Rotating Coordinate Systems

The central idea is that different coordinate systems give descriptions of the same point,

$$\vec{r} = x e_1 + y e_2 + z e_3 = \bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3$$

The idea here is that $\bar{e}_1, \bar{e}_2, \bar{e}_3$ is a rotating frame and these coordinate systems share a common origin. Furthermore, we have in mind the coordinates x, y, z of some moving object so generally these are also functions of time. Time is time so differentiate,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[\bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3\right]$$

$$\frac{d\vec{r}}{dt} = \underbrace{\frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3}_{\mathrm{I}} + \underbrace{\bar{x}\frac{d\bar{e}_1}{dt} + \bar{y}\frac{d\bar{e}_2}{dt} + \bar{z}\frac{d\bar{e}_3}{dt}}_{\mathrm{II}}$$

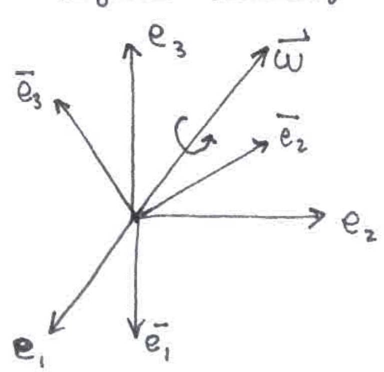
Note that in contrast the fixed, time independent frame e_1, e_2, e_3 has the simple familar form,

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[x e_1 + y e_2 + z e_3\right]$$

$$= \underbrace{\frac{dx}{dt}e_1 + \frac{dy}{dt}e_2 + \frac{dz}{dt}e_3}_{\vec{V}_S}$$

$\vec{V}_S$ the velocity relative to the fixed coordinate system S

In contrast, $\vec{V}_{\bar{S}} = \frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3$ is velocity relative to the rotating coordinate system $\bar{S}$. The terms in II give the velocity of the $\bar{S}$-frame itself. It can be shown that $\frac{d\bar{e}_j}{dt} = \vec{\omega}\times\bar{e}_j$ for $j = 1,2,3$ where $\vec{\omega}$ is the angular velocity of the rotating frame



(imagine $\bar{e}_1, \bar{e}_2, \bar{e}_3$ rotate about the axis $\vec{\omega}$)

Here II $\longrightarrow \vec{\omega}\times\underbrace{(\bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3)}_{\vec{r}}$

(using $\frac{d\bar{e}_j}{dt} = \vec{\omega}\times\bar{e}_j$ for $j = 1,2,3$)

Setting aside the question of why $\frac{d\bar{e}_j}{dt} = \vec{\omega} \times \bar{e}_j$, we find

$$\frac{d\vec{r}}{dt} = \frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3 + \bar{x}\frac{d\bar{e}_1}{dt} + \bar{y}\frac{d\bar{e}_2}{dt} + \bar{z}\frac{d\bar{e}_3}{dt}$$

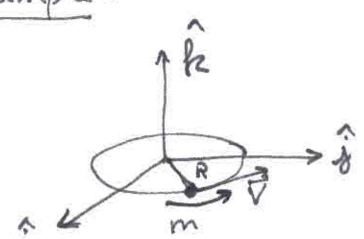
$$= \frac{d\bar{x}_1}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3 + \bar{x}(\vec{\omega}\times\bar{e}_1) + \bar{y}(\vec{\omega}\times\bar{e}_2) + \bar{z}(\vec{\omega}\times\bar{e}_3)$$

$$= \underbrace{\frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3}_{\vec{V}_{\bar{S}}} + \vec{\omega}\times(\underbrace{\bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3}_{\vec{r}})$$

We find $\boxed{\vec{V}_S = \vec{V}_{\bar{S}} + \vec{\omega}\times\vec{r}}$

This assumed a <u>common</u> origin.

<u>Example:</u>



Let $\bar{S}$ rotate at constant velocity $\vec{\omega}$ about the $\hat{k}$ vector. Consider point m fixed at point in $\bar{S}$ frame; $\vec{r}(t) = \langle \underbrace{R\cos t, R\sin t}, 0\rangle$

oops. I just set $\omega = 1$ see ✱

$$\vec{V}_s = \omega\hat{k}\times(R\cos t\,\hat{i} + R\sin t\,\hat{j})$$
$$= \omega R\cos t(\hat{k}\times\hat{i}) + \omega R\sin t(\hat{k}\times\hat{j})$$
$$= (\omega R\cos t)\hat{j} - (\omega R\sin t)\hat{i}$$
$$= \omega R\,(\underbrace{\langle -\sin t, \cos t\rangle}_{\text{the tangential velocity}})$$

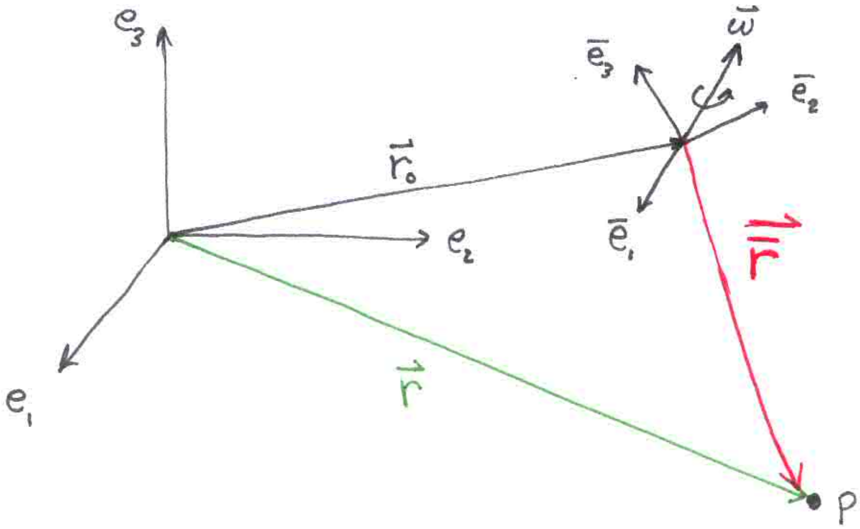
$\underline{\frac{d\vec{r}}{dt} = \langle -R\cos t, R\sin t\rangle}$

I have shown that $\vec{V} = \vec{\omega}\times\vec{r}$ is true in this special case.

✱ $\vec{r}(t) = \langle R\cos\omega t, R\sin\omega t, 0\rangle$

$$\frac{d\vec{r}}{dt} = R\omega\langle -\sin\omega t, \cos\omega t, 0\rangle = \omega\hat{k}\times\vec{r}$$

(I forgot the ω above)



rotating coordinate system $\bar{S}$ with non-matching origin relative to fixed inertial frame $S : \{e_1, e_2, e_3\}$

$$\vec{r} = x e_1 + y e_2 + z e_3$$

$$\vec{\bar{r}} = \bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3$$

From picture we see that the variable point P has

$$\vec{r} = \vec{r}_0 + \vec{\bar{r}}$$

Almost same calculation goes through,

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}_0}{dt} + \frac{d}{dt}\left(\bar{x}\bar{e}_1 + \bar{y}\bar{e}_2 + \bar{z}\bar{e}_3\right)$$

$$= \frac{d\vec{r}_0}{dt} + \underbrace{\frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3}_{} + \bar{x}\frac{d\bar{e}_1}{dt} + \bar{y}\frac{d\bar{e}_2}{dt} + \bar{z}\frac{d\bar{e}_3}{dt}$$

$$\boxed{\vec{V}_S = \frac{d\vec{r}_0}{dt} + \vec{V}_{\bar{S}} + \vec{\omega}\times\vec{\bar{r}}}$$

This formula relates the velocity measured relative to a fixed vs. rotating frame. The first two terms should be familar from freshman mechanics.

$$\frac{d\vec{r}_0}{dt} = \text{velocity of moving frame's origin}$$

$$\vec{V}_{\bar{S}} = \text{velocity relative to moving frame}$$

However, the $\vec{\omega}\times\vec{\bar{r}}$ is probably new to you in this context.

Comparing acceleration in fixed frame S verses rotating $\bar{S}$.

Continuing with the notation from the previous page,

$$\frac{d\vec{r}}{dt} = \vec{V}_s = \frac{d\vec{r}_o}{dt} + \frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3 + \vec{\omega}\times\vec{\bar{r}}$$

Now differentiate again, note this calculation is very much like the one we just completed,

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}_o}{dt^2} + \frac{d^2\bar{x}}{dt^2}\bar{e}_1 + \frac{d^2\bar{y}}{dt^2}\bar{e}_2 + \frac{d^2\bar{z}}{dt^2}\bar{e}_3 + \frac{d\bar{x}}{dt}\frac{d\bar{e}_1}{dt} + \frac{d\bar{y}}{dt}\frac{d\bar{e}_2}{dt} + \frac{d\bar{z}}{dt}\frac{d\bar{e}_3}{dt} + \frac{d}{dt}\left(\vec{\omega}\times\vec{\bar{r}}\right)$$

$$= \vec{a}_o + \vec{a}_{\bar{S}} + \frac{d\bar{x}}{dt}(\vec{\omega}\times\bar{e}_1) + \frac{d\bar{y}}{dt}(\vec{\omega}\times\bar{e}_2) + \frac{d\bar{z}}{dt}(\vec{\omega}\times\bar{e}_3) + \frac{d\vec{\omega}}{dt}\times\vec{\bar{r}} + \vec{\omega}\times\frac{d\vec{\bar{r}}}{dt}$$

$$= \vec{a}_o + \vec{a}_{\bar{S}} + \vec{\omega}\times\left(\frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3\right) + \frac{d\vec{\omega}}{dt}\times\vec{\bar{r}} + \vec{\omega}\times\frac{d\vec{\bar{r}}}{dt}$$

$$\boxed{\vec{a}_S = \vec{a}_o + \vec{a}_{\bar{S}} + 2\vec{\omega}\times\vec{V}_{\bar{S}} + \frac{d\vec{\omega}}{dt}\times\vec{\bar{r}} + \vec{\omega}\times(\vec{\omega}\times\vec{\bar{r}})}$$

The notation $\vec{a}_o = \frac{d^2\vec{r}_o}{dt^2}$ gives acceleration of origin of $\bar{S}$ relative to S.

<u>Thought Experiment</u>: Suppose you took measurements relative to $\bar{S}$ and <u>assumed</u> it was an interstial frame. What "fake" forces would you encounter? Assuming Newton's 2nd Law,

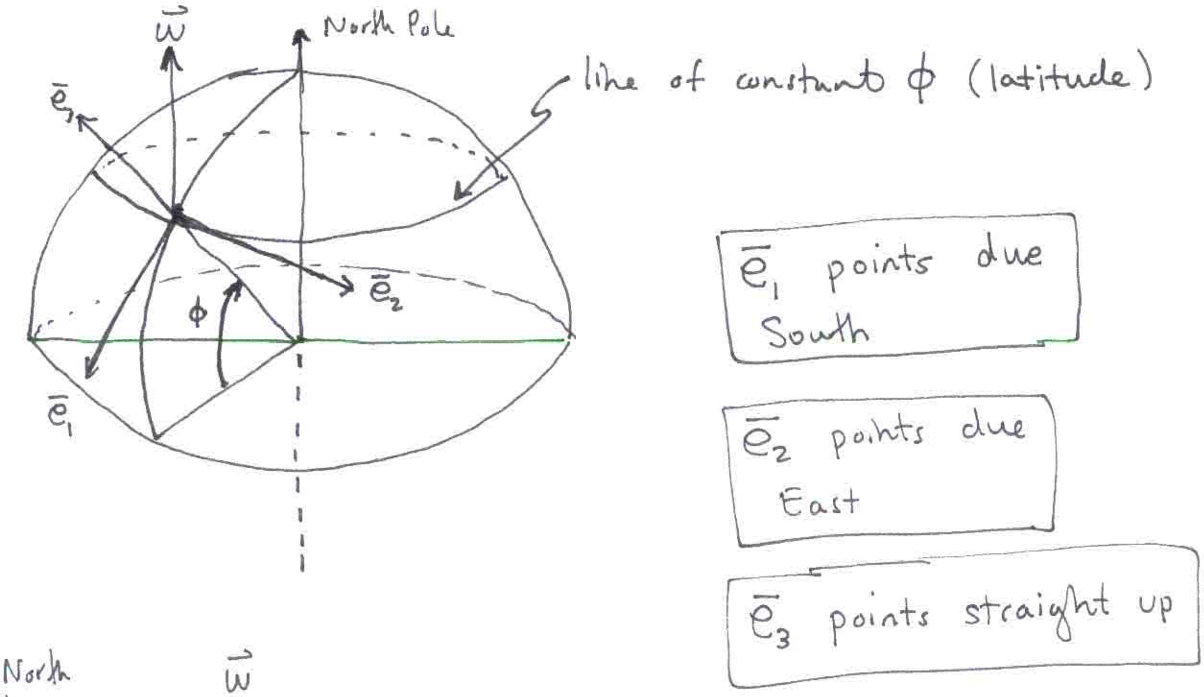
$$m\vec{a}_{\bar{S}} = m\vec{a}_S - m\vec{a}_o - 2m\vec{\omega}\times\vec{V}_{\bar{S}} - m\frac{d\vec{\omega}}{dt}\times\vec{\bar{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{\bar{r}})$$

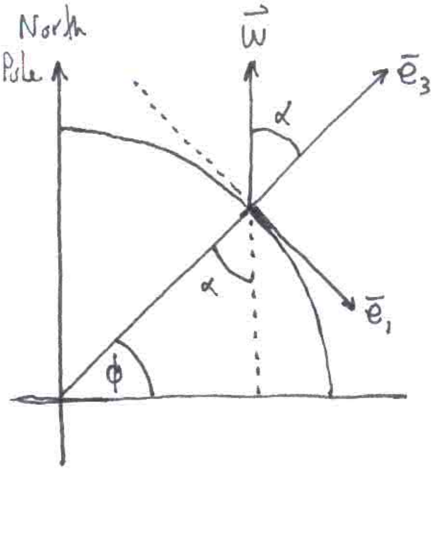
$$\boxed{m\vec{a}_{\bar{S}} = \vec{F}_{net} - m\vec{a}_o - 2m\vec{\omega}\times\vec{V}_{\bar{S}} - m\frac{d\vec{\omega}}{dt}\times\vec{\bar{r}} - m\vec{\omega}\times(\vec{\omega}\times\vec{\bar{r}})}$$

$\vec{F}_{net}$	$-m\vec{a}_o$	$-2m\vec{\omega}\times\vec{V}_{\bar{S}}$	$-m\frac{d\vec{\omega}}{dt}\times\vec{\bar{r}}$	$-m\vec{\omega}\times(\vec{\omega}\times\vec{\bar{r}})$
real forces like springs or gravity	rectilinear acceleration of frame $\bar{S}$	Coriolis Force	no-name but you've felt this on a tilt-a-whirl as it stops or starts	Centrifugal force.

Rotating Frame of Reference we live in

We've done almost all the math. Think about it, the earth is essentially moving at constant velocity relative to solar system over a time of <u>minutes</u>. So we can reasonable put a <u>fixed</u> coordinate system S at the center of the earth. Moreover, modulo earthquakes & tidal waves, the rotation of the earth is nearly constant in magnitude. This means the $\frac{d\vec{\omega}}{dt}$ term vanishes. Let's set up a rotating frame of reference (borrowed from McComb's "Dynamics and Relativity" Oxford Press.)





Continuing, we find the following Deq$^{\underline{n}}$ of motion near surface of earth at Latitude ϕ, either ignore or lump $\vec{\omega}\times(\vec{\omega}\times\vec{r})$ into the $-mg\bar{e}_3$ term,

$$m\left(\frac{d^2\bar{x}}{dt^2}\bar{e}_1 + \frac{d^2\bar{y}}{dt^2}\bar{e}_2 + \frac{d^2\bar{z}}{dt^2}\bar{e}_3\right) = \underbrace{-mg\bar{e}_3}_{\text{gravity}} - \underbrace{2m\vec{\omega}\times\vec{V}_{\bar{S}}}_{\text{Coriolis Force}}$$

Work out the Coriolis term,

$$\vec{\omega} = \omega e_3 = \omega(e_3\cdot\bar{e}_1)\bar{e}_1 + \omega\underbrace{(e_3\cdot\bar{e}_2)}_{0}\bar{e}_2 + \omega(e_3\cdot\bar{e}_3)\bar{e}_3$$

$$\Rightarrow \underbrace{\vec{\omega} = \omega\cos(\alpha+\tfrac{\pi}{2})\,\bar{e}_1 + \omega\cos\alpha\,\bar{e}_3}$$

See picture on previous page it's clear the angles are $\alpha+\frac{\pi}{2}$ and α between e_3 & $\bar{e}_1$ & e_3 & $\bar{e}_3$ respective.

However, $\alpha = \frac{\pi}{2}-\phi$ thus

$$\cos(\alpha+\tfrac{\pi}{2}) = \cos(\pi-\phi) = -\cos\phi$$

$$\cos\alpha = \cos(\tfrac{\pi}{2}-\phi) = +\sin\phi$$

We find that $\underline{\vec{\omega} = -\omega\cos\phi\,\bar{e}_1 + \omega\sin\phi\,\bar{e}_3}$.

I did this so we can use $\bar{e}_1\times\bar{e}_2 = \bar{e}_3$ etc... in the following,

$$\begin{aligned}
\vec{\omega}\times\vec{V}_{\bar{S}} &= (-\omega\cos\phi\,\bar{e}_1 + \omega\sin\phi\,\bar{e}_3)\times\left(\frac{d\bar{x}}{dt}\bar{e}_1 + \frac{d\bar{y}}{dt}\bar{e}_2 + \frac{d\bar{z}}{dt}\bar{e}_3\right)\\
&= -\omega\cos\phi\frac{d\bar{y}}{dt}\bar{e}_3 + \omega\cos\phi\frac{d\bar{z}}{dt}\bar{e}_2 + \omega\sin\phi\frac{d\bar{x}}{dt}\bar{e}_2 - \omega\sin\phi\frac{d\bar{y}}{dt}\bar{e}_1\\
&= \left(-\omega\sin\phi\frac{d\bar{y}}{dt}\right)\bar{e}_1 + \left(\omega\cos\phi\frac{d\bar{z}}{dt} + \omega\sin\phi\frac{d\bar{x}}{dt}\right)\bar{e}_2 - \omega\cos\phi\frac{d\bar{y}}{dt}\bar{e}_3
\end{aligned}$$

Putting this together with Newton's Law,

$$\boxed{\begin{aligned}
\bar{e}_1 &: \quad m\frac{d^2\bar{x}}{dt^2} = 2m\omega\sin\phi\frac{d\bar{y}}{dt}\\
\bar{e}_2 &: \quad m\frac{d^2\bar{y}}{dt^2} = -2m\omega\cos\phi\frac{d\bar{z}}{dt} - 2m\omega\sin\phi\frac{d\bar{x}}{dt}\\
\bar{e}_3 &: \quad m\frac{d^2\bar{z}}{dt^2} = 2m\omega\cos\phi\frac{d\bar{y}}{dt} - mg
\end{aligned}}$$

Remark: since $m, \omega, \cos\phi$ are all constants for a given problem we can solve this by reduction of order to a 6×6 nonhomog. matrix problem!

Approximate Sol^n for Coriolis Problem

Compared to mg the terms with $2m\omega$ are proportionally smaller. If we consider throwing an object vertically it stands to reason only $\frac{d\bar{z}}{dt}$ is nontrivial, the Coriolis force will create some nonzero $\frac{d\bar{x}}{dt}, \frac{d\bar{y}}{dt}$ as time progresses, but those terms are small. Hence we can solve:

$$m\frac{d^2\bar{x}}{dt^2} = 0$$

$$m\frac{d^2\bar{y}}{dt^2} = -2m\omega\cos\phi\,\frac{d\bar{z}}{dt}$$

$$m\frac{d^2\bar{z}}{dt^2} = -mg$$

We may simply integrate the $\bar{x}$ & $\bar{z}$ eq^n's to find

$$\boxed{\begin{aligned}\bar{x}(t) &= x_o \\ \bar{z}(t) &= z_o - \quad t - \tfrac{1}{2}gt^2\end{aligned}}$$

Same as usual. The interesting feature is the Eastward drift captured in the $\bar{y}$ - eq^n.

Hence $\frac{d\bar{z}}{dt} = \bar{V}_o - gt$. This gives,

$$m\frac{d^2\bar{y}}{dt^2} = -2m\omega\cos\phi\left[\bar{V}_o - gt\right]$$

$$\Rightarrow \frac{d^2\bar{y}}{dt^2} = -2\omega\cos\phi\,\bar{V}_o + (2\omega\cos\phi\, g)\,t$$

$$\Rightarrow \frac{d\bar{y}}{dt} = -2\omega\cos\phi\,\bar{V}_o\,t + \omega\cos\phi\, g\,t^2$$

(assumed $\frac{d\bar{y}}{dt}(0) = 0$.)

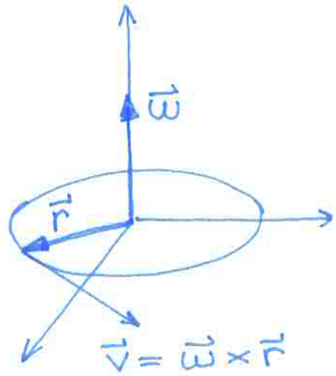
$$\Rightarrow \boxed{\bar{y}(t) = \left(\tfrac{1}{3}g\omega\cos\phi\right)t^3 - \left(\omega\cos\phi\,\bar{V}_o\right)t}$$

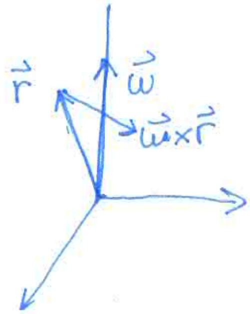
Or, if we drop a mass so $\bar{V}_o = 0$ we have

$$\underbrace{\boxed{\bar{y}(t) = \tfrac{1}{3}(g\omega\cos\phi)\,t^3}}_{\text{Coriolis drift goes east in Northern Hemisphere.}}$$

(Notice the $-2m\vec{\omega}\times\vec{V}_{\bar{S}}$ points opposite direction below equator.)

Concerning why $\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$ (I used $\frac{de_j}{dt} = \vec{\omega}\times e_j$ before).





$A_{\vec{\omega}}$ rotates at constant angular velocity ω about $\frac{\vec{\omega}}{\omega} = \hat{n}$

$$A_{\vec{\omega}}(t) = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{r}(t) = A_{\vec{\omega}}(t)\,\vec{r}_0 \rightsquigarrow \vec{r}(t) = [\cos\omega t\, x_0 - \sin\omega t\, y_0, \sin\omega t\, x_0 + \cos\omega t\, y_0, z$$

$$\frac{d\vec{r}}{dt} = \frac{dA}{dt}\vec{r}_0 = \begin{bmatrix} -\omega\sin\omega t & -\omega\cos\omega t & 0 \\ \omega\cos\omega t & -\omega\sin\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$= \begin{bmatrix} -\omega x_0\sin\omega t - \omega y_0\cos\omega t \\ \omega x_0\cos\omega t - \omega y_0\sin\omega t \\ 0 \end{bmatrix}$$

$$= \underline{\omega[-x_0\sin\omega t - y_0\cos\omega t,\ x_0\cos\omega t - y_0\sin\omega t,\ 0]}$$

$$\vec{\omega} = \omega[0,0,1]^T \rightsquigarrow \vec{\omega}\times\vec{r} = \omega\hat{k}\times(x\hat{i} + y\hat{j} + z_0\hat{k})$$

$$= \omega x\hat{j} - \omega y\hat{i}$$

$$= \underline{[-\omega x_0\sin\omega t - \omega y_0\cos\omega t,\ \omega x_0\cos\omega t - \omega y_0\sin\omega t,\ 0]}$$

$$\therefore\ \vec{v} = \vec{\omega}\times\vec{r}$$

Remark: this proof is almost general but it needs a little work....

(Just for fun, this is unfinished)

$$A^T A = I$$

$$A = e^{Bt}, \quad \frac{dA}{dt} = Be^{Bt}$$

$$\frac{dA^T}{dt} A + A^T \frac{dA}{dt} = 0$$

Assume $\gamma(t) = e^{Bt}$ then $\gamma(0) = I$ & $\gamma'(0) = B$.
If $\gamma^T \gamma = I$ then $\frac{d\gamma^T}{dt}(t)\, \gamma(t) + \gamma^T(t) \frac{d\gamma}{dt}(t) = 0$

$\therefore\ B^T + B = 0 \longrightarrow B^T = -B$.

$$\Rightarrow B = \begin{bmatrix} 0 & b_3 & b_2 \\ -b_3 & 0 & b_1 \\ -b_2 & -b_1 & 0 \end{bmatrix} = b_3 \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\epsilon_{ij3}} + b_2 \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{\epsilon_{ij2}} + b_1 \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{\epsilon_{ij1}}$$

Let $(J_k)_{ij} = \epsilon_{ijk}$.

Claim: $R_{\vec{\omega}} = \exp(\vec{J} \cdot \vec{\omega})$

$$R_{\omega\hat{k}} = \exp(\omega J_3)$$

Umm....
$$\begin{aligned} \frac{dR_{\vec{\omega}}}{dt} &= \frac{d}{dt} \exp(t(\vec{J}\cdot\vec{\omega})) \\ &= \exp(t\vec{J}\cdot\vec{\omega}) \frac{d}{dt}(t(\vec{J}\cdot\vec{\omega})) \\ &= \underline{\vec{J}\cdot\vec{\omega}} \exp(t\vec{J}\cdot\vec{\omega}) \\ &= \epsilon_{ijk}\omega_k R_{\vec{\omega}} \end{aligned}$$

$$\begin{aligned} \vec{r}(t) &= R_{\vec{\omega}}(t)\, \vec{r}_0 \\ \frac{d\vec{r}}{dt} &= \frac{dR_{\vec{\omega}}}{dt}(t)\, \vec{r}_0 \\ &= (\omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3) R_{\vec{\omega}}(t) \vec{r}_0 \\ &= (\omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3) \vec{r}(t). \\ &= (\omega_1, \omega_2, \omega_3) \times \vec{r}(t) \end{aligned}$$

$$J_1 \vec{r} = \epsilon_{ij1} r_j = \epsilon_{231} r_3 + \epsilon_{321} r_2 = r_3 - r_2 = z - y.$$