Method of Frobenius : §8.6 and §8.7

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<u>Remark:</u> the method presented by the text avoids certain pathological issues. If you wish to learn more you might look at "Advanced Engineering Mathematics" by Peter V. O'Neil. It presents a more general version that treats cases we ignore for now.

§8.6 #1 Classify each singular point of the following DEq$^{\underline{n}}$ as <u>regular</u> or <u>irregular</u>

$$(x^2-1)y'' + xy' + 3y = 0$$

Write in standard form,

$$y'' + \underbrace{\left(\frac{x}{x^2-1}\right)}_{P(x)} y' + \underbrace{\left(\frac{3}{x^2-1}\right)}_{q(x)} y = 0$$

(using notation of Def$^{\underline{n}}$ (3) pg. 488)

Since $x^2-1 = (x+1)(x-1)$ we have two singular points, $x_0 = -1$ and $x_0 = 1$. Observe:

$$(x+1)P(x) = \frac{x}{x-1} \quad \text{and} \quad (x+1)^2 q(x) = \frac{3(x+1)}{x-1}$$

Hence $x_0 = -1$ is a regular singular point since $(x+1)P(x)$ and $(x+1)^2 q(x)$ are analytic at $x_0 = -1$. Likewise,

$$(x-1)P(x) = \frac{x}{x+1} \quad \text{and} \quad (x-1)^2 q(x) = \frac{3(x-1)}{x+1}$$

are analytic at $x_0 = 1$ thus $x_0 = 1$ is also a <u>regular</u> <u>singular</u> pt.

§8.6 #3 Classify singular pts. of $(x^2+1)z'' + 7x^2 z' - 3xz = 0$

$$z'' + \left(\frac{7x^2}{x^2+1}\right)z' - \left(\frac{3x}{x^2+1}\right)z = 0$$

Since $x^2+1 = (x+i)(x-i)$ clearly $x = \pm i$ are singular points.

Moreover,

$$\left.\begin{aligned} (x \pm i)\left[\frac{7x^2}{(x+i)(x-i)}\right] &= \frac{7x^2}{(x \mp i)} \\ (x \pm i)^2\left[\frac{-3x}{(x+i)(x-i)}\right] &= \frac{-3x(x \pm i)}{(x \mp i)} \end{aligned}\right\} \text{both analytic at } x_0 = \mp i$$

Thus both $x_0 = i$ and $x_0 = -i$ are <u>regular</u> <u>singular</u> <u>points</u>.

§ 8.6 # 13 Find the indicial equation and the exponents for the specified singularity of

$$(x^2-x-2)^2 z'' + (x^2-4)z' - 6xz = 0 \quad \text{at } x = 2$$

To begin put the DEqⁿ in standard form so we can identify P and q for the eqⁿ,

$$z'' + \underbrace{\left[\frac{x^2-4}{(x^2-x-2)^2}\right]}_{P(x)} z' - \underbrace{\left[\frac{6x}{(x^2-x-2)^2}\right]}_{q(x)} z = 0$$

$$P_o \equiv \lim_{x\to 2}\left((x-2)\frac{x^2-4}{(x^2-x-2)^2}\right) = \lim_{x\to 2}\left(\frac{\cancel{(x-2)^2}(x+2)}{(x+1)^2\cancel{(x-2)^2}}\right) = \frac{2+2}{(2+1)^2} = \frac{4}{9}.$$

$$q_o \equiv \lim_{x\to 2}\left(\cancel{(x-2)^2}\left(\frac{-6x}{(x+1)^2\cancel{(x-2)^2}}\right)\right) = \lim_{x\to 2}\left(\frac{-6x}{(x+1)^2}\right) = \frac{-12}{9} = \frac{-4}{3}.$$

Hence we find the following indicial equation (relative to the regular singular point $x_o = 2$)

$$r(r-1) + P_o r + q_o = 0 \qquad \text{(general form)}$$

$$r^2 - r + \frac{4}{9}r - \frac{4}{3} = 0$$

$$\boxed{r^2 - \frac{5}{9}r - \frac{4}{3} = 0} \longleftarrow \underline{\text{indicial equation}}$$

$$9r^2 - 5r - 12 = 0$$

$$r = \frac{5 \pm \sqrt{25 - 4(9)(-12)}}{18} = \underbrace{\boxed{\frac{5 \pm \sqrt{457}}{18} = r}}_{\text{"exponents"}}$$

Remark: this is just a warm-up for the later interesting problems (#19 ⟶ #47). The text simply wishes to make certain definitions clear before we attempt the entire method.

§8.6 #19 Use the method of Frobenius to find at least four nonzero terms in the series expansion about $x=0$,

$$9x^2y'' + 9x^2y' + 2y = 0 \quad (\text{assume } x > 0)$$

We wish to solve $y'' + y' + \frac{2}{9x^2}y = 0$. Identify that $P(x) = 1$ and $q(x) = 2/9x^2$. Note,

$$p_o = \lim_{x\to 0}[x(1)] = 0 \quad \text{while} \quad q_o = \lim_{x\to 0}\left(x^2\frac{2}{9x^2}\right) = \frac{2}{9}.$$

We seek a sol$^{\underline{n}}$ of the form

$$W(r,x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

However, r must satisfy indicial eq$^{\underline{n}}$:

$$r(r-1) + p_o r + q_o = 0 \Rightarrow r^2 - r + \frac{2}{9} = 0$$

$$\Rightarrow r = \frac{1 \pm \sqrt{1 - 8/9}}{2} = \frac{1 \pm 1/3}{2}$$

$$\Rightarrow \underline{r = \frac{2}{3} \text{ or } \frac{1}{3}}.$$

We should use the larger root (it turns out there is always a sol$^{\underline{n}}$ for the larger root, however the smaller not necessarily so.)

$$W(2/3, x) = \underbrace{\sum_{n=0}^{\infty} a_n x^{n+2/3}}$$

It finally hits me! There are <u>not</u> necessarily power series. This is quite a departure from §8.3–8.4.

Let's continue now that I've pointed out the obvious, we should return to the given differential eq$^{\underline{n}}$ to avoid working with rational functions.

§8.6 #19 Continued

Given $w(r,x) = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow w'(r,x) = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$

and $w''(r,x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)x^{n+r-2}$. We wish to

solve $9x^2y'' + 9x^2y' + 2y = 0$ with $w(r,x)$. Substituting,

$$9x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + 9x^2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + 2\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

We deduced $r = 2/3$ on the previous page,

$$\sum_{n=0}^{\infty} 9(n+\tfrac{2}{3})(n-\tfrac{1}{3})a_n x^{n+r} + \sum_{n=0}^{\infty} 9(n+2/3)a_n x^{n+r+1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty}\Big((3n+2)(3n-1)a_n + (9(n-1)+6)a_{n-1} + 2a_n\Big)x^{n+r} +$$

$$+\Big(9(\tfrac{2}{3})(\tfrac{-1}{3})a_0 + 2a_0\Big)x^r = 0$$

x^r: $-2a_0 + 2a_0 = 0$ (consistent, but useless)

x^{1+r}: $10a_1 + 6a_0 + 2a_1 = 0 \Rightarrow 6a_0 = -12a_1 \Rightarrow \underline{a_1 = \frac{-1}{2}a_0}$.

x^{2+r}: $40a_2 + 15a_1 + 2a_2 = 0 \Rightarrow a_2 = \frac{-15}{42}a_1 = \frac{-5}{14}\left(\frac{-a_0}{2}\right) = \underline{\frac{5}{28}a_0 = a_2}$

x^{3+r}: $88a_3 + 24a_2 + 2a_3 = 0 \Rightarrow a_3 = \frac{-24}{90}a_2 = \frac{-4}{15}\left(\frac{5}{28}a_0\right) = \underline{\frac{-1}{21}a_0 = a_3}$.

Let's assemble the sol$^{\underline{n}}$,

$$\boxed{w(2/3, x) = a_0\left[x^{2/3} - \frac{1}{2}x^{5/3} + \frac{5}{28}x^{8/3} - \frac{1}{21}x^{11/3} + \cdots\right]}$$

<u>Remark</u>: this is <u>not</u> a general sol$^{\underline{n}}$. We still need another calculation to find the 2$^{\underline{nd}}$ linearly independent sol$^{\underline{n}}$. The method for that 1/2 of the Frobenius method is given in §8.7. (see §8.7 #1 for completion of this problem)

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§8.6 #23) Solve $x^2 z'' + (x^2+x)z' - z = 0$ by the Method of Frobenius. (find the first four terms in a sol$^{\underline{n}}$ centered at $x=0$)

Observe that $P(x) = \frac{x^2+x}{x^2}$ and $q(x) = \frac{-1}{x^2}$. We calculate the exponents; $P_0 = \lim_{x\to 0}\left[x\left[\frac{x^2+x}{x^2}\right]\right] = \lim_{x\to 0}\left[\frac{x^3}{x}+\frac{x^2}{x^2}\right] = 1$, also $q_0 = \lim_{x\to 0}\left[x^2\left(\frac{-1}{x^2}\right)\right] = -1$. We find the indicial equation,

$$r(r-1) + P_0 r + q_0 = r^2 - r + r - 1 = r^2 - 1 = 0 \quad \therefore \underline{r = \pm 1}.$$

We'll use $r=1$, assume $w(1,x) = \sum_{n=0}^{\infty} a_n x^{n+1}$ is a sol$^{\underline{n}}$,

$$x^2 \sum_{n=0}^{\infty} a_n (n+1) n x^{n-1} + (x^2+x)\sum_{n=0}^{\infty} a_n (n+1) x^n - \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty}(n+1)a_n\left(x^{n+2} + x^{n+1}\right) - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

switch index on $\sum_{n=0}^{\infty}(n+1)a_n x^{n+2} = \sum_{j=1}^{\infty} j a_{j-1} x^{j+1}$

(this means I need to separate the $n=0$ case in the next equation,

$$\left(0(1)a_0 + (0+1)a_0 - a_0\right)x + \sum_{n=1}^{\infty}\left(\left[n(n+1)+(n+1)-1\right]a_n + n a_{n-1}\right)x^{n+1} = 0$$

The $n=0$ term is consistent but not interesting, $n=1,2,3,\dots$ yield useful conditions on the coefficients a_n; $\left[(n+1)^2 - 1\right]a_n + n a_{n-1} = 0$,

$\underline{n=1}$) $3a_1 + a_0 = 0 \Rightarrow \underline{a_1 = \frac{-1}{3}a_0}$.

$\underline{n=2}$) $8a_2 + 2a_1 = 0 \Rightarrow a_2 = \frac{-2}{8}\left(\frac{-a_0}{3}\right) = \underline{\frac{1}{12}a_0 = a_2}$.

$\underline{n=3}$) $15a_3 + 3a_2 = 0 \Rightarrow a_3 = \frac{-3}{15}\left(\frac{1}{12}a_0\right) = \underline{\frac{-1}{60}a_0 = a_3}$.

Consequently,

$$\boxed{w(1,x) = z = a_0\left(x - \frac{1}{3}x^2 + \frac{1}{12}x^3 - \frac{1}{60}x^4 + \cdots\right)}$$

<u>Remark</u>: an algorithm is pretty clear here: ① find P_0 & q_0 ② solve indicial eq$^{\underline{n}}$ ③ Pick larger exponent and substitute sol$^{\underline{n}}$ of the form $w(r,x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ into DEq$^{\underline{n}}$ ④ relable sums as needed to match all terms, may need to separate a few low values. ⑤ equate coefficients and solve for $a_1, a_2, \dots$

§8.6 #25 Find general form of a sol^{n} using the method of Frobenius on $4x^2y'' + 2x^2y' - (x+3)y = 0$. Center sol^{n} at $x=0$ and assume $x>0$ — PH-111

Identify that $P(x) = \frac{2x^2}{4x^2} = \frac{1}{2}$ and $q(x) = \frac{-x-3}{4x^2}$. Calculate $P_0 = \lim_{x\to 0}\left(x\,\frac{1}{2}\right) = 0$ & $q_0 = \lim_{x\to 0}\left(x^2\left(\frac{-x-3}{4x^2}\right)\right) = \frac{-3}{4}$. We find the indicial eq^{n}: $r(r-1) + P_0 r + q_0 = 0 \Rightarrow r^2 - r - \frac{3}{4} = 0 \therefore r = \frac{1 \pm \sqrt{1+3}}{2}$ which simplifies to $r = \frac{1\pm 2}{2} = \frac{3}{2}$ or $\frac{-1}{2}$. We'll assume $W(3/2, x)$ is a solution of the form $W(3/2, x) = \sum_{n=0}^{\infty} a_n x^{n+3/2}$. We substitute this into the DEq^{n},

$$4x^2 \sum_{n=0}^{\infty} a_n\left(n+\tfrac{3}{2}\right)\left(n+\tfrac{1}{2}\right)x^{n-\frac{1}{2}} + 2x^2\sum_{n=0}^{\infty} a_n(n+3/2)x^{n+\frac{1}{2}} - \sum_{n=0}^{\infty} a_n\left(x^{n+5/2} + 3x^{n+3/2}\right) = 0$$

$$\underbrace{\sum_{n=0}^{\infty}(2n+3)(2n+1)a_n x^{n+3/2}}_{} + \sum_{n=0}^{\infty}(2n+3)a_n x^{n+5/2} - \sum_{n=0}^{\infty} a_n x^{n+5/2} - \sum_{n=0}^{\infty} 3a_n x^{n+3/2} = 0$$

$$\underbrace{3a_0x^{3/2} + \sum_{n=1}^{\infty}(2n+3)(2n+1)a_n x^{n+3/2} = 3a_0x^{3/2} + \sum_{j=0}^{\infty}(2(j+1)+3)(2(j+1)+1)a_{j+1}x^{j+5/2}}_{\text{let } j+5/2 = n+3/2,\ n = j+1}$$

now this matches the other terms with $x^{n+5/2}$ we need to convert the last term by the same shift,

$$\sum_{n=0}^{\infty} 3a_n x^{n+3/2} = 3a_0x^{3/2} + \sum_{n=1}^{\infty} 3a_n x^{n+3/2} = 3a_0x^{3/2} + \sum_{j=1}^{\infty} 3a_{j+1}x^{j+5/2}$$

Observe that the terms with $x^{3/2}$ cancel. We find,

$$\sum_{n=1}^{\infty}\Big((2(n+1)+3)(2(n+1)+1)a_{n+1} + (2n+3)a_n - a_n - 3a_{n+1}\Big)x^{n+5/2} = 0$$

$$\sum_{n=1}^{\infty}\Big(\big[(2n+5)(2n+3) - 3\big]a_{n+1} + \big[2n+2\big]a_n\Big)x^{n+5/2} = 0$$

$$\left[(2n+5)(2n+3)-3\right]a_{n+1} + 2(n+1)a_n = 0$$

We wish to find the general formula for a_n. We work out terms until a pattern emerges from the recursion,

$\underline{n=0}$ $(15-3)a_1 + 2a_0 = 0 \longrightarrow a_1 = \frac{-2}{12}a_0 = \frac{-1}{6}a_0$

$\underline{n=1}$ $((7)(5)-3)a_2 + 4a_1 = 0 \longrightarrow a_2 = \frac{-4}{32}a_1 = \left(\frac{-1}{8}\right)\left(\frac{-1}{6}\right)a_0$

$\underline{n=2}$ $((9)(7)-3)a_3 + 6a_2 = 0 \longrightarrow a_3 = \frac{-6}{60}a_2 = \left(\frac{-1}{10}\right)\left(\frac{-1}{8}\right)\left(\frac{-1}{6}\right)a_0$

$\underline{n=3}$ $[(11)(9)-3]a_4 + 8a_3 = 0 \longrightarrow a_4 = \frac{-96}{8}a_3 = \left(\frac{-1}{12}\right)\left(\frac{-1}{10}\right)\left(\frac{-1}{8}\right)\left(\frac{-1}{6}\right)a_0$

The pattern is clear,

$$a_n = \frac{(-1)^n}{\underbrace{(2n+4)(2n+2)(2n)\cdots(6)}_{n\text{-factors}}}$$

$$= \frac{(-1)^n}{2^n(n+2)(n+1)n\cdots 3}$$

$$= \frac{(-1)^n}{2^{n-1}(n+2)!}$$

Therefore, we find a sol$^{\underline{n}}$ to the diff. eq$^{\underline{n}}$ $4x^2y'' + 2x^2y' - (x+3)y = 0$,

$$\boxed{y = W(3/2, x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n-1}(n+2)!}x^{n+3/2}}$$

<u>Remark:</u> to be rigorous we could do an inductive argument to prove our formula for a_n is true for all n. I'm usually content with the method used above, but if your uncertain of this sort of formula it's good to remember <u>induction</u> can prove such assertions.

<u>Remark:</u> the method of Frobenius goes beyond the Cauchy-Euler equations. Notice this problem was not "equidimensional" yet we could still solve it. Also, again I mention our sol$^{\underline{n}}$ is not a power series eithe

§8.6 # 41 To make an expansion about the "point at infinity" it is customary to change variables to $z = 1/x$ then expand about $z = 0$. To say ∞ is a regular singular point means that $z = 0$ is a regular singular point for the transformed DEq$^{\underline{n}}$. Solve $x^3 y'' - x^2 y' - y = 0$ via the Method of Frobenius expanded about the point at ∞. (Find 4 nontrivial terms)

Let $z = 1/x$ then $\frac{dy}{dz} = \frac{dx}{dz}\frac{dy}{dx} = \frac{-1}{z^2}\frac{dy}{dx}$ furthermore,

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left[\frac{-1}{z^2}\frac{dy}{dx}\right] = \frac{2}{z^3}\frac{dy}{dx} - \frac{1}{z^2}\frac{d}{dz}\left[\frac{dy}{dx}\right] = \frac{2}{z^3}\frac{dy}{dx} - \frac{1}{z^2}\left(\frac{-1}{z^2}\frac{d^2y}{dx^2}\right)$$

Hence, $\underbrace{\frac{dy}{dx} = -z^2\frac{dy}{dz}}_{(*)}$. and $\underbrace{\frac{d^2y}{dz^2} = \frac{2}{z^3}\frac{dy}{dx} + \frac{1}{z^4}\frac{d^2y}{dx^2}}$

$$\frac{d^2y}{dz^2} = \frac{2}{z^3}\left(-z^2\frac{dy}{dz}\right) + \frac{1}{z^4}\frac{d^2y}{dx^2}$$

$$\underbrace{z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz} = \frac{d^2y}{dx^2}}_{(**)}.$$

Using $(*)$ and $(**)$ we find, (also $x = 1/z$)

$$x^3\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - y = 0$$

$$\frac{1}{z^3}\left[z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz}\right] - \frac{1}{z^2}\left[-z^2\frac{dy}{dz}\right] - y = 0$$

$$zy'' + 2y' + y' - y = 0$$

$$\underline{zy'' + 3y' - y = 0}.$$

[technically y is not the same y as we began with. Now y is a function of z]

Identify $P(z) = \frac{3}{z}$ and $q(z) = \frac{-1}{z}$, hence $P_o = \lim_{z\to 0}\left(\frac{3}{z}\right)z = 3$ and $q_o = \lim_{z\to 0}\left(-\frac{z^2}{z}\right) = 0$.

The indicial eq$^{\underline{n}}$ (for Frobenius centered about $z=0$)

$$r(r-1) + P_o r + q_o = r^2 - r + 3r = r(r+2) = 0 \quad \therefore \underline{r=0 \text{ or } r=-2}$$

We'll use $r = 0$

§8.6 #41 Continued:

Suppose $W(0,z) = \sum_{n=0}^{\infty} a_n z^n$ solves $zy'' + 3y' - y = 0$,

$$\sum_{n=0}^{\infty} n(n-1)a_n z^{n-1} + 3\sum_{n=0}^{\infty} na_n z^{n-1} - \sum_{n=0}^{\infty} a_n z^n = 0$$

$\left(\begin{array}{ll} \text{Let } j-1 = n & \text{then switch} \\ n=0 \longrightarrow j=1 & \text{back to } n \end{array}\right)$.

$$\sum_{n=1}^{\infty}\Big(n(n-1)a_n + 3na_n - a_{n-1}\Big)z^{n-1} + \underbrace{0(-1)a_0 z^{-1} + 3\cdot 0\cdot a_0 z^{-1}}_{n=0 \text{ trivial terms}} = 0$$

$$\sum_{n=1}^{\infty}\Big(n(n+2)a_n - a_{n-1}\Big)z^{n-1} = 0$$

We find,

$\underline{n=1}$ $3a_1 - a_0 = 0 \quad \therefore \quad \underline{a_1 = \frac{1}{3}a_0}$.

$\underline{n=2}$ $8a_2 - a_1 = 0 \quad \therefore \quad a_2 = \frac{1}{8}\left(\frac{1}{3}a_0\right) = \underline{\frac{1}{24}a_0 = a_2}$.

$\underline{n=3}$ $15a_3 - a_2 = 0 \quad \therefore \quad a_3 = \frac{1}{15}\left(\frac{1}{24}a_0\right) = \underline{\frac{1}{360}a_0 = a_3}$.

Thus,

$$W(0,z) = a_0\left(1 + \frac{1}{3}z + \frac{1}{24}z^2 + \frac{1}{360}z^3 + \cdots\right)$$

Which reverts back to,

$$\boxed{y(x) = a_0\left(1 + \frac{1}{3x} + \frac{1}{24x^2} + \frac{1}{360x^3} + \cdots\right)}$$

Remark: we have only found half of the general sol$^{\underline{n}}$ for these 2$^{\underline{nd}}$ order ODEs. In § 8.7 we'll see how to obtain the other half of the general sol$^{\underline{n}}$. I have purposefully chosen matching problems.

- §8.6 #19 AND §8.7 #1 go together
- §8.6 #23 AND §8.7 #5 go together