# Fourier Series : § 10-3

**§10.3#1** Let $f(x) = x^3 + \sin(ax)$. Prove $f$ is an odd function.

Notice $f(-x) = (-x)^3 + \sin(a(-x)) = -x^3 - \sin(ax) = -f(x) \quad \forall x \in \mathbb{R}$.
Thus $f$ is an odd function.

**§10.3#5** Let $f(x) = e^{-x}\cos(3x)$. Show $f$ is neither even nor odd

Consider $f(-x) = e^{-(-x)}\cos(3(-x)) = e^{x}\cos(3x) \neq \pm f(x)$ thus $f$ is neither even nor odd. It may be worth mentioning that $e^x = \cosh(x) + \sinh(x)$ decomposes $e^x$ into it's even and odd components. In fact, we can do the same for $f(x)$,

$$f(x) = \underbrace{\cosh(x)\cos(3x)}_{\text{even}} - \underbrace{\sinh(x)\cos(3x)}_{\text{odd.}}$$

You can check me on the algebra of these claims.

**§10.3#7** Prove the following: (a.) for even $f, g$ the product $fg$ is even. (b.) for odd fncts $f, g$ the product $fg$ is even. (c.) if $f$ even and $g$ is odd fnct then $fg$ is an odd fnct.

(a.) Let $f, g$ be even fncts. Consider,

$$\begin{aligned}(fg)(-x) &= f(-x)\,g(-x) && : \text{def}^{\text{n}} \text{ of product fnct.}\\ &= f(x)\,g(x) && : f, g \text{ assumed even.}\\ &= (fg)(x) && : \text{def}^{\text{n}} \text{ of product fnct.}\end{aligned}$$

Thus $fg$ is even.

(b.) Let $f, g$ be odd fncts. Consider

$$\begin{aligned}(fg)(-x) &= f(-x)\,g(-x) && : \text{def}^{\text{n}} \text{ of product fnct.}\\ &= (-f(x))(-g(x)) && : f, g \text{ assumed odd.}\\ &= (fg)(x) && : \text{def}^{\text{n}} \text{ of product fnct.}\end{aligned}$$

Thus $fg$ is even.

(c.) Let $f$ be even and $g$ odd. Consider,

$$\begin{aligned}(fg)(-x) &= f(-x)\,g(-x) && : \text{def}^{\text{n}} \text{ of product fnct.}\\ &= f(x)(-g(x)) && : f \text{ even, } g \text{ odd}\\ &= -(fg)(x) && : \text{def}^{\text{n}} \text{ of product fnct.}\end{aligned}$$

Hence $fg$ is odd.

$\boxed{\S 10.3 \# 9}$ Find Fourier Series for $f(x) = x$ on $-\pi < x < \pi$

Clearly $f$ is continuous on $[-\pi, \pi]$ hence $T = \pi$ and the Fourier Integrals (9) & (10) work out to (I skip details of IBP below)

$\underline{n \neq 0}$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = \frac{1}{\pi}\left(\frac{1}{n}x\sin(nx) + \frac{1}{n}\cos(nx)\right)\Big|_{-\pi}^{\pi}$$
$$= \frac{1}{\pi}\left(\frac{1}{n}\cos(n\pi) - \cos(-\pi)\right)$$
$$= 0.$$

(well duh. $x\cos(nx)$ is an odd fnct. of course this integral is going to be zero!)

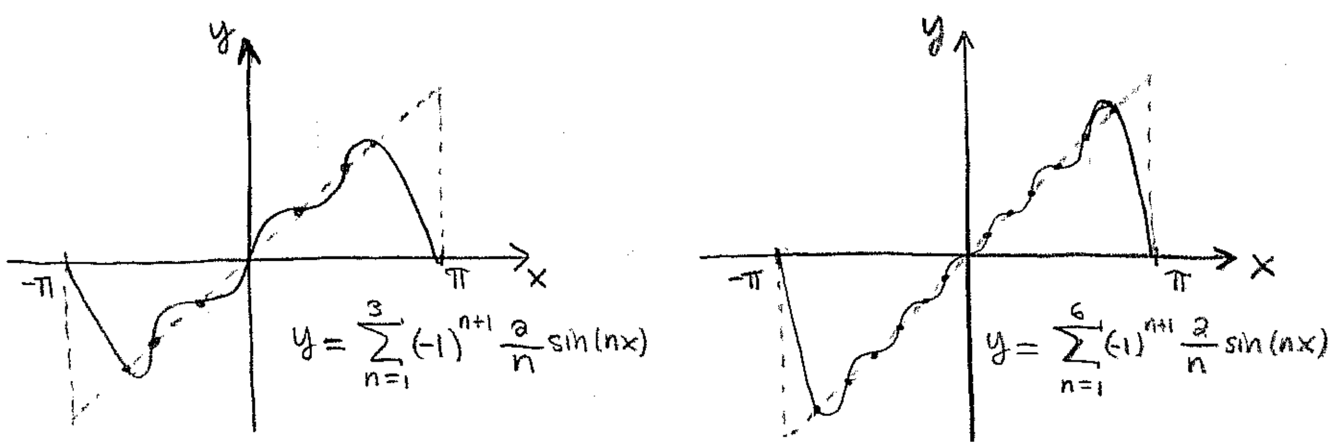
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left(\frac{-1}{n}x\cos(nx) + \frac{1}{n}\sin(nx)\right)\Big|_{-\pi}^{\pi}$$
$$= \frac{-1}{n\pi}\left(\pi\cos(n\pi) + \pi\cos(-n\pi)\right)$$
$$= \frac{-2\cos(n\pi)}{n}$$
$$= \frac{2}{n}(-1)^{n+1}$$

; note $\cos(0) = \cos(2\pi) = \cos(4\pi) = 1$
$\cos(\pi) = \cos(3\pi) = -1$
thus $\cos(n\pi) = (-1)^n$
think about it.

Finally $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0.$ (odd fnct integral trick.)

Thus we find

$$\boxed{f(x) \sim \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \frac{2}{5}\sin(5x) - \frac{1}{3}\sin(6x) + \cdots}$$




The more terms we take the closer to $y = x$ for $-\pi < x < \pi$.
The graph will repeat if we look beyond $-\pi < x < \pi$. (Sawtooth)

$\boxed{\S 10.3 \# 11}$ Let $f(x) = \begin{cases} 1 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$ find Fourier expansion on $-2 < x < 2$

We need to calculate $a_n$ and $b_n$,

$$a_o = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\int_{-2}^{0} dx + \frac{1}{2}\int_{0}^{2} x\,dx = 1 + \frac{1}{2}\left(\frac{4}{2}\right) = \underline{2 = a_o}.$$

Suppose $n \geq 1$ in what follows,

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{0} \cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right)\Big|_{-2}^{0} + \frac{1}{\cancel{2}}\left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\right|_{0}^{2}$$

$$= \frac{1}{n\pi}\Big(\sin(0) - \sin(-n\pi)\Big) + \frac{2}{(n\pi)^2}\Big[2\sin(n\pi) + \cos(n\pi) - 0 - \cos(0)\Big]$$

$$= \frac{2}{(n\pi)^2}\Big[\cos(n\pi) - 1\Big]$$

$$= \underline{\frac{2}{n^2\pi^2}\Big[(-1)^n - 1\Big]}.$$

Next calculate $b_n$,

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{0} \sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2} x\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{-1}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0} + \frac{1}{2}\left[\frac{-2}{n\pi}x\cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi x}{2}\right)\right|_{0}^{2}$$

$$= \frac{-1}{n\pi}\Big(1 - \cos(n\pi)\Big) + \frac{-1}{n\pi}\Big(2\cos(n\pi)\Big) + \frac{2}{n^2\pi^2}\Big(\sin(n\pi) - \sin(0)\Big)$$

$$= \frac{1}{n\pi}\Big[-\cos(n\pi) - 1\Big]$$

$$= \underline{\frac{1}{n\pi}\Big[(-1)^{n+1} - 1\Big]}.$$

Thus, the Fourier expansion is

$$\boxed{f(x) \sim 1 + \sum_{n=1}^{\infty}\left\{\frac{2}{n^2\pi^2}\Big[(-1)^n - 1\Big]\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{n\pi}\Big[(-1)^{n+1} - 1\Big]\sin\left(\frac{n\pi x}{2}\right)\right\}}$$

We use $\sim$ rather than $=$ to indicate the representation may only converge on a subset of dom$(f)$.

$\boxed{\text{§10.3 \#13}}$ Find Fourier Expansion of $f(x) = x^2$ on $-1 < x < 1$ $\boxed{\text{PH-130}}$

Since $f$ is an even function we will find $b_n = 0$ for all $n \geq 1$.
Consider $n = 0$,

$$a_0 = \int_{-1}^{1} x^2\, dx = \tfrac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3}.$$

Let $n \geq 1$ and calculate,

$$a_n = \int_{-1}^{1} x^2 \cos(n\pi x)\, dx$$

<u>Pause</u>: let's integrate. Use integration by parts, twice,

$$\int \underbrace{x^2}_{u} \underbrace{\cos(n\pi x)\, dx}_{dV} = \frac{x^2}{n\pi}\sin(n\pi x) - \int \underbrace{\frac{1}{n\pi}\sin(n\pi x)}\, 2x\, \underbrace{dx}_{dV}$$

$$= \frac{x^2 \sin(n\pi x)}{n\pi} - \frac{2x}{n\pi}\left(\frac{-\cos n\pi x}{n\pi}\right) + \int \frac{-\cos n\pi x}{n\pi}\left(\frac{2}{n\pi}dx\right)$$

$$= \frac{x^2 \sin(n\pi x)}{n\pi} + \frac{2x}{n^2\pi^2}\cos(n\pi x) - \frac{2}{(n\pi)^3}\sin(n\pi x) + C$$

Returning to the calculation of $a_n$,

$$a_n = \left\{\left(\frac{x^2}{n\pi} - \frac{2}{(n\pi)^3}\right)\sin(n\pi x) + \frac{2x}{(n\pi)^2}\cos(n\pi x)\right\}\Bigg|_{-1}^{1}$$

$$= \frac{2}{(n\pi)^2}\cos(n\pi) + \frac{2}{(n\pi)^2}\cos(-n\pi)$$

$$= \underline{\frac{4}{n^2\pi^2}(-1)^n}.$$

Consequently,

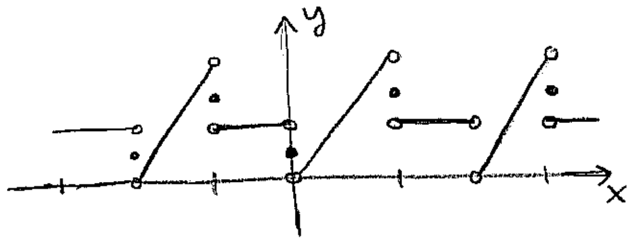
$$\boxed{f(x) \sim \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \cos(n\pi x)}$$

§10.3 #19 and #21) We are asked to give the function to which the Fourier series converges for #11 and #13

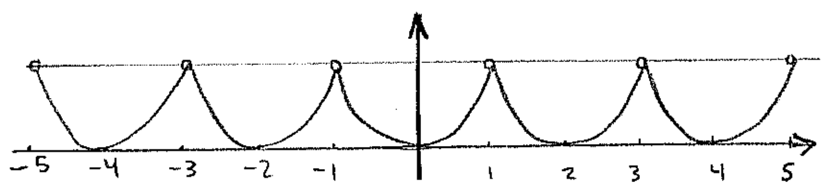
#11) $f(x) = \begin{cases} 1 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$

has Fourier Series which converges to the function $g(x)$ with period $T = 4$ and $g(x) = f(x)$ for $-2 < x < 2$. The endpts / discont. pts of $x = 0$ and $x = \pm 2$ are given by the average of the fnct. to the left & right of the discontinuity. Thus

$$g(0) = \tfrac{1}{2}(1 + 0) = \tfrac{1}{2}$$
$$g(2) = \tfrac{1}{2}(1 + 2) = \tfrac{3}{2}$$
$$g(-2) = \tfrac{1}{2}(1 + 2) = \tfrac{3}{2}$$




#13) $f(x) = x^2$ for $-1 < x < 1$




Notice the points of possible discontinuity $x = \pm 1, \pm 3, \pm 5, \ldots$ match up from left and right thus $g(x) = x^2$ for $-1 \le x \le 1$ and $g$ has period 2.

Remark: the Fourier Expansion's averaging of the left/right values near a point of discontinuity is not terribly important to most questions. Often we can ignore a set of points of "measure zero" especially where an integral is involved.

§10.3#29 In §8.8 it was shown that the Legendre polynomials $P_n(x)$ are <u>orthogonal</u> on $[-1,1]$ with respect to weight function $w(x) = 1$. Given that

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

find the 1st 3 coefficients in the Legendre polynomial expansion $f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \cdots$ for the function $f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$

Here "orthonormal" $P_m$ means $\int_{-1}^{1} P_m P_n \, dx = \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$

We can use integration to pick off components. It's a generalization of the dot-product idea from Calc. III,

$$\int_{-1}^{1} P_m(x) \left[ a_0 P_0(x) + a_1 P_1(x) + \cdots \right] dx = a_0 \delta_{m0} + a_1 \delta_{m1} + a_2 \delta_{m2} + \cdots$$

We only have "orthogonal", this means $\|P_m\| \neq 1$ necessarily,

$$\|P_0\|^2 a_0 = \int_{-1}^{1} P_0(x) f(x) dx = \int_{-1}^{1} f(x)\, dx = -1 + 1 = 0 \Rightarrow \underline{a_0 = 0}.$$

Likewise,

$$\|P_1\|^2 a_1 = \int_{-1}^{1} P_1(x) f(x) dx = \int_{-1}^{0} -x \, dx + \int_{0}^{1} x \, dx = \left.\frac{-x^2}{2}\right|_{-1}^{0} + \left.\frac{x^2}{2}\right|_{0}^{1} = \frac{1}{2} + \frac{1}{2} = \underline{1 = a_1}$$

$$\|P_2\|^2 a_2 = \int_{-1}^{1} P_2(x) f(x) dx = \int_{-1}^{0} \left(\frac{1}{2} - \frac{3}{2}x^2\right) dx + \int_{0}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) dx$$
$$= \left(\frac{x}{2} - \frac{1}{2}x^3 \right|_{-1}^{0} + \left(\frac{1}{2}x^3 - \frac{x}{2}\right|_{0}^{1}$$
$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = 0 \Rightarrow \underline{a_2 = 0}.$$

Calculate $\|P_2\|^2 = \int_{-1}^{1} P_2(x) P_2(x) dx = \int_{-1}^{1} x^2 dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3}$.

Consequently $a_1 = \frac{2}{3}$ and so

$$\underline{f(x) \cong \frac{2}{3} x + \cdots}$$

<u>Remark</u>: I'm using discussion in text on pgs. 624–625. Orthogonal polynomials are natural extensions of Fourier Idea.