vec{r}(0) = \vec{r}_o$

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \langle F_o, 0, 0\rangle + \langle 0, -F_o, F_o\rangle + \langle 0, 0, 3F_o\rangle$$

$$\Rightarrow \vec{F}_{net} = \langle F_o, -F_o, 4F_o\rangle = m\vec{a}$$

$$\Rightarrow \vec{a} = \frac{F_o}{m}\langle 1, -1, 4\rangle \leftarrow \underline{\text{constant}}\ \underline{\text{acceleration}}$$

Integrate once and again to find

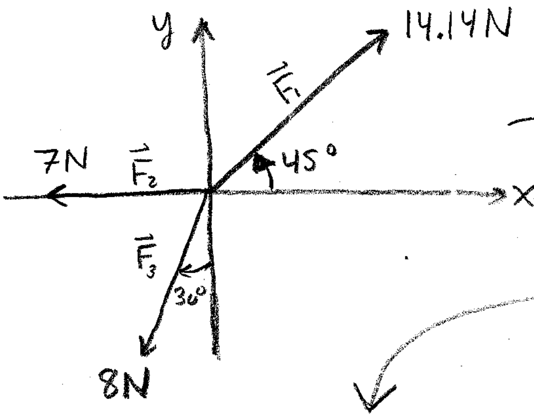
$$\boxed{\vec{V}(t) = \vec{V}_o + \frac{tF_o}{m}\langle 1,-1,4\rangle \ \ \& \ \ \vec{r}(t) = \vec{r}_o + t\vec{V}_o + \frac{t^2F_o}{2m}\langle 1,-1,4\rangle}$$

Comment: the vector notation hides much, for example the previous example gives,

$$\left\{\begin{array}{l} V_x(t) = V_{ox} + \frac{F_o t}{m} \\ V_y(t) = V_{oy} - \frac{F_o t}{m} \\ V_z(t) = V_{oz} + \frac{4F_o t}{m} \end{array}\right\} \longleftrightarrow \underline{\vec{V}(t) = \vec{V}_o + \frac{tF_o}{m}\langle 1, -1, 4\rangle}$$

$$\left\{\begin{array}{l} x(t) = x_o + V_{ox} t + \frac{F_o t^2}{2m} \\ y(t) = y_o + V_{oy} t - \frac{F_o t^2}{2m} \\ z(t) = z_o + V_{oz} t + \frac{4F_o t^2}{2m} \end{array}\right\} \longleftrightarrow \underline{\vec{r}(t) = \vec{r}_o + t\vec{V}_o + \frac{t^2 F_o}{2m}\langle 1, -1, 4\rangle}$$

Example: Given forces pictured below act on mass m which is initially at rest find motion. Assume particle starts at origin



$$\left\{\begin{array}{l} \vec{F}_1 = 14.14\,N\langle \cos 45°, \sin 45°\rangle \\ \vec{F}_2 = 7N\langle -1, 0\rangle \\ \vec{F}_3 = 8N\langle -\sin 30°, -\cos 30°\rangle \end{array}\right\}$$

(N)	x	y
F_1	10	10
F_2	-7	0
F_3	-4	-6.93
F_{net}	-1	3.07

$$\vec{F}_{net} = \langle -1, 3.07\rangle F_o \quad , \underline{F_o = 1N}$$

$$m\vec{a} = \langle -1, 3.07\rangle F_o$$

$$\vec{a} = \frac{F_o}{m}\langle -1, 3.07\rangle \leftarrow \underline{\text{constant!}}$$

$$\vec{V}(t) = \underbrace{\vec{V}_o}_{0} + \frac{tF_o}{m}\langle -1, 3.07\rangle$$

Hence $\underline{\vec{v}(t) = \frac{tF_o}{m}\langle -1, 3.07\rangle}$ and $\underline{\vec{r}(t) = \frac{t^2F_o}{2m}\langle -1, 3.07\rangle}$.

Or, if you prefer,

$$\boxed{\begin{array}{ll} V_x(t) = \left(\frac{-F_o}{m}\right)t & \text{and} \quad x(t) = \frac{-t^2F_o}{2m} \\ V_y(t) = \left(\frac{3.07F_o}{m}\right)t & \quad\quad\quad y(t) = \frac{3.07F_o t^2}{2m} \end{array}}$$

Calculus I used

① FTC for $\int$ of space curves : $\int_a^b \frac{d\vec{r}}{dt}\,dt = \vec{r}(b) - \vec{r}(a)$.

Proof : recall FTC from calculus I ; $\int_a^b \frac{df}{dt}\,dt = f(b) - f(a)$.

$$\begin{aligned}
\int_a^b \frac{d\vec{r}}{dt}\,dt &= \int_a^b \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt \\
&= \left\langle \int_a^b \frac{dx}{dt}\,dt, \int_a^b \frac{dy}{dt}\,dt, \int_a^b \frac{dz}{dt}\,dt \right\rangle \quad \text{FTC times three.} \\
&= \langle x(b)-x(a), y(b)-y(a), z(b)-z(a) \rangle \\
&= \langle x(b), y(b), z(b) \rangle - \langle x(a), y(a), z(a) \rangle \\
&= \vec{r}(b) - \vec{r}(a). //
\end{aligned}$$

② I also claimed if $\frac{d\vec{C}}{dt} = 0$ then $\int_a^b \vec{C} f(t)\,dt = \vec{C} \int_a^b f(t)\,dt$.

Proof : $\int_a^b \vec{C} f(t)\,dt = \int_a^b \langle c_1, c_2, c_3 \rangle f(t)\,dt$, $\vec{C} = \underbrace{\langle c_1, c_2, c_3 \rangle}_{\text{constants!}}$

$$\begin{aligned}
&= \int_a^b \langle c_1 f, c_2 f, c_3 f \rangle\,dt \\
&= \left\langle \int_a^b c_1 f\,dt, \int_a^b c_2 f\,dt, \int_a^b c_3 f\,dt \right\rangle \quad \text{prop. of integral.} \\
&= \left\langle c_1 \int_a^b f\,dt, c_2 \int_a^b f\,dt, c_3 \int_a^b f\,dt \right\rangle \\
&= \langle c_1, c_2, c_3 \rangle \int_a^b f(t)\,dt \\
&= \vec{C} \int_a^b f(t)\,dt. //
\end{aligned}$$

③ the vertex of a parabola described by $y = f(x) = A(x-h)^2 + k$ is at (h,k) and that gives max/min value for y.

$$f(x) = A(x-h)^2 + k \implies \frac{df}{dx} = 2A(x-h) \implies \frac{d^2f}{dx^2} = 2A.$$

Hence, $f'(h) = 0$ makes $(h, f(h)) = (h,k)$ a critical point and $f''(h) = A \neq 0$ makes $f(h)$ the min/max by 2nd der. test. //