$\underbrace{\text{BOUNDARY VALUE PROBLEMS}}_{\text{"BVP"}}$

**§10.2#1** Solve the BVP $y''-y=0$ on $0<x<1$ where $y(0)=0,\ y(1)=-4$

We find $y''-y=0$ has Characteristic Eq$^{\underline{n}}$ $\lambda^2-1=(\lambda-1)(\lambda+1)=0$ thus $\lambda_1=1,\ \lambda_2=-1$. I like to write the sol$^{\underline{n}}$ as $y=c_1\cosh(x)+c_2\sinh(x)$ in this case. Apply the boundary conditions,

$$y(0)=0=c_1\cosh(0)+c_2\sinh(0)=c_1 \quad \therefore \quad \underline{c_1=0}.$$

$$y(1)=-4=c_1\cosh(1)+c_2\sinh(1) \Rightarrow \underline{c_2=\frac{-4}{\sinh(1)}}.$$

Thus $\boxed{y=\frac{-4\sinh(x)}{\sinh(1)}}$ (my answer is better than the text's ☺.)

**§10.2#5** Solve the BVP $y''-y=1-2x$, for $0<x<1$ and $y(0)=0,\ y(1)=1+e$

We found $y_h=c_1\cosh(x)+c_2\sinh(x)$ in #1. Notice $g(x)=1-2x$ suggests we try $y_p=A+Bx$ (clearly no overlap with $y_h$) Since $y_p''=0$ we need $-y_p=1-2x$ hence $-A-Bx=1-2x \Rightarrow \underline{A=-1}$ and $\underline{B=2}$. The general sol$^{\underline{n}}$ follows:

$$\underline{y=c_1\cosh(x)+c_2\sinh(x)-1+2x}$$

Now we attempt to fit the given Boundary Condition (BC)

$$y(0)=0=c_1-1 \quad \therefore \quad \underline{c_1=1}$$

$$y(1)=\underbrace{1+e=\cosh(1)+c_2\sinh(1)-1+2}$$

$$e-\cosh(1)=c_2\sinh(1) \longrightarrow \underline{c_2=\frac{e-\cosh(1)}{\sinh(1)}}.$$

Let me translate $c_2$ back to exponential notation, (trying to check against text)

$$c_2=\frac{e-\cosh(1)}{\sinh(1)}=\frac{e-\frac{1}{2}(e+\frac{1}{e})}{\frac{1}{2}(e-\frac{1}{e})}=\frac{2e^2-e^2-1}{e^2-1}=\frac{e^2-1}{e^2-1}=1=c_2$$

Hence, $\boxed{y=\cosh(x)+\sinh(x)+2x-1=e^x+2x-1}$

<u>Remark:</u> §10.2#1 is solved cleanly via the hyperbolic formulation of $y_h = c_1 \cosh(x) + c_2 \sinh(x)$. Probably §10.2 #5 would be easier to solve in the manifestly exponential notation $y_h = c_1 e^x + c_2 e^{-x}$. Which should we use at which time? It's a question of style, I obviously have no answer.

**§10.2#9** Find eigenvalues and eigenfunctions which provide solutions to the BVP: $y'' + \lambda y = 0$, $0 < x < \pi$, $y(0) = 0$, $y'(\pi) = 0$

**λ>0** I'll use $r$ for the Char. Eq$^{\underline{n}}$ since $\lambda$ is already used,

$$r^2 + \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda} \Rightarrow \underline{y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)}.$$

We need BC's to hold,

$$y(0) = c_1 \cos(0) + c_2 \sin(0) = 0 \Rightarrow \underline{c_1 = 0}.$$

$$y'(\pi) = \underbrace{-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}_{\substack{\text{zero} \\ \text{since } c_1 = 0}} + \underbrace{c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)}_{\text{must be zero}} = 0$$

$c_2 = 0$ gives trivial sol$^{\underline{n}}$

need $\cos(\sqrt{\lambda}\pi) = 0$

$$\Rightarrow \sqrt{\lambda}\pi = (n + \tfrac{1}{2})\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow \sqrt{\lambda} = n + \tfrac{1}{2}$$

$$\Rightarrow \lambda = \left(n + \tfrac{1}{2}\right)^2 \text{ for } n \in \mathbb{Z}.$$

$$\Rightarrow \underline{\lambda = \tfrac{1}{4}(2n+1)^2 \text{ for } n \in \mathbb{Z}.}$$

We should also consider the cases $\lambda = 0$ and $\lambda < 0$. It's conceivable there are also sol$^{\underline{n}}$s to the BVP for these cases,

**λ=0** $y'' = 0 \Rightarrow y = c_1 + c_2 x$. Notice that the BC's $y(0) = 0$ and $y'(\pi) = 0 \Rightarrow \underline{c_1 = 0}$ and $\underline{c_2 = 0}$

Hence $\lambda = 0$ yields the <u>trivial</u> sol$^{\underline{n}}$ $\boxed{y = 0 \text{ for } \lambda = 0}$

trivial sol$^{\underline{n}}$ is not considered an "eigenfnct."

§10.2 # 9 Continued

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$\lambda < 0$) $y'' + \lambda y = 0$ has sol$^{n}$'s governed by Char. Eq$^{n}$ $y = c_1 \cosh \sqrt{-\lambda}\,x + c_2 \sinh \sqrt{-\lambda}\,x$. To fit the BC's we need,

$$y(0) = 0 = c_1.$$
$$y'(\pi) = 0 = \sqrt{-\lambda}\, c_1 \sinh(\sqrt{-\lambda}\,\pi) + \sqrt{-\lambda}\, c_2 \cosh(\sqrt{-\lambda}\,\pi)$$
$$\Rightarrow \sqrt{-\lambda}\, c_2 \cosh(\sqrt{-\lambda}\,\pi) = 0$$

But $\cosh(\theta) \geq 1 \;\; \forall \theta \in \mathbb{R}$ and $\sqrt{-\lambda} \neq 0$ since $\lambda < 0$. We must conclude $c_2 = 0$. $\Rightarrow \boxed{y = 0}$

To conclude, the BVP $y'' + \lambda y = 0$, $0 < x < \pi$ subject to BC's $y(0) = 0$ and $y'(\pi) = 0$ has sol$^{n}$'s iff the eigenvalues $\lambda$ are of the form $\lambda = \frac{1}{4}(2n+1)^2$ for some $n \in \mathbb{Z}$ and the corresponding nontrivial eigenfunction sol$^{n}$'s are

$$\boxed{y_n(x) = c_n \sin\left[\left(n + \tfrac{1}{2}\right)x\right]}$$

Remark: when we solve $y'' + \lambda y = 0$ we're really solving a whole family of DEq$^{n}$'s which are indexed by $\lambda$. When we impose the BC's $y(0) = 0$, $y'(\pi) = 0$ we find that only part of the family is consistent with the BC. We call such $\lambda$ "eigenvalues", but technically the DEq$^{n}$ $y'' + \lambda y = 0$ in operator notation is $(D^2 + \lambda)[y] = 0$. Note the eigenvalues of $D$ would be $\pm i\sqrt{\lambda}$ and the corresponding eigenfunctions are precisely $y_n(x)$ boxed above. My point? The text calling $\lambda$ an "eigenvalue" bugs me. In fact, $\lambda$ is simply associated to a particular set of eigenvalues $(\pm i\sqrt{\lambda})$.

§10.2 #13 Solve the BVPs: $y'' + \lambda y = 0$, $0 < x < \pi$ subject to the BC's $y(0) - y'(0) = 0$, $y(\pi) = 0$

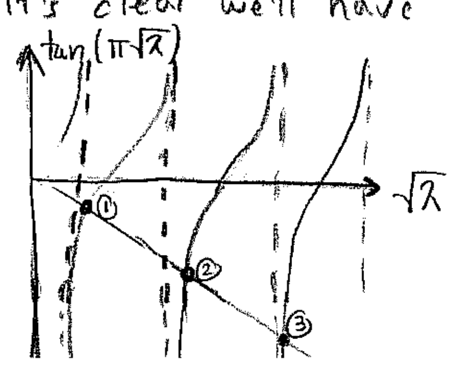
$\lambda = 0$ $y'' = 0 \Rightarrow y = c_1 + c_2 x$ then BC's yield $c_1 - c_2 = 0$, $c_1 + c_2\pi = 0$
Hence $c_1 = c_2$ and $c_1(1+\pi) = 0$ thus $\underline{c_1 = 0}$ and $\underline{c_2 = 0}$. Only trivial sol$^{\underline{n}}$ here.

$\lambda < 0$ $y'' + \lambda y = 0 \Rightarrow r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}$ define $\alpha = \sqrt{-\lambda} \in \mathbb{R}$
then write
$$y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$
$$y' = c_1 \alpha \sinh \alpha x + c_2 \alpha \cosh \alpha x$$
Apply BC's,
$$y(0) - y'(0) = c_1 - c_2\alpha = 0$$
$$y(\pi) = c_1 = 0 \Rightarrow c_2 = c_1/\alpha = 0 \therefore \underline{y = 0, \text{ only trivial sol}^{\underline{n}}}.$$

$\lambda > 0$ $y'' + \lambda y = 0 \Rightarrow r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$, let $\beta = \sqrt{\lambda}$
we find sol$^{\underline{n}}$'s of the form $y = c_1 \cos \beta x + c_2 \sin \beta x$. Now apply BC's and look for nontrivial sol$^{\underline{n}}$'s
$$y(0) - y'(0) = c_1 - \beta c_2 = 0 \Rightarrow c_1 = \beta c_2$$
$$y(\pi) = c_1 \cos(\beta\pi) + c_2 \sin(\beta\pi) \Rightarrow c_2\Big(\beta\cos(\beta\pi) + \sin(\beta\pi)\Big) = 0$$
$$\Rightarrow \tan(\beta\pi) = -\beta$$

"Eigenvalues" $\lambda_n$ are sol$^{\underline{n}}$'s of the transcendental eq$^{\underline{n}}$ $\tan(\sqrt{\lambda_n}\,\pi) = -\sqrt{\lambda_n}$ where $\lambda_n$ has corresponding sol$^{\underline{n}}$'s of the form
$$\boxed{y_n(x) = c_n\left[\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\,x) + \sin(\sqrt{\lambda_n}\,x)\right]}$$

Note, it is appropriate to index the sol$^{\underline{n}}$'s by $n$ since graphically it's clear we'll have ctbly many sol$^{\underline{n}}$'s




① has $\tan(\sqrt{\lambda_1}\,\pi) = -\sqrt{\lambda_1}$
② has $\tan(\sqrt{\lambda_2}\,\pi) = -\sqrt{\lambda_2}$
etc... have to use numerical methods to get actual explicit values.

§10.2 #15 Solve the heat flow problem:

(1) $\frac{\partial u}{\partial t}(x,t) = \beta \frac{\partial^2 u}{\partial x^2}(x,t)$, $0 < x < L$, $t > 0$

(2) $u(0,t) = u(L,t) = 0$, $t > 0$

(3) $u(x,0) = f(x)$

in the case $\beta = 3$, $L = \pi$ and $f(x) = \sin(x) - 6\sin(4x)$

On pgs. 606–608 the text derives the general sol$^{n}$ $u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\beta t\left(\frac{n\pi}{L}\right)^2} \sin\left(\frac{n\pi x}{L}\right)$ where the coefficients $C_n$ are the coefficients of the Fourier expansion of $f(x)$; $f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$. In a nutshell, a Fourier expansion is a sum of trig-fncts. Here the $f(x)$ is <u>already</u> given as a very simple Fourier series

$$f(x) = \sin(x) - 6\sin(4x) = c_1 \sin\left(\frac{\pi x}{\pi}\right) + c_4 \sin\left(\frac{4\pi x}{\pi}\right) + \cdots$$

Clearly $c_1 = 1$ and $c_4 = -6$ while all other $C_n = 0$. The infinite sum collapses to two terms, $n = 1, 4$, we find

$$\boxed{u(x,t) = e^{-3t}\sin(x) - 6e^{-48t}\sin(4x)}$$

<u>Remark:</u> § 10.3–10.4 discuss how to take a function and find a sum of sine and/or cosine functions which represent it. A Fourier Series is a <u>trigonometric series</u> in the same sense a Taylor series is a <u>power series</u>. § 10.2 #15 has $f(x)$ which is already a trig-series so no work really needs to be done to find $C_n$. It's a lot like being asked to find the Maclaurin series expansion for $f(x) = x^3 + 3x^2 + 7$. ($x^3 + 3x^2 + 7$ is the Maclaurin series for $f(x)$).

§10.2 #21 Solve the vibrating string problem:

(16) $\dfrac{\partial^2 u}{\partial t^2} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$ , $0 < x < L$ , $t > 0$

(17) $u(0,t) = u(L,t) = 0$ , $t \geq 0$ (ends of string fixed in place)

(18) $u(x,0) = f(x)$ , $0 \leq x \leq L$ (initial shape of string)

(19) $\dfrac{\partial u}{\partial t}(x,0) = g(x)$ , $0 \leq x \leq L$ (initial velocities of string)

Given $\alpha = 3$, $L = \pi$ and $f(x) = 6\sin(2x) + 2\sin(6x)$, $g(x) = 11\sin(9x) - 14\sin(15x)$

Fortunately, the text has found the general sol$^{\underline{n}}$ on pgs. 610–612. We need only identify the $a_n$, $b_n$ from $f(x)$ and $g(x)$, see (27) of text:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = \underbrace{a_1\sin(x) + a_2\sin(2x) + \cdots = 6\sin(2x) + 2\sin(6x)}_{\underline{a_2 = 6 \quad \text{and} \quad a_6 = 2}}.$$

See (27) and $B_n = \left(\frac{n\pi\alpha}{L}\right) b_n$

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi\alpha}{L}\right) b_n \sin\left(\frac{n\pi x}{L}\right) = 3b_1\sin(x) + 6b_2\sin(2x) + \cdots + 3nb_n\sin(nx) + \cdots$$

We were given $g(x) = \underbrace{11\sin(9x)}_{n=3} - \underbrace{14\sin(15x)}_{n=15}$ we find $b_3, b_{15} \neq 0$ but all other $b_n = 0$. Moreover,

$$11 = 27b_9 \Rightarrow \underline{b_9 = \frac{11}{27}}. \quad \text{and} \quad -14 = 45b_{15} \Rightarrow \underline{b_{15} = \frac{-14}{45}}.$$

Now plug these into the general sol$^{\underline{n}}$ (24)

$$u(x,t) = \sum_{n=1}^{\infty} \left[ a_n\cos(3nt) + b_n\sin(3nt) \right] \sin(nx)$$

$$\boxed{u(x,t) = \underbrace{6\cos(6t)\sin(2x)}_{n=2} + \underbrace{2\cos(18t)\sin(6x)}_{n=6} + \frac{11}{27}\sin(27t)\sin(9x) - \frac{14}{45}\sin(45t)\sin(15x)}$$

**Remark:** It would be nice to rewrite this sol$^{\underline{n}}$ in the form $u = y_1 + y_2 + y_3 + y_4$ where $y_i = \cos(w_i t + k_i x)$. (I think this is possible)

§10.2#23 Find formal sol$^{\underline{n}}$ for heat flow problem with $\beta = 2$ and $L = 1$ given $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(n\pi x)$

Observe that comparing $f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(n\pi x)$ we find $c_n = 1/n^2$. Consequently, using text's sol$^{\underline{n}}$ (11)

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-2\pi^2 n^2 t} \sin(nx)}$$

This is a "formal sol$^{\underline{n}}$" since we are not certain that $u, u_t, u_{xx}$ are in fact <u>convergent</u> Fourier Series on $0 < x < L$.

§10.2#27 Suppose the sol$^{\underline{n}}$ $u(r,\theta) = R(r)T(\theta)$ for $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. Find DEq$^{\underline{n}}$'s which $R$ and $T$ must satisfy as a consequence of the given PDE. We use notation $T' = \frac{dT}{d\theta}$ and $R' = \frac{dR}{dr}$.

Let $u = RT$ then $\frac{\partial u}{\partial r} = \frac{\partial R}{\partial r}T + R\frac{\partial T}{\partial r}$ but $\frac{\partial T}{\partial r} = 0$ since $T = T(\theta)$. Hence $\frac{\partial u}{\partial r} = \frac{\partial R}{\partial r}T = R'T$. Likewise $\frac{\partial u}{\partial \theta} = RT'$.

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}(R'T) = \left(\frac{\partial R'}{\partial r}\right)T = R''T$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}(RT') = R\left(\frac{\partial T'}{\partial \theta}\right) = RT''$$

The key in the calculations above is that $\frac{\partial T}{\partial r} = 0$ and $\frac{\partial R}{\partial \theta} = 0$ by assumption. Substituting yields,

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0$$

Divide the eq$^{\underline{n}}$ by $RT$ to find,

$$\frac{R''T}{RT} + \frac{\frac{1}{r}R'T}{RT} + \frac{\frac{1}{r^2}RT''}{RT} = 0 \Rightarrow \underbrace{\frac{-T''}{T}}_{\text{function of } \theta \text{ only}} = \underbrace{\frac{r^2}{R}\left(R'' + \frac{1}{r}R'\right)}_{\text{function of } r \text{ only}}$$

Therefore, $\exists$ an eigenvalue(s) $\lambda$ such that $\lambda = \frac{-T''}{T} = \frac{r^2}{R}\left(R'' + \frac{1}{r}R'\right)$. Thus

$$\boxed{\begin{aligned} T'' + \lambda T &= 0 \\ r^2R'' + rR' - \lambda R &= 0 \end{aligned}}$$

$\boxed{\text{§10.2 \# 29}}$ Assume $u_t = \beta(u_{xx} + u_{yy})$ has a sol$^{\underline{n}}$ of the form $u(x,y,t) = \mathcal{X}(x)\mathcal{Y}(y)T(t)$. Find ODE's which $\mathcal{X}$, $\mathcal{Y}$ and $T$ must solve if $u$ is to solve the given PDE.

We use short hand, $\frac{\partial \mathcal{X}}{\partial x} = \mathcal{X}'$, $\frac{\partial \mathcal{Y}}{\partial y} = \mathcal{Y}'$, $\frac{\partial T}{\partial t} = T'$ and we assume $T = T(t)$, $\mathcal{Y} = \mathcal{Y}(y)$ and $\mathcal{X} = \mathcal{X}(x)$ meaning $T$ is a fnct. of only $t$ etc... It follows

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\mathcal{X}\mathcal{Y}T\right) = \overbrace{\mathcal{X}\mathcal{Y}}^{\text{constant in } t}\frac{\partial T}{\partial t} = \mathcal{X}\mathcal{Y}T'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\mathcal{X}\right)\right)\overbrace{\mathcal{Y}T}^{\text{const. in } x} = \mathcal{X}''\mathcal{Y}T$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\mathcal{Y}\right)\right)\overbrace{\mathcal{X}T}^{\text{const. in } y} = \mathcal{Y}''\mathcal{X}T$$

Substitute these calculations into the PDE,

$$\mathcal{X}\mathcal{Y}T' = \beta\left(\mathcal{X}''\mathcal{Y}T + \mathcal{Y}''\mathcal{X}T\right)$$

Divide by $\mathcal{X}\mathcal{Y}T$, notice the nice cancellations,

$$\underbrace{\frac{T'}{T}}_{\text{function of } t \text{ only}} = \underbrace{\beta\left(\frac{\mathcal{X}''}{\mathcal{X}}\right) + \beta\left(\frac{\mathcal{Y}''}{\mathcal{Y}}\right)}_{\text{function of } x, y \text{ alone}} \Rightarrow \underbrace{K}_{\text{constant}} = \frac{T'}{T} = \beta\left(\frac{\mathcal{X}''}{\mathcal{X}} + \frac{\mathcal{Y}''}{\mathcal{Y}}\right)$$

Hence $T' = KT$ and

$$\underbrace{\frac{K}{\beta} - \frac{\mathcal{X}''}{\mathcal{X}}}_{\text{function of } x \text{ only}} = \underbrace{\frac{\mathcal{Y}''}{\mathcal{Y}}}_{\text{function of } y \text{ only}} = \underbrace{J}_{\text{constant}}.$$

Consequently, $\mathcal{Y}'' = J\mathcal{Y}$ and $\frac{K}{\beta}\mathcal{X} - \mathcal{X}'' = J\mathcal{X}$.

Summarizing,

$$\boxed{\begin{aligned} T' - KT &= 0 \\ \mathcal{Y}'' - J\mathcal{Y} &= 0 \\ \mathcal{X}'' + (J - K/\beta)\mathcal{X} &= 0 \end{aligned}}$$

(the text replaces $K$ with $K\beta$ but that is equivalent to the sol$^{\underline{n}}$ I offer here).

<u>Remark</u>: Cartesian Coordinate PDE $\Rightarrow$ constant coeff ODE / Eigenproblem.
Polar Coordinate PDE $\Rightarrow$ Cauchy - Euler ODE / Eigenproblem.

§10.2 # 33. When temperature $u$ in a wire reaches a steady state then $u = u(x)$ and $u$ satisfies Laplace's Eq$^{\underline{n}}$ $\partial^2 u/\partial x^2 = 0$.
a) Find steady state sol$^{\underline{n}}$ when $u(0) = 50$ while $u(L) = 50$.
b) Find steady state sol$^{\underline{n}}$ when $u(0) = 10$ while $u(L) = 40$

a.) $\dfrac{\partial^2 u}{\partial x^2} = 0$ and $u = u(x) \Rightarrow u'' = 0$ is an ODEq$^{\underline{n}}$ in $x$

$$\Rightarrow u(x) = c_1 + c_2 x$$

$$u(0) = 50 \Rightarrow c_1 = 50$$
$$u(L) = c_1 + Lc_2 = 50 \Rightarrow c_2 = 0 \quad \therefore \quad \boxed{u(x) = 50}$$

b.) Again the same general sol$^{\underline{n}}$ holds $u(x) = c_1 + c_2 x$

thus $u(0) = 10 = c_1$

$$u(L) = 40 = 10 + c_2 L \Rightarrow c_2 = 30/L \quad \therefore \quad \boxed{u(x) = 10 + \frac{30}{L} x}$$

Remark: this Problem illustrates the Maximum Principle for Laplace's Eq$^{\underline{n}}$ (see pg. 674). For $D: 0 < x < L$ we have $\partial D = \{0, L\}$ and clearly the maximum of $u$ is attained on the boundary. §10.7 is devoted to studying Laplace's Eq$^{\underline{n}}$ in various dimensions and contexts.