HOMEWORK 10 : §10.1 # 7 : CALCULUS II : (STEWART 6th Ed.)

§10.1 # 7

(a.) What can we say about a solⁿ of $\frac{dy}{dx} = -y^2$ just from looking at the DEqⁿ?

Notice $y^2 \geq 0$ thus $-y^2 \leq 0$ and so we find either $y = 0$ for all x or $y \neq 0$ but y is decreasing.

(b.) Verify that all members of $y = \frac{1}{x+c}$ are solⁿ's of $y' = -y^2$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x+c}\right) = \frac{-1}{(x+c)^2}\frac{d}{dx}(x+c) = \frac{-1}{(x+c)^2} = -y^2.$$

Thus, $y = 1/(x+c)$ is a solⁿ of $y' = -y^2$.

(c.) Is there a solⁿ of $y' = -y^2$ that is not in the family of curves given in (b)?

YES. It's not hard to guess $y = 0$ is a solⁿ. Moreover, once the guess is made it's easy to verify that $y' = -y^2$ if $y \equiv 0$.

identically zero. (it's zero for all x)

Notice

$$\frac{dy}{dx} = -y^2$$

$$\Rightarrow \underbrace{\frac{dy}{-y^2} = dx}_{} \Rightarrow \frac{1}{y} = x + c \Rightarrow y = \frac{1}{x+c}$$

this step assumes $y \neq 0$, we lose the $y = 0$ solⁿ in the sep. of variables calculation. Solⁿ's like this are called "exceptional"

(d.) Find solⁿ of $y' = -y^2$ such that $y(0) = 0.5$.

$$0.5 = \frac{1}{0+c} \Rightarrow \frac{1}{2} = \frac{1}{c} \Rightarrow \underline{c = 2}. \Rightarrow \underline{y = \frac{1}{x+2}}.$$

§10.3 # 1

$$\frac{dy}{dx} = \frac{y}{x} \overset{①}{\Rightarrow} \int \frac{dy}{y} = \int \frac{dx}{x} \Rightarrow \boxed{\ln|y| = \ln|x| + C}.$$

(implicit sol^{n})

We can solve for y here. Take exp of both sides,

$$e^{\ln|y|} = e^{\ln|x|+C} = e^{\ln|x|}e^{C} \Rightarrow |y| = e^{C}|x|$$

$$\Rightarrow y = \pm e^{C} x$$

$$\Rightarrow \boxed{y = kx \quad \text{for } k \neq 0}.$$

Notice $\boxed{y=0}$ is also a sol^{n} here, we lost it in step ①.

§10.3 # 2

$$\frac{dy}{dx} = \frac{\sqrt{x}}{e^{y}} \Rightarrow \int e^{y}\,dy = \int \sqrt{x}\,dx$$

$$\Rightarrow e^{y} = \frac{2}{3}x^{3/2} + C$$

$$\Rightarrow \boxed{y = \ln\left(\frac{2}{3}x^{3/2} + C\right)}$$

§10.3 # 8

$$\frac{dy}{d\theta} = \frac{e^{y}\sin^2\theta}{y\sec\theta} \Rightarrow \underbrace{\int ye^{-y}\,dy}_{\text{I}} = \underbrace{\int \frac{\sin^2\theta}{\sec\theta}\,d\theta}_{\text{II}}$$

I: $\int \underbrace{y}_{u}\underbrace{e^{-y}dy}_{dv} = -ye^{-y} + \int e^{-y}\,dy = \underline{-ye^{-y} - e^{-y} + C_1}$.

II: $\int \frac{\sin^2\theta}{\sec\theta}\,d\theta = \int (\sin^2\theta)\cos\theta\,d\theta = \int u^2\,du = \underline{\frac{1}{3}\sin^3\theta + C_2}$.

Thus, equating I & II and including an arbitrary constant,

$$\boxed{-ye^{-y} - e^{-y} = \frac{1}{3}\sin^3\theta + C}.$$

(I can't find a nice explicit sol^{n} here.)

§10.3 # 11 Solve $\frac{dy}{dx} = \frac{x}{y}$ given $y(0) = -3$.

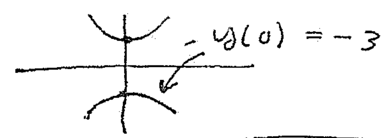
$$\frac{dy}{dx} = \frac{x}{y} \implies \int y\,dy = \int x\,dx$$

$$\tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C$$

$$y(0) = -3 \implies \tfrac{1}{2}(9) = \tfrac{1}{2}(0) + C \quad \therefore \ C = \tfrac{9}{2}$$

$$\implies \tfrac{1}{2}y^2 = \tfrac{1}{2}y \implies \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + \tfrac{9}{2}$$

$$\text{or} \quad \underbrace{y^2 - x^2 = 9.}_{\text{hyperbola}}$$



<u>We want the lower branch</u> $\implies \boxed{y = -\sqrt{9 + x^2}}$

§10.3 #12 Solve $y' = (y\cos x)/(1+y^2)$ given $y(0) = 1$

$$\frac{dy}{dx} = \frac{y\cos(x)}{1+y^2}$$

$$\implies \underbrace{\int \left(\frac{1+y^2}{y}\right) dy}_{\text{I}} = \underbrace{\int \cos(x)\,dx}_{\text{II.}}$$

Clearly II. $= \sin(x) + C_1$. Consider,

(I.) $\int \left(\frac{1+y^2}{y}\right) dy = \int \left(\frac{1}{y} + y\right) dy = \ln|y| + \frac{1}{2}y^2 + C_2$

Thus,

$$\underline{\ln|y| + \tfrac{1}{2}y^2 = \sin(x) + C} \quad \text{(implicit general sol}^{\text{n}}\text{)}$$

Apply the initial condition,

$$y(0) = 1 : \quad \ln(1) + \tfrac{1}{2} = \sin(0) + C \quad \therefore \ \underline{C = 1/2}.$$

$$\therefore \ \boxed{\ln|y| + \tfrac{1}{2}y^2 = \sin(x) + \tfrac{1}{2}}$$

$\boxed{\text{§10.3 \#16}}$ Solve $xy' + y = y^2$ given $y(1) = -1$

$$x\frac{dy}{dx} = y^2 - y$$

$$\underbrace{\int \frac{dy}{y^2-y}}_{\text{I.}} = \underbrace{\int \frac{dx}{x}}_{\text{II}}$$

$$\text{II} = \ln|x| + C_2.$$

Notice $\frac{1}{y^2-y} = \frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$. We find A, B as usual: $1 = A(y-1) + By \Rightarrow \underline{1 = B}$, or $1 = -A \therefore \underline{A = -1}$.

Hence,

$$\text{I.}: \int \frac{dy}{y^2-y} = \int\left(\frac{-1}{y} + \frac{1}{y-1}\right)dy = \underline{-\ln|y| + \ln|y-1| + C}.$$

Equating I & II yields,

$$-\ln|y| + \ln|y-1| = \ln|x| + C$$

$$\Rightarrow \ln\left|\frac{y-1}{y}\right| = \ln|x| + C$$

$$\Rightarrow \ln\left|1 + \frac{1}{y}\right| = \ln|x| + C$$

$$\Rightarrow |1 - 1/y| = e^c|x| \Rightarrow \underbrace{1 - \frac{1}{y} = \pm e^c x}$$

$$\Rightarrow \frac{1}{y} = 1 + kx$$

$$\Rightarrow \underline{y = \frac{1}{1+kx}}.$$

Using the given $y(1) = -1 \Rightarrow -1 = \frac{1}{1+k} \Rightarrow \underline{k = -2}$.

$$\therefore \boxed{y = \frac{1}{1-2x}}$$

$\boxed{\text{§10.3 \# 22) Solve } xy' = y + xe^{y/x} \text{ by subst. } v = y/x}$

If $v = y/x$ then $y = vx$ thus $y' = v'x + v$. Hence,

$$\underbrace{x(v'x+v) = vx + xe^{v}}$$

substituted v for y wherever possible using $y = vx$ and $y' = v'x + v$.

$$x^2 v' + \cancel{vx} = \cancel{vx} + xe^{v}$$

$$x^2 \frac{dv}{dx} = xe^{v}$$

$$\Rightarrow \int e^{-v}\,dv = \int \frac{1}{x}\,dx$$

$$-e^{-v} = \ln|x| + C \quad \Rightarrow \quad -e^{-\frac{y}{x}} = \ln|x| + c$$

$$\Rightarrow \exp\left(\frac{-y}{x}\right) = k - \ln|x|$$

$$\Rightarrow \frac{-y}{x} = \ln(k - \ln|x|)$$

$$\Rightarrow \boxed{y = -x\ln(k - \ln|x|)}.$$

$\boxed{\text{§10.3\#30) } y^2 = kx^3 \text{ gives a family of curves. Find the O.T. for this family and sketch a few}}$

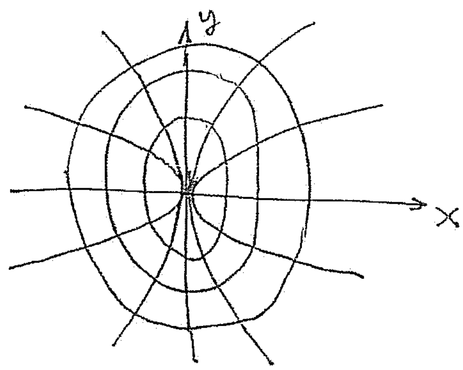
If $y^2 = kx^3$ then $2\ln(y) = \ln(k) + 3\ln(x)$

hence these curves satisfy $\frac{2}{y}\frac{dy}{dx} = \frac{3}{x} \Rightarrow \left.\frac{dy}{dx}\right|_{curve} = \frac{3y}{2x}$

Orthogonal Traj will have $\left.\frac{dy}{dx}\right|_{O.T.} = \frac{-2x}{3y}$ hence solve

$$\int 3y\,dy = \int -2x\,dx \Rightarrow \frac{3}{2}y^2 = -x^2 + C \Rightarrow \boxed{\frac{y^2}{2/3} + x^2 = k}$$

the O.T. are ellipses!



(hopefully you found nicer pictures via Mathematica or some such technology.)

§10.5 #8 | Solve $x^2y' + 2xy = \cos^2 x$

$$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{1}{x^2}\cos(x) \quad : \text{ put into } \underline{\text{standard}}\ \underline{\text{form}}$$

Calculate int. factor, $\mu = \exp\left(\int \frac{2}{x}\,dx\right) = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2$.
Multiply by $\mu = x^2$ on the standard form DE$_q^n$,

$$\underbrace{x^2\frac{dy}{dx} + 2xy}_{} = \cos(x)$$

(integrate both sides w.r.t. x)

$$\frac{d}{dx}\left(x^2 y\right) = \cos(x) \implies x^2 y = \sin(x) + C$$

$$\therefore \boxed{y = \frac{1}{x^2}\sin(x) + \frac{C}{x^2}.}$$

§10.5 #10 | Solve $y' + y = \sin(e^x)$

observe $P = 1$ and the given DE$_q^n$ is already in standard form,

$$\mu = \exp\left(\int 1\,dx\right) = \exp(x) = e^x \Leftarrow \text{ the integrating factor.}$$

Multiply DE$_q^n$ $\mu = e^x$,

$$\underbrace{e^x\frac{dy}{dx} + e^x y}_{} = e^x\sin(e^x)$$

$$\frac{d}{dx}\left(e^x y\right) = e^x\sin(e^x) \implies e^x y = \int e^x \sin(e^x)\,dx$$

$$\implies e^x y = \int \sin(u)\,du, \quad u = e^x$$

$$\implies e^x y = -\cos(e^x) + C$$

$$\therefore \boxed{y = -e^{-x}\cos(e^x) + Ce^{-x}}$$

$\boxed{\S 10.5 \# 17}$ Solve $\frac{dV}{dt} - 2tV = 3t^2e^{t^2}$, $V(0) = 5$

Calculate $\mu = \exp\left(\int -2t\,dt\right) = \exp(-t^2)$. Multiply by the DE$_q$ by $\mu = e^{-t^2}$,

$$\underbrace{e^{-t^2}\frac{dV}{dt} - 2te^{-t^2}V}_{} = 3t^2e^{t^2}e^{-t^2} = 3t^2$$

$$\frac{d}{dt}\left[e^{-t^2}V\right] = 3t^2$$

Integrate w.r.t. t both sides,

$$e^{-t^2}V = t^3 + C$$

$$\underline{V(t) = t^3e^{t^2} + Ce^{t^2}}.$$

$$V(0) = 5 = 0 + C \therefore C = 5 \Rightarrow \boxed{V(t) = t^3e^{t^2} + 5e^{t^2}.}$$

$\boxed{\S 10.5 \# 18}$ Solve $2xy' + y = 6x$, $x > 0$ and $y(4) = 20$

$$\frac{dy}{dx} + \left(\frac{1}{2x}\right)y = 3 \Rightarrow \mu = \exp\left(\int \frac{dx}{2x}\right) = e^{\frac{1}{2}\ln|x|} = e^{\ln|x|^{1/2}} = \sqrt{x}$$

assuming $x > 0$ as was given.

Multiply the DE$_q^n$ by $\mu = \sqrt{x}$,

$$\underbrace{\sqrt{x}\frac{dy}{dx} + \frac{1}{2\sqrt{x}}y}_{} = 3\sqrt{x}$$

$$\frac{d}{dx}\left[\sqrt{x}\,y\right] = \sqrt{x}$$

$$\sqrt{x}\,y = 3\left(\frac{2}{3}x^{3/2}\right) + C$$

$$\underline{y(x) = 2x + C/\sqrt{x}}.$$

$$y(4) = 2(4) + C/\sqrt{4} = 20 \Rightarrow \frac{C}{2} = 12 \Rightarrow \underline{C = 24}.$$

$$\therefore \boxed{y(x) = 2x + \frac{24}{\sqrt{x}}}$$

§10.5 # 23 The Bernoulli DEq$^{\underline{n}}$ $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$ is linear if $n=0$ or $n=1$. However, if $n>1$ then we can transform it to a linear DEq$^{\underline{n}}$ of the form

$$\frac{du}{dx} + (1-n)P(x)\,u = (1-n)\,Q(x)$$

given the substitution $u = y^{1-n}$

Let $u = y^{1-n}$ then $y = u^{\frac{1}{1-n}}$ if $n \neq 1$.

Differentiate w.r.t. x ;

$$\frac{dy}{dx} = \frac{1}{1-n}\,u^{\left(\frac{1}{1-n}-1\right)}\frac{du}{dx} = \left(\frac{1}{1-n}\right)u^{\frac{1-(1-n)}{1-n}}\frac{du}{dx}$$

Hence,

$$\frac{dy}{dx} = \left(\frac{1}{1-n}\right)u^{\frac{n}{1-n}}\frac{du}{dx}$$

Substitute into the Bernoulli DEq$^{\underline{n}}$,

$$\left(\frac{1}{1-n}\right)u^{\frac{n}{1-n}}\frac{du}{dx} + P(x)\,u^{\frac{1}{1-n}} = Q(x)\left(u^{\frac{1}{1-n}}\right)^n$$

$$\frac{du}{dx} + (1-n)\,u^{\left(\frac{1}{1-n}-\frac{n}{1-n}\right)}P(x) = (1-n)\,Q(x)$$

Note that $\frac{1}{1-n} - \frac{n}{1-n} = \frac{1-n}{1-n} = 1$. Hence, $\underline{\frac{du}{dx} + (1-n)\,u\,P = (1-n)\,Q}$ //

§10.5 # 26 Substitute $u = y'$ to solve the following DEq$^{\underline{n}}$:

$$xy'' + 2y' = 12x^2 \implies xu' + 2u = 12x^2$$

$$\implies \frac{du}{dx} + \frac{2}{x}u = 12x$$

$$\implies \mu = \exp\left(\int \frac{2dx}{x}\right) = \exp\left(2\ln|x|\right) = \exp\left(\ln|x|^2\right) = x^2.$$

$$x^2\frac{du}{dx} + 2xu = 12x^3$$

Thus, $\frac{d}{dx}\left(x^2u\right) = 12x^3 \implies x^2u = 3x^4 + C_1$

$$\implies u = \frac{3x^4+C_1}{x^2} = \frac{dy}{dx}$$ now integrate $\underline{\underline{\text{again}}}$

§10.5 #26 Continued

$$\frac{dy}{dx} = 3x^2 + \frac{C_1}{x^2}$$

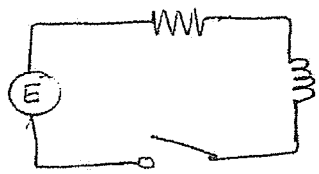
Integrate both sides w.r.t. x,

$$\boxed{y(x) = x^3 - \frac{C_1}{x} + C_2}$$

Remark: we have two arbitrary constants since this was a 2^{nd} order DEq^{t}.

§10.5 # 27

Fig. 4



$$L\frac{dI}{dt} + RI = E(t)$$

Solve given that
$E(t) = 40V$ (constant)
$L = 2H$ and $R = 10\Omega$
and $I(0) = 0A$.

(a.)

$$2\frac{dI}{dt} + 10I = 40$$

$$\frac{dI}{dt} + 5I = 20 \quad , \quad \mu = \exp\left(\int 5\,dt\right) = e^{5t}$$

$$\underbrace{e^{5t}\frac{dI}{dt} + 5e^{5t}I}_{} = 20e^{5t} \qquad \text{(multiplied by } \mu\text{)}$$

$$\frac{d}{dt}\left[e^{5t}I\right] = 20e^{5t}$$

$$e^{5t}I = \frac{20}{5}e^{5t} + C_1$$

$$I(t) = 4 + C_1e^{-5t}$$

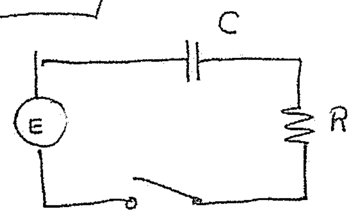
$I(0) = 4 + C_1 = 0 \quad \therefore \quad C_1 = -4$.

$$\therefore \quad \boxed{I(t) = 4 - 4e^{-5t}.}$$

(b.) $I(0.1) = 4 - 4e^{-5(0.1)} \cong \boxed{1.57 \cong I(0.1)}$

[both in units of Amps.]

§10.5 #29



$$RI + \frac{Q}{C} = E(t)$$

However, $I = dQ/dt$ thus

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Suppose $R = 5\,\Omega$, $C = 0.05\,F$, $E(t) = 60V$ and $Q(0) = 0$.
Find $Q(t)$ and $I(t)$

$$5\frac{dQ}{dt} + \frac{1}{0.05}Q = 60$$

$$\frac{dQ}{dt} + \frac{1}{0.25}Q = 12$$

$$\frac{dQ}{dt} + 4Q = 12 \Rightarrow \mu = e^{\int 4dt} = e^{4t}.$$

$$e^{4t}\frac{dQ}{dt} + 4e^{4t}Q = 12e^{4t}$$

$$\frac{d}{dt}\left[e^{4t}Q\right] = 12e^{4t}$$

$$e^{4t}Q = 3e^{4t} + C_1$$

$$\boxed{Q(t) = 3 + C_1e^{-4t}} \Rightarrow Q(0) = 0 = 3 + C_1$$

$$\therefore \underline{C_1 = -3}.$$

$$Q(t) = 3 - 3e^{-4t}$$

$$I(t) = \frac{dQ}{dt} = \frac{d}{dt}\left(3 - 3e^{-4t}\right)$$

$$\therefore \boxed{I(t) = 12e^{-4t}}$$

Remark: as $t \to \infty$ notice that $Q \to 3$ and $I \to 0$. For large times all the voltage is dropped across the capacitor which is like an open circuit once charged.

Homework 13 : Calculus II : §10.4 # 7, 8 (Stewart 6th Ed.)

§10.4 #7 Consider population of 1000 people.

Let R = # who have heard rumor

N = # who haven't heard rumor

y = fraction who've heard rumor $= \dfrac{R}{1000}$

We have $R + N = 1000$ and $R = 1000\,y$.

$$\frac{dR}{dt} = k\left(\frac{R}{1000}\right)\left(\frac{1000-R}{1000}\right)$$

$$\Rightarrow \quad 1000\,\frac{dy}{dt} = k\,y\,(1-y)$$

$$\Rightarrow \quad \frac{dy}{dt} = M\,y\,(1-y)$$

(answer to part a.)

(b.) Solve it.

Recall (or observe) $\dfrac{1}{y(1-y)} = \dfrac{1}{y-y^2} = \dfrac{1-y+y}{y(1-y)} = \dfrac{1}{y} + \dfrac{1}{1-y}$

$$\int \frac{dy}{y(1-y)} = \int M\,dt \longrightarrow \int\left(\frac{1}{y} - \frac{1}{y-1}\right)dy = \int M\,dt$$

$$\ln|y| - \ln|y-1| = Mt + C_1$$

$$\ln\left|\frac{y}{y-1}\right| = Mt + C_1$$

$$\ln\left|\frac{y-1}{y}\right| = -Mt - C_1$$

$$1 - \frac{1}{y} = k\,e^{-Mt}$$

$$\therefore \quad \boxed{y = \frac{1}{1-ke^{-Mt}}}$$

§ 10.4 # 7 Continued

At 8AM, 80 people have heard rumor
At noon, 500 people have heard the rumor
At what time will 900 people have heard rumor?

Let $t=0$ be 8AM and $t=4$ be noon.

$$y(0) = \frac{80}{1000} = \frac{1}{1-k} \Rightarrow 1000 = 80 - 80k$$

$$\therefore k = \frac{-920}{80} = \underline{-11.5 = k}$$

$$y(4) = \frac{500}{1000} = \frac{1}{1-ke^{-4m}}$$

$$2 = 1 + 11.5e^{-4m} \Rightarrow \frac{1}{11.5} = e^{-4m}$$

$$\Rightarrow 11.5 = e^{4m}$$

$$\Rightarrow m = \frac{1}{4}\ln(11.5)$$

$$\Rightarrow \underline{m \cong 0.6106}$$

$$y(t) = \frac{1}{1+11.5e^{-0.6106t}}$$

Find t s.t. $y(t) = \frac{900}{1000} = \frac{1}{1+11.5e^{-0.6106t}}$

$$1000 = 900 + 900(11.5)e^{-0.6106t}$$

$$100 = 900(11.5)\exp(-0.6106t)$$

$$\frac{1}{9(11.5)} = \frac{1}{\exp(0.6106t)}$$

$$t = \frac{1}{0.6106}\ln(9(11.5)) \cong 7.5984$$

$$t = 8AM + 7.598 \text{ Hours}$$

$$\longrightarrow \boxed{t = 3:35:24 \text{ PM}}$$

(I leave § 10.4 # 8 to you to consider)