

Dr. Cook } Monday } 10:30-
Wednesday am

Dr. Mavinga } Tuesday } 9:00 am
Thursday

~~METHODS~~ Differential Equation

definition: $F(y^{(n)}, y^{(n-1)}, \dots, y', y, x) = 0$
is an n^{th} order ODE in dep. variable y & indep. variable x .

$G(y_i^{(j_i)}, x_k) = 0$, for $j_i = 0, 1, 2, \dots, s_i$ for $k = 1, 2, \dots, n$
then $G(y_i^{(j_i)}, x_k) = 0$ is partial Diff Egm (PDE) in
dep. variable y_1, y_2, \dots, y_s
order $\max\{j_i \mid i = 1, 2, \dots, s\}$

Ex: $U_{xx} + U_t = 0$, of order 2.

Ex: $U_{xx} + U_{yy} = 0$ Coefficient Function

• $\vec{F}(y^{(n)}, \dots, y', y, x) = 0$ is linear iff $\exists P_n, P_{n-1}, \dots, P_0$ $(n^{\text{th}}$ order linear ODE)
s.t $\vec{F} = 0 \Leftrightarrow P_n y^{(n)} + P_{n-1} y^{(n-1)} + \dots + P_1 y' + P_0 y = g(x)$
 $\underbrace{L[y]}_{\text{linear operator}} = (P_n D^n + P_{n-1} D^{n-1} + \dots + P_1 D + P_0) [y]$

①

Furthermore, if $g(x) \equiv 0$

then $L[y] = 0$ is a homogeneous ODE.

Theorem: $L[y] = g$ where $\bigcap_{i=0}^n \text{dom}(P_i) = U \subseteq \mathbb{R}$ and $P_i : \text{dom}(P_i) \rightarrow \mathbb{R}$ are analytic (maybe smooth) $U \subseteq \text{Dom}(g)$, where $L[y] = g$ is n^{th} order, then the solution for which $x_0 \in U$ and

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

$\exists!$ a solution $y = y_h + y_p$ where $y_h = \sum_{i=1}^n c_i y_i$, $L[y_i] = 0 \quad \forall i = 1, n$ and $L[y_p] = g$.

Moreover, any solution has this form -

The construction of y_h and y_p are not unique.

We say $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n + y_p$

forms the "general sol"

and $\{y_1, y_2, \dots, y_n\}$ is LI set of functions

called the fundamental solⁿ set

n=1 $y' = f(x, y)$ ↓ exp. variable
↓
exact solution.

Ex: $\frac{dy}{dx} = \frac{\sin(x)}{\cos(y)} \rightarrow \cos y \, dy = \sin x \, dx$
 $\Rightarrow \boxed{\sin y = -\cos x + C}$ implicit sol¹

Ex: $\frac{dy}{dx} + \frac{2}{x}y = \sin(x^3)$

$P = \exp\left(\int P \, dx\right) = \exp\left(2 \int \frac{1}{x} \, dx\right) = \exp(2 \ln|x|) = |x|^2 = x^2 (x \neq 0)$

$\Rightarrow x^2 \frac{dy}{dx} + 2xy = \sin(x^3) x^2$

$\Rightarrow (yx^2)' = x^2 \sin(x^3)$

$\Rightarrow yx^2 = -\frac{1}{3} \cos(x^3) + C$

$\Rightarrow \boxed{y(x) = -\frac{1}{3x^2} \cos(x^3) + \frac{C}{x^2}}$

Ex: $x \, dx + y \, dy = 0$ (Pfaffian form) $(d(f(x, y)) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy)$

$\Rightarrow d\left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) = 0$

$\Rightarrow \frac{1}{2}x^2 + \frac{1}{2}y^2 = C$

Pfaff's theorem: Given any ODE, $\frac{dy}{dx} = f(x, y)$

a.k.a $f(x, y) \, dx - dy = 0, \exists I \text{ s.t.}$

$\int f \, dx - \int dy = 0$ is exact.

meaning $\exists G \text{ s.t. } dG = 0$ hence $G(x, y) = C$ is the sol. (3)

Theorem: If $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ then $\frac{dy}{dx} = f(x, y)$ has solution through (x_0, y_0)

If $\frac{\partial f}{\partial y}$, f are continuous in rectangle R containing (x_0, y_0) then $\exists!$ solution of $\frac{dy}{dx} = f(x, y)$ w/ $y(x_0) = y_0$.

$$\underline{\text{Ex}}: \frac{dy}{dx} = \sqrt{y} \Rightarrow \frac{dy}{\sqrt{y}} = dx$$

$$\Rightarrow 2\sqrt{y} = x + C$$

$$y = \left(\frac{x}{2} + C\right)^2$$

$$y(0) = 0, \text{ then } y_1 = \frac{1}{4}x^2$$

$$y_2(x) \equiv 0, x \geq 0.$$

not satisfying theorem (1) since $\frac{\partial f}{\partial x}$ is not continuous.

Ex: Substitution:

$$r^2 = x^2 + y^2$$

$$\Rightarrow 2rdr = 2xdx + 2ydy, \tan^2 \theta = \frac{y}{x}$$

$$(2xdx + 2ydy) - \frac{dy}{x} - \frac{y}{x^2} dx = 0 \Rightarrow \sec^2 \theta = \frac{dy}{x} - \frac{y}{x^2} dx$$

$$\Rightarrow \ln dr - \frac{1}{2} \sec^2 \theta d\theta = 0$$

$$\Rightarrow \boxed{r^2 = \tan \theta + C}$$

(4)

$$\underline{f(x,y)dx + g(x,y)dy = 0} \quad | \quad (*)$$

How to understand parametrically?

Suppose $G(x,y) = C$ is the solution to $(*)$

Parametric solution is $x = \phi(t), y = \psi(t)$

$$G(\phi(t), \psi(t)) = C$$

$$\frac{\partial G}{\partial x}(\phi(t), \psi(t)) \frac{d\phi}{dt} + \frac{\partial G}{\partial y}(\phi(t), \psi(t)) \frac{d\psi}{dt} = 0$$

$$\frac{\partial G}{\partial x} \frac{dx}{dt} + \frac{\partial G}{\partial y} \frac{dy}{dt} = 0 \quad (x = \phi, y = \psi)$$

$$\nabla G \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = 0$$

$$\nabla G = \langle G_x, G_y \rangle \quad (G_x = f, G_y = g)$$

$$\therefore \langle f, g \rangle \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle}_{\text{tangent vector to the soln}} = 0$$

\curvearrowleft tangent vector to the solution

⑤

Koeffizient Problem (Homogeneous Problem)

$L[y] = 0$, $L \in \mathbb{R}[D]$, order of $L = n$.

$$L = D^n + a_{n-1} D^{n-1} + \dots + a_2 D^2 + a_1 D + a_0 (\#)$$

Notation: $D[f](x) = \frac{df}{dx} = f'(x)$.

$$(c f)(x) = c f(x)$$

$$(g(x) f)(x) = g(x) f(x)$$

$T = xD$, calculate $T^2[f]$

$$\begin{aligned} T^2[f] &= T[T[f]] \\ &= T[x f'] \\ &= x D[x f'] \\ &= x (f' + x f'') \\ &= x f' + x^2 f'' \\ &= (xD + x^2 D^2)[f] \end{aligned}$$

$$\underline{\text{Theorem}} : (D+\alpha)(D+\beta) = \overline{(D+\beta)(D+\alpha)}^m$$

$\forall \alpha, \beta \text{ and } m \in \mathbb{N}$

Moral of Story : we can factor the L for the constant coefficient problem.

$$\underline{\text{Ex}} : D^2 + 3D + 2 = (D+1)(D+2)$$

$$\begin{aligned} \underline{\text{Ex}} : D^4 - 16 &= (D^2 - 4)(D^2 + 4) \\ &= (D-2)(D+2)(D^2 + 4) \end{aligned}$$

thus T.T.4 says

$$L[y] = (L_1 \circ L_2 \circ \dots \circ L_r)[y] = 0$$

If $L_j[y_j] = 0$ then $[y_j] = 0$

Final Cases:

1) $(D-\lambda)[y] = 0$, note $y = e^{\lambda x}$ solves this.

2) $(D-\lambda)^2[y] = 0$

note $y_1 = e^{\lambda x}$ still work!

$$y_2 = xe^{\lambda x}$$

$$\therefore y = c_1 y_1 + c_2 y_2 \dots$$

$(D - \lambda^*)^m [y] = 0$ has solution
 $e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \dots, x^{m-1} e^{\lambda x}.$ $\{y_1, y_2, \dots, y_m\}$

Theorem: If L is a real operator defining $L[y] = 0$
 a differential equation

$z = \operatorname{Re}(z) + i \operatorname{Im}(z)$ solves $[y] = 0.$

then $L[\operatorname{Re}(z)] = 0 \notin L[\operatorname{Im}(z)] = 0$

- for (A) complex factors come in conjugate pairs

$$\lambda_1, \lambda_1^*, \lambda_2, \lambda_2^*, \lambda_3, \lambda_3^*, \dots$$

\downarrow \downarrow
2 real soln 2 real soln

Ex: $y'' + y = 0$

$$(D^2 + 1)[y] = 0$$

$$(D - i)(D + i)[y] = 0$$

$$y = e^{ix} = \cos x + i \sin x \rightarrow y = C_1 \cos x + C_2 \sin x$$

$$y = e^{-ix} = \cos x - i \sin x \rightarrow y = \bar{C}_1 \cos x + \bar{C}_2 (-\sin x)$$

Ex: $y'' - y = 0$

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

$$\Rightarrow y = C_1 e^x + C_2 e^{-x} = \bar{C}_1 \cosh x + \bar{C}_2 \sinh x$$

⑧

$$\underline{\text{Ex}} \quad y^{(4)} + 4y'' + 4y = 0$$

$$D^4 + 4D^2 + 4 = 0$$

$$(D^2 + 2)^2 = 0$$

$$y = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x) + C_3 x \cos(\sqrt{2}x) + C_4 x \sin(\sqrt{2}x)$$

$$\underline{\text{Ex}}: \quad y''' + 4y'' + 5y' = 0$$

$$D(D^2 + 4D + 5)y = 0$$

$$D((D+2)^2 + 1)y = 0 \quad \rightarrow y = C + C_1 e^{-2x} \cos x + C_2 e^{-2x} \sin$$

$$\underline{\text{Ex}}: \quad x^2 y'' + y = 0 \quad (\text{Cauchy - Euler})$$

$$\text{Guess: } y = x^R$$

$$y'' = R(R-1)x^{R-2}$$

$$R(R-1)x^R + x^R = 0$$

$$x^R (R(R-1) + 1) = 0$$

$$R^2 - R + 1 = 0 \Rightarrow (R - \frac{1}{2})^2 + \frac{3}{4}$$

$$R = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$y = C_1 \sqrt{x} \cos \frac{\sqrt{3}}{2}x + C_2 \sqrt{x} \sin \frac{\sqrt{3}}{2}x$$

⑨

May 13, 13

Variation of Parameters

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_{n-1} y' + a_n y = g(x)$$

& we have fundamental solution set $\{y_1, y_2, \dots, y_n\}$ (LI)
 $\nexists L[y_j] = 0 \quad \forall j$

$$(\forall x \in U) \begin{cases} c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0 \\ c_1 y'_1 + c_2 y'_2 + \dots + c_n y'_n = 0 \\ \vdots \\ c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \dots + c_n y_n^{(n-1)} = 0 \end{cases} \Rightarrow \vec{c} = \vec{0}$$

$$M(y_1, y_2, \dots, y_n) = \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y'_1 & y'_2 & \dots & y'_n \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}$$

We have $M\vec{c} = \vec{0}$ has only $\vec{c} = 0$ sol^{*}

$$\Rightarrow \det(M) \neq 0$$

$$W = \det M \quad (\text{Wronskian})$$

$$= k e^{\left(-\frac{a_1}{a_0}x\right)}$$

$$(\text{Abel's f-La})$$



"Theorem": $ay'' + by' + cy = g$

and y_1, y_2 solve the above homogenous case.

$$\left\{ \begin{array}{l} ay_1'' + by_1' + cy_1 = 0 \\ ay_2'' + by_2' + cy_2 = 0 \end{array} \right.$$

$$W = y_1 y_2' - y_2 y_1' = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

$$\frac{dW}{dx} = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1''$$

$$= y_1 \left(-\frac{b}{a} y_2'' - \frac{c}{a} y_2 \right) - y_2 \left(-\frac{b}{a} y_1'' - \frac{c}{a} y_1 \right)$$

$$= -\frac{b}{a} W$$

$$\Rightarrow W = K e^{-\frac{b}{a} x}$$

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Undetermined Coefficient:

$$\underline{\text{Ex}}: \quad y'' + y = t^2$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h = C_1 \cos t + C_2 \sin t$$

$$t: \quad \hookrightarrow \quad y_p = At^2 + Bt + C$$

$$\therefore \quad y_p'' = 2At$$

$$y_p + y_p'' = At^2 + (2A+B)t + C = t^2$$

$$\begin{cases} A = 1 \\ 2A + B = 0 \\ C = 0 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -2 \\ C = 0 \end{cases}$$

$$y = C_1 \cos t + C_2 \sin t + t^2 - 2t$$

$$\underline{\text{Ex}}: \quad y'' - y = e^t$$

$$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow y_h = C_1 e^t + C_2 e^{-t}$$

$$(D^2 - 1)[y] = e^t$$

$$\Rightarrow (D-1)(D+1)[y] = 0$$

$$\Rightarrow (D-1)^2(D+1)[y] = 0 \Rightarrow y = C_1 e^t + C_2 t e^t + C_3 e^{-t}$$

$$y_p'' = (C_2(1+t)e^t)' = C_2(2+t)$$
(12)

$$y_p'' - y_p = e^t$$

$$c_2 [2+t-t] e^t = e^t$$

$$\Rightarrow c_2 = \frac{1}{2}$$

$$y = G e^t + c_2 e^{-t} + \frac{1}{2} t e^t$$

Guess: $y_p = v_1 y_1 + v_2 y_2 + v_3 y_3 + \dots + v_n y_n$

where v_1, v_2, \dots, v_n are unknown functions.
(We require $L[y_p] = g$)

$$y_p' = v_1 y_1' + \dots + v_n y_n' + v_1' y_1 + v_2' y_2 + \dots + v_n' y_n$$

Exercise: Let $\mathbb{R}^{n \times n}$ be the set of matrices.

$$\text{w/ } \|A\| = \text{Trace}(A^T A)$$

$$\text{Define } D_\delta(A_0) = \{B \in \mathbb{R}^{n \times n} \mid \|B - A\| < \delta\}$$

a) Prove its ~~of $\mathbb{R}^{n \times n}$~~ if its Metric Space

b) Let $f: \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ be defined by

$$f(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

Prove: f is well-defined $\forall A \in \mathbb{R}^{n \times n}$

i.e Prove the series converges.

Hint: It can be argued that $\mathbb{R}^{n \times n}$ is complete $\xrightarrow{\text{Cauchy} \Rightarrow \text{con.}}$ (13)

Def: If M is a Metric Space then M is complete iff Cauchy
 \Rightarrow convergent.

Ex1: $(\mathbb{Q}, |q| = \sqrt{q^2})$ is not complete.

Ex 12.10) (Roman)

Suppose $\vec{x}_k = (x_{k1}, x_{k2}, x_{k3}, \dots, x_{kn}) \in \mathbb{R}^n$, Cauchy sequence in \mathbb{R}^n .
 Then $d(x_k, x_m)^2 = \sum_{i=1}^n (x_{ki} - x_{mi})^2 \rightarrow 0$ as $m, k \rightarrow \infty$

For each $\epsilon > 0$, $\exists N > 0$, $\forall k, m > N \Rightarrow \left| \sum_{i=1}^n (x_{ki} - x_{mi})^2 \right| < \epsilon$

$(x_{ki} - x_{mi})^2 \rightarrow 0$ for $m, k \rightarrow \infty$

For $\epsilon > 0$, $\exists N$, $| (x_{ki} - x_{mi})^2 | \leq \left| \sum_{j=1}^n (x_{kj} - x_{mj})^2 \right| / \epsilon$

$\Rightarrow \{x_{ki}\}_{k=1}^{\infty}$ converges in \mathbb{R}

$\lim_{m \rightarrow \infty} (x_{mi}) = x_{oi} \in \mathbb{R}$

$\Rightarrow \{\vec{x}_k\}$ converges.

\rightarrow Complete!

Limits and Continuity in $V, \|\cdot\|$

Assume V is a n -dimensional vector space w/ norm $\|\cdot\|: V \rightarrow \mathbb{R}$ s.t [this make V a normed linear space]

1) $\|cv\| = |c| \|v\|$

2) $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$

3) $\|v+w\| \leq \|v\| + \|w\|$

We can build a metric from $\|\cdot\|$ as follows.

$$d(P, Q) = \|Q - P\| \quad \forall P, Q \in V$$

You can define $B_\delta(x_0) = \{x \in V \mid d(x, x_0) < \delta\}$

and $U \subseteq V$ is open $\exists \delta > 0$ s.t $B_\delta(x_0) \subseteq U \forall x_0 \in U$

If V is also complete, i.e. each Cauchy sequence \Rightarrow converges, then V is a Banach Space (Complete normed linear space)

Def: If $f: V \rightarrow W$ where V and W are Banach spaces.

$$\lim_{x \rightarrow x_0} f(x) = y_0 \quad \text{iff}$$

$$\forall \epsilon > 0. \exists \delta > 0 \text{ s.t } x \in V \text{ w/ } 0 < \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$$

when $f(x_0) = y_0 \Rightarrow f$ is continuous at x_0 .

when f is continuous at each $x \in V \Rightarrow f$ is cont on V
i.e. f is continuous on domain V

Theorem: 1) $\lim (f+g) = \lim f + \lim g$

2) $\lim (cf) = c \lim f$.

3) If $f, g: V \rightarrow \mathbb{R}$ then fg is defined

$$\lim fg = (\lim f)(\lim g)$$

provided $\lim f$ and $\lim g$ exists.

4) $\lim \frac{f}{g} = \frac{\lim f}{\lim g}$, $\lim g \neq 0$

5) $\lim_{x \rightarrow x_0} \|f\| = \|\lim_{x \rightarrow x_0} f\| (\checkmark)$

①, ②, ③ $f: V \rightarrow W$; V, W are banach space.

Comment: If $f, g: V \rightarrow W$ and W permits some product $*: W \times W \rightarrow \tilde{W}$ it usually the case true that (defining $f * g$ point wise)

$$\lim_{x \rightarrow x_0} (f * g) = (\lim_{x \rightarrow x_0} f) * (\lim_{y \rightarrow y_0} g)$$

Here \tilde{W} could be $W, \mathbb{R}, \mathbb{C}$, etc...

Dot, cross, matrix products all repeat the matrix space structure built on $\mathbb{R}^n, \mathbb{R}^3, \mathbb{R}^{m \times n}$

Ex: $V = \mathbb{R}^{n \times n}$ then

$\|A\| = \text{Trace}(AA^T)$ defines V as a Normed linear space and you can prove $A, B : W \rightarrow \mathbb{R}^n$ then $\lim_{x \rightarrow x_0} [A(x)B(x)] = [\lim_{x \rightarrow x_0} A(x)][\lim_{x \rightarrow x_0} B(x)]$.

Remark: As we proved the other day :

$$A, B \in \mathbb{R}^{n \times n} \Rightarrow \|AB\| \leq \|A\|\|B\|$$

this makes the complete, normed, linear space $\mathbb{R}^{n \times n}$ also an algebra (wrt matrix multiplication)

\Rightarrow It is a Banach Algebra.

Rehearsal: $f : V \rightarrow W$, Banach Spaces.

and $\{f_1, f_2, \dots, f_n\} = \beta_W$ forms a basis for W

then $\lim_{x \rightarrow x_0} f(x) = y_0$ if $\lim_{x \rightarrow x_0} [f(x)]_j = [y_0]_j$

$$\text{where } f(x) = \sum_{j=1}^n [f(x)]_j f_j$$

like-wise for y_0

The limit of a vector-valued function is the vector of the limit for its components

Proof: $\xrightarrow{(\Rightarrow)}$ $\lim_{x \rightarrow x_0} f(x) = y$,

Consider $[f(x)]_j$. Need to show $\lim_{x \rightarrow x_0} [f(x)]_j = [y_j]$

$$|[f(x)]_j| = \frac{1}{\|f_j\|} \| [f(x)]_j f_j \| \leq \frac{1}{\|f_j\|} \|f(x)\|$$

Let $\varepsilon > 0$. $\exists \delta > 0$ s.t. $\forall x$ satisfying $0 < |x - x_0| < \delta$
 $\Rightarrow \|f(x) - y_0\| < \|f_j\| \varepsilon$.

Consider

$$|[f(x)]_j - y_j| = \frac{1}{\|f_j\|} \| [f(x)]_j f_j - (y_j) f_j \| \leq \frac{1}{\|f_j\|} \|f(x) - y_0\| \leq \frac{1}{\|f_j\|} \|f_j\| \varepsilon = \varepsilon$$

→ Done.

(\Leftarrow) $\lim_{x \rightarrow x_0} [f(x)]_j = [y_j] \quad \forall j = \overline{1, n}$

Let $\varepsilon > 0$. Choose $\delta = \min(\delta_1, \delta_2, \dots, \delta_n)$ s.t
 $0 < |x - x_0| < \delta_j \Rightarrow |[f(x)]_j - [y_j]| < \frac{\varepsilon}{n \|f_j\|}$

$$\|f(x) - y_0\| = \left\| \sum_{j=1}^n ([f(x)]_j - [y_j]) f_j \right\| < \sum_{j=1}^n \|([f(x)]_j - [y_j]) f_j\| = \sum_{j=1}^n \| [f(x)]_j - [y_j] \| \cdot \frac{\varepsilon}{n \|f_j\|} = \varepsilon$$

Theorem: If $f \circ g$ is defined and limit below exist then

$$\lim_{x \rightarrow x_0} [f(g(x))] = f(\lim_{\substack{x \rightarrow x_0 \\ \text{if } y_0}} g(x))$$

(We assume continuity of f at y_0 here)

Proof:

Given

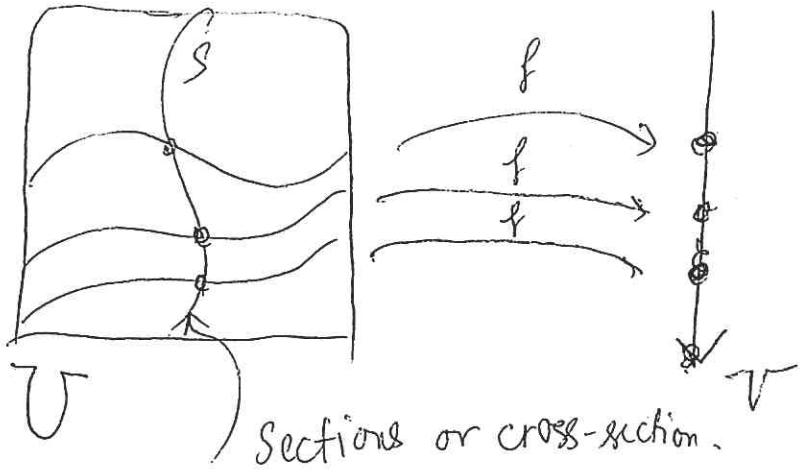
General Comment: U, V sets.

$$f: U \longrightarrow V$$

$$\{ f^{-1}\{v\} / v \in V \}$$

$x_1 \sim x_2$ iff $\exists y \in V$ s.t. $x_1, x_2 \in f^{-1}\{y\}$
defines an equivalence relation

the partition of \sim is that which chops U into fibers.



To make f a bijection:

- (1) reduce V by replacing $V = \text{Range}(f) = f(U)$
- (2) Replace U w/ S where $S \cap f^{-1}\{y\}$ is a singleton
for each $y \in V$