§10.4 #5 Let $f(x) = -1$ for $0 < x < 1$. Compute the Fourier sine series for $f(x)$.

The idea is simply to extend f to an odd function f_o on $-1 < x < 1$ and compute the Fourier series of that. It will necessarily only have sine terms due to the odd character of f_o. We don't have to find f_o but that's what motivates,

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{n\pi x}{T}\right)dx$$

$$= 2\int_0^1 -\sin(n\pi x)\,dx$$

$$= \frac{2}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$= \frac{2}{n\pi}\left[\cos(n\pi) - 1\right] \implies \boxed{f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[(-1)^n - 1\right]\sin(n\pi x)}$$

We can simplify the boxed answer by noting n even yields trivial terms, thus simplify to case $n = 2k-1$ for $k \geq 1$ and then $(-1)^n - 1 = (-1)^{2k-1} - 1 = (-1)^{-1} - 1 = -2$ thus

$$\boxed{f(x) \sim \sum_{k=1}^{\infty} \frac{-4}{2k-1}\sin\left[(2k-1)\pi x\right]}$$

§10.4 #13 Find Fourier cosine series for e^x on $0 < x < 1$

We calculate $a_0 = 2\int_0^1 e^x dx = 2(e-1)$. Then for $n \geq 1$

$$a_n = 2\int_0^1 e^x\cos(n\pi x)\,dx$$

$$= \frac{2e\cos(n\pi)}{n^2\pi^2+1} + \frac{2en\pi\sin(n\pi)}{n^2\pi^2+1} - \frac{2}{n^2\pi^2+1}$$

$$= \frac{2e(-1)^n - 2}{n^2\pi^2+1}$$

Thus $$\boxed{e^x = e - 1 + \sum_{n=1}^{\infty} \frac{2(e(-1)^n - 1)}{n^2\pi^2+1}\cos(n\pi x)}$$

Remark: It is hopefully becoming clear that integration is key to find Fourier Expansions. Straight-forward, but tedious.

§10.4 #17 Solve the heat flow problem

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 < x < \pi,\ t > 0 \quad \text{and}$$

$$\left.\begin{aligned} &u(x,0) = f(x) = 1 - \cos(2x) \quad \text{for} \quad 0 < x < \pi \\ &\text{and} \quad u(0,t) = u(\pi,t) = 0 \quad \text{for} \quad t > 0 \end{aligned}\right\} \text{Boundary Conditions (BC's)}$$

The Fourier cosine series for $f(x)$ is given, however the sol$^{\underline{n}}$ to the heat flow problem (1)-(2)-(3) on pg. 606-609 is naturally given in terms of the Fourier sine series, thus we compute the Fourier sine series, Let $n \in \mathbb{N}$,

$$\cos A \sin B = \tfrac{1}{2}(e^{iA}+e^{-iA})\tfrac{1}{2i}(e^{iB}-e^{-iB}) = \tfrac{1}{4i}(e^{i(A+B)} - e^{-i(A+B)}) + -\tfrac{1}{4i}(e^{i(A-B)} - e^{-i(A-B)}) = \tfrac{1}{2}\sin(A+B) - \tfrac{1}{2}\sin(A-B)$$

Let $A = 2x$, $B = nx$

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi (1-\cos(2x))\sin(nx)\,dx \\
&= \frac{2}{\pi}\int_0^\pi \sin(nx)\,dx - \frac{2}{\pi}\int_0^\pi \cos(2x)\sin(nx)\,dx \\
&= \frac{-2}{n\pi}\cos(nx)\Big|_0^\pi - \frac{1}{\pi}\int_0^\pi \{\sin[(2+n)x] - \sin[(2-n)x]\}\,dx \\
&= \frac{-2}{n\pi}[\cos(n\pi) - 1] - \frac{1}{\pi}\int_0^\pi \sin((2+n)x)\,dx + \frac{1}{\pi}\int_0^\pi \sin((2-n)x)\,dx \\
&= \frac{2}{n\pi}[1-(-1)^n] + \frac{1}{(2+n)\pi}\cos((2+n)x)\Big|_0^\pi - \frac{1}{(2-n)\pi}\cos((2-n)x)\Big|_0^\pi \\
&= \frac{2}{n\pi}[1-(-1)^n] + \frac{1}{(2+n)\pi}[\cos(2\pi+n\pi) - 1] - \frac{1}{(2-n)\pi}[\cos(2\pi - n\pi) - 1] \\
&= \frac{2}{n\pi}[1-(-1)^n] + \frac{1}{(2+n)\pi}[(-1)^n - 1] - \frac{1}{(2-n)\pi}[(-1)^n - 1]
\end{aligned}$$

If n is even then $1-(-1)^n = 0$ thus $n = 2k-1$ for $k \in \mathbb{N}$ yield the nontrivial terms, note $1-(-1)^{2k-1} = 1+1 = 2$.

$$b_{2k-1} = \frac{4}{n\pi} - \frac{2}{(2+n)\pi} + \frac{2}{(2-n)\pi} \qquad (n = 2k-1)$$

Thus $b_{2k-1} = \dfrac{1}{\pi}\left[\dfrac{4}{2k-1} - \dfrac{2}{2+2k-1} + \dfrac{2}{2-2k+1}\right]$

Consequently the sol$^{\underline{n}}$ (noting $\beta = 5$ and using Eq$^{\underline{n}}$ (11) on p. 608),

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \frac{2}{\pi}\left[\frac{2}{2k-1} - \frac{1}{2k+1} - \frac{1}{2k-3}\right] e^{-5(2k-1)^2 t}\sin[(2k-1)x]}$$

(we rewrote Eq$^{\underline{n}}$ (11) as to sum over odd n which is conveniently described by summing $k = 1, 2, \ldots$ and replacing n with $2k-1$.)