§10.5 #3 Find a formal sol^n to the BVP on $0<x<\pi,\ t>0$

(*) $\boxed{\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}}$ and $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0,\ t>0$

Where $u(x,0) = x$ on $0<x<\pi$

Remark: §10.1, 10.2, 10.3, 10.4 were more or less "plug & chug". Now we solve the problems from start to finish w/o relying on a complete derivation of the problem from the text. Notice this is not the same as the heat-flow problem (1)-(2)-(3) solved on 606-609, and even if it was the time has come to derive it for yourself.

Step 1. Assume $u(x,t) = X(x)\,T(t)$,

$$\left.\begin{array}{l}\frac{\partial u}{\partial t} = XT' \\ \frac{\partial^2 u}{\partial x^2} = X''T\end{array}\right\} \longrightarrow \underbrace{XT' = 3X''T}$$

divide by XT to find $\frac{T'}{T} = \frac{3X''}{X}$

Step 2. Let $K = \frac{T'}{T} = \frac{3X''}{X}$ notice these must be constant since T'/T is fnct. of t while $3X''/X$ is fnct of x only. We find two ODE's that must be solved,

$$\boxed{\begin{array}{l} T' - KT = 0 \\ 3X'' - KX = 0 \end{array}} \text{ — } (**)$$

Step 3. Apply the given BC's to (**). This will place strict conditions on the constant K. We have to examine all possible values for K. Since we divided by XT we already assumed we're looking for nontrivial sol^{n}s.

Step 3 Continued:

If $K = 0$ then $T'' = 0$ & $\Xi'' = 0$ thus $T(t) = A + Bt$ and $\Xi(x) = C + Dx$. Note the BC's state

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(\pi, t) = 0 \quad \text{for all } t > 0$$

$$\Rightarrow \Xi'(0)T(t) = 0 \quad \text{and} \quad \Xi'(\pi)T(t) = 0 \quad \text{for all } t > 0$$

We assume $T(t) \neq 0$ for at least some $t > 0$ thus we require $\underline{\Xi'(0) = 0}$ and $\underline{\Xi'(\pi) = 0}$ to satisfy the BC's.

We find $\Xi'(0) = D_1 = 0 \therefore \boxed{\Xi_0(x) = A.} \leftarrow K = 0$ case.

If $K > 0$ then (**) states $3\Xi'' - K\Xi = 0$ yields $3\lambda^2 - K = 0 \Rightarrow \lambda = \pm\sqrt{K/3} = \pm\alpha$ (defines α)

Sol^{n}'s are $\Xi(x) = A\cosh\alpha x + B\sinh\alpha x$. Note the arguments given in $K = 0$ case still hold here thus we require $\Xi'(0) = 0$ and $\Xi'(\pi) = 0$;

$$\Xi'(0) = A\alpha\sinh(0) + B\alpha\cosh(0) = 0 \Rightarrow \underline{B = 0}.$$

$$\Xi'(\pi) = A\alpha\sinh(\pi) = 0 \Rightarrow \underline{A = 0}. \quad \therefore \boxed{\text{no nontrivial sol}^{n}\text{'s for } K > 0 \text{ case}}$$

If $K < 0$ then (**) states $3\Xi'' - K\Xi = 0 \Rightarrow 3\lambda^2 - K = 0$ thus $\lambda = \pm\sqrt{K/3} = \pm i\sqrt{-K/3} = \pm i\gamma$ where $\gamma \in \mathbb{R}$ is defined by the eq^{n} just stated. General sol^{n}'s of (**) are $\Xi(x) = A\cos\gamma x + B\sin\gamma x$. We need the general sol^{n} to fit the BC's, this means $\Xi'(0) = \Xi'(\pi) = 0$.

$$\Xi'(0) = -A\gamma\sin(0) + B\gamma\cos(0) = 0 \Rightarrow \underline{B = 0}.$$

$$\Xi'(\pi) = -A\gamma\sin(\gamma\pi) = 0$$

For nontrivial sol^{n} we need $A \neq 0$ hence $\sin(\gamma\pi) = 0$. It follows $\gamma\pi = n\pi$ for some $n \in \mathbb{Z}$. By definition I have $\gamma > 0$ thus $n \in \mathbb{N}$. Thus there are <u>many</u> sol^{n}'s in the $K < 0$ case, we index them by $n \in \mathbb{N}$,

$$\boxed{\Xi_n(x) = C_n\cos(nx) \quad \text{for arbitrary } n.}$$

Step 3 Continued:

We've examined all cases for K and it turns out that only $K=0$ and $K<0$ yield nontrivial sol^{n}'s to (*) which satisfy the BC's. Note $n=0$ fits formula ($A=c_0$),

$$\underline{X_n(x) = c_n \cos(nx)}.$$

Now, we turn to the other half of (**), $T' - KT = 0$. When $K=0$ we find $T'=0$ thus $T(t) = \text{const.}$ When $K<0$ we have $\gamma = \sqrt{-K/3} = n \longrightarrow \frac{-K}{3} = n^2 \longrightarrow \underline{K = -3n^2}$.

Thus $T' + 3n^2 T = 0 \Rightarrow \underline{T_n(t) = \tilde{c}_n e^{-3n^2 t}}$. Thus we have sol^{n}'s $u_n(x,t) = X_n(x)\,T_n(t)$ of the form,

$$\underline{u_n(x,t) = c_n \cos(nx)\, e^{-3n^2 t}}.$$

(family of sol^{n}'s to (*) which satisfy BC's $u_x(0,t) = u_x(\pi,t) = 0$,)

Step 4: We cannot satisfy $u(x,0) = x$ with a single member of the family of sol^{n}'s above. However, for an appropriate choice of Fourier coefficients a_n the sol^{n}

$$\underline{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\, e^{-3n^2 t}}$$

will satisfy the initial condition $u(x,0) = x$. We calculate the coefficients via the usual integrals (see pg. 635, $T=\pi$)

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx \;:\; (\text{use IBP, } dV = \cos(nx)dx,\ u = x)$$

$$= \frac{2}{\pi}\left(\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n}\,dx \right)$$

$$= \left(\frac{2\cos(nx)}{n^2}\right)\Big|_0^{\pi}$$

$$= \underline{\frac{2}{n^2}\left[(-1)^n - 1\right] = a_n} \quad (\text{for } n \geq 1)$$

The case $n=0$ must be dealt with separately,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\,\frac{\pi^2}{2} = \underline{\pi = a_0}.$$

Step 4 Continued:

We can simplify the formula for a_n if we notice $a_{2k} = 0$ for any $k \in \mathbb{N}$ since $(-1)^n - 1 = 1 - 1 = 0$ when n is even. Thus we can rewrite the solⁿ as a sum over $k = 1, 2, 3, \dots$ where n is replaced by $2k-1$ and $(-1)^n - 1 = -2$. Hence,

$$\boxed{u(x,t) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{-4}{(2k-1)^2} \cos\left[(2k-1)x\right] e^{-3(2k-1)^2 t}}$$

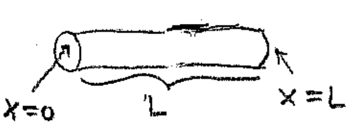
provides a <u>formal</u> solⁿ to (*) together with it's BC's and IC $u(x,0) = x$. It is "formal" since we have no proof that the series above converges, or that u_{xx}, u_t have convergent Fourier series.

Remark: Solving a heat-equation (or wave-eqⁿ) etc... problem from start to finish often takes several pages of work. If the problem matches one we've already solved then we can refer to earlier work. However, I do expect you understand the steps leading to the solⁿ. I can easily change the problem just a bit and ask you to solve a problem for which there does not exist a prepackaged solⁿ.

Observation: To model temperature u at position x along a rod of length L we found the heat-flow PDE predicts $u(x,t)$:

$$\underbrace{\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}}_{\text{one-dimensional heat flow eq}^{\underline{n}}}$$

- → $u(0,t) = 0$, $u(L,t) = 0$ → Fourier Sine Series Solⁿ.
- → $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$ → Fourier Cosine Series Solⁿ
- → other BC's → Some combination, a.k.a. the "a_n" and "b_n" both non zero. (we've not seen this yet)



PH-139

§10.5 #7 Solve $\boxed{\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}}$ —(*) , $0 < x < \pi$, $t > 0$

for BC's $u(0,t) = 5$ and $u(\pi, t) = 10$, $t > 0$

and IC $u(x,0) = \sin(3x) - \sin(5x)$, $0 < x < \pi$

I'll follow the logic of Ex. 2 on pg. 642–643. We suppose $\exists$ sol$^{\underline{n}}$

$$u(x,t) = V(x) + W(x,t)$$

Notice that

$$\frac{\partial u}{\partial t} = \frac{\partial W}{\partial t} \quad \text{since } \frac{\partial}{\partial t}(V) = 0 \text{ as } V \text{ is a fnct. of only } x.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 W}{\partial x^2}$$

Recall we wish to solve $u_t = 2u_{xx}$ this becomes

$$W_t = 2V_{xx} + 2W_{xx}$$

If we impose $V_{xx} = 0$ and $W_t = 2W_{xx}$ then $u = V + W$ will solve (*). The reason to introduce V and W is so that W can cover the transient sol$^{\underline{n}}$ while V allows for the desired steady state sol$^{\underline{n}}$. In particular the idea is for V to allow $u(0,t) = 5$ and $u(\pi, t) = 10$. Notice

$$u(0,t) = V(0) + W(0,t) = 5$$
$$u(\pi,t) = V(\pi) + W(\pi,t) = 10$$

Notice $W(0,t) = W(\pi, t) = 0$ and $V(0) = 5$, $V(\pi) = 10$ will give $u(x,t)$ the required BC's.

<u>Sol$^{\underline{n}}$ of Steady State Part:</u>

We require $V''(x) = 0$ and $V(0) = 5$ and $V(\pi) = 10$.

$\lambda^2 = 0 \Rightarrow V(x) = Ax + B$ and $V(0) = B = 5$, $V(\pi) = A\pi + 5 = 10$

Hence $\underbrace{\underline{V(x) = \frac{5}{\pi}x + 5}}$.

this is half of our proposed sol$^{\underline{n}}$
notice we still need to resolve
the form of the transient sol$^{\underline{n}}$ $W(x,t)$

We need $u(x,0) = \sin(3x) - \sin(5x)$ this suggests $u(x,0) = v(x) + w(x,0) = \sin(3x) - \sin(5x)$. Thus the initial condition on $w(x,t)$ is found to be:

$$w(x,0) = \sin(3x) - \sin(5x) - \frac{5}{\pi}x - 5 \equiv f(x)$$

Thus we are faced with solving the standard (1)-(2)-(3) heat-flow problem for $w(x,t)$ where

$$\boxed{w_t = 2w_{xx} \quad \text{and} \quad \underbrace{w(0,t) = w(\pi,t) = 0}_{B.C's} \quad \text{and} \quad \underbrace{w(x,0) = f(x)}_{IC's}}$$

This problem reduces to finding the Fourier sine series for $f(x)$, I expect you could derive this but I'll forgo the details this time (see 606-609 for gory details). We propose that $\exists$ coefficients C_n such that $f(x) = \sum_{n=1}^{\infty} C_n \sin(nx)$,

Clearly $\sin(3x), \sin(5x)$ contribute to C_3 and C_5. Consider $g(x) = \frac{-5}{\pi}x - 5$ separately, $g(x) = \sum_{n=1}^{\infty}{}' \, \bar{C}_n \sin(nx)$

We can find $\bar{C}_n$ by the usual integrals (see p. 635)

$$\bar{C}_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{-5x}{\pi} - 5\right)\sin(nx)\,dx$$

(this problem is for $0<x<L$ we use "Fourier Sine Series" for $[0,\pi]$ (a.k.a half-range sine expansion))

$$= \frac{-10}{\pi^2}\int_0^{\pi} x\sin(nx)\,dx - \frac{10}{\pi}\int \sin(nx)\,dx$$

$$= \frac{-10}{\pi^2}\left[\frac{-1}{n}x\cos nx + \frac{1}{n}\sin nx\right]\Bigg|_0^{\pi} + \frac{10}{n\pi}\cos nx\Big|_0^{\pi}$$

$$= \frac{10}{n\pi^2}\pi\cos(n\pi) + \frac{10}{n\pi}[\cos(n\pi) - 1]$$

$$= \frac{10}{n\pi}\left[2(-1)^n - 1\right]$$

Thus,

$$f(x) = \sin(3x) - \sin(5x) + \sum_{n=1}^{\infty}\frac{10}{n\pi}\left[2(-1)^n - 1\right]\sin(nx)$$

We find $C_n = \frac{10}{n\pi}\left[2(-1)^n - 1\right] + \underbrace{\delta_{n,3} - \delta_{n,5}}_{\text{adds or subtracts 1 in } n=3 \text{ or } n=5 \text{ case}}$

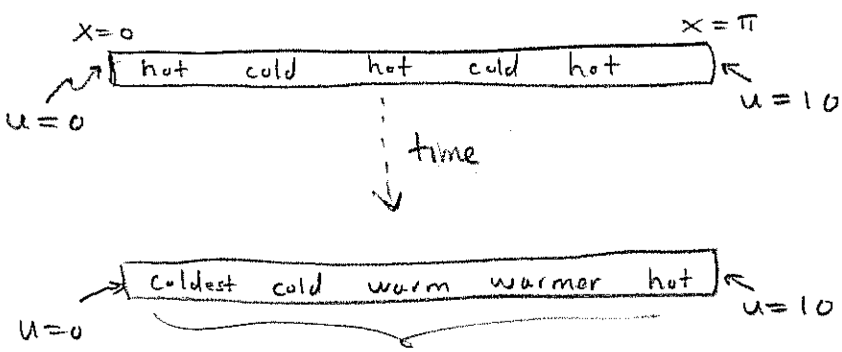
§10.5 # 7 Continued

We find $w(x,t) = \sum_{n=1}^{\infty} c_n e^{-2n^2 t} \sin(nx)$ where c_n is given on last page. Recall that the sol^{n} to (*) is denoted $u(x,t)$ and $u(x,t) = v(x) + w(x,t)$, the sol^{n} is

$$\boxed{\begin{aligned} u(x,t) = {} & \frac{5x}{\pi} + 5 - \frac{30}{\pi} e^{-2t} \sin x + \frac{5}{\pi} e^{-8t} \sin 2x + \\ & + \left(1 - \frac{10}{\pi}\right) e^{-18t} \sin 3x + \frac{5}{2\pi} e^{-32t} \sin 4x + \\ & - \left(1 + \frac{6}{\pi}\right) e^{-50t} \sin(5x) + \sum_{n=6}^{\infty} \frac{10}{n\pi}\left[2(-1)^n - 1\right] e^{-2n^2 t} \sin(nx) \end{aligned}}$$

The sol^{n} above has $u(x,t) \longrightarrow \frac{5x}{\pi} + 5$ as $t \longrightarrow \infty$. All the terms with $\exp(-\beta n^2 t)$ will vanish as $t \longrightarrow \infty$.

<u>Remark:</u> physically this sol^{n} is interesting we have a rod or bar where the ends are held at fixed, but unequal temperatures, then whatever the initial temp. distribution is it will dissipate and resolve into $\frac{5x}{\pi} + 5$ after a long time.



linear increase in temp. w.r.t x.

§ 10.5 # 15 Find formal sol$^{\underline{n}}$ to $\boxed{\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}}$ — (*)
on $0 < x < \pi$, $0 < y < \pi$, $t > 0$ subject to the
BC's $\frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(\pi, y, t) = 0$ for $0 < y < \pi$, $t > 0$
and $u(x, 0, t) = u(x, \pi, t) = 0$ for $0 < x < \pi$, $t > 0$
and Initial Condition $u(x, y, 0) = \cos 6x \sin 4y - 3 \cos x \sin 11y$

Step 1: Assume $u(x, y, t) = X(x) Y(y) T(t)$ we find (*) yields

$$XYT' = X''YT + XY''T$$

Step 2:
$$\underbrace{\frac{T'}{T}}_{\text{fnct. of } t \text{ only}} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{fnct. of } x, y \text{ only}} \Rightarrow \frac{T'}{T} = K = \frac{X''}{X} + \frac{Y''}{Y}$$

$\uparrow$ constant

Thus $T' = KT$ and $\underbrace{K - \dfrac{Y''}{Y}}_{\text{fnct of } y \text{ alone}} = \underbrace{\dfrac{X''}{X}}_{\text{fnct of } x} = J$ $\leftarrow$ constant

Hence $X'' = JX$ while $KY - Y'' = JY$.

To summarize,

$$\boxed{\begin{aligned} &\text{I.} \quad T' - KT = 0 \\ &\text{II.} \quad X'' - JX = 0 \\ &\text{III.} \quad Y'' + (J - K)Y = 0 \end{aligned}}$$

the given PDE (*) reduces to this family of ODEs. We'll find both J and K can only take on certain values due to the constraint of BC's.

Step 3: Translate BC's in terms of $u(x, y, t)$ to corresponding conditions for $X(x)$, $Y(y)$, $T(t)$,

$$\frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(\pi, y, t) = 0 \Rightarrow \underline{X'(0) = X'(\pi) = 0}. \quad \text{IV}$$

$$u(x, 0, t) = u(x, \pi, t) = 0 \Rightarrow \underline{Y(0) = Y(\pi) = 0}. \quad \text{V}$$

§10.5 #15 Continued:

BC (IV) places strict conditions on J from II; $X'' - JX = 0$ for $0 < x < \pi$, with $X'(0) = 0$ and $X'(\pi) = 0$. This is the same as Step 3. of §10.5#3 we'll find $J=0$ and $J<0$ yield nontrivial sol^n's, for $n = 0, 1, 2, 3, \ldots$

$$X_n(x) = C_n \cos(nx) \quad \text{and} \quad J = -n^2 \quad \text{VI}$$

no 3 in our current problem see $\boxed{\text{PH-136-137}}$

Now turn back to III notice our sol^n's to II require that J have the special form $J = -n^2$ for $J \in \mathbb{N} \cup \{0\}$. We wish to solve,

$$Y'' - (n^2 + K)Y = 0 \quad \text{with} \quad \underbrace{Y(0) = Y(\pi) = 0}_{\text{V}}$$

This is the (1)-(2)-(3) heat-flow problem which is solved 606 - 609 in text. We need $n^2 + K < 0$ in order to get nontrivial sol^n's. Let us work it out,

$$\lambda^2 - n^2 - K = 0 \Rightarrow \lambda_n = \pm\sqrt{n^2+K}$$
$$\Rightarrow \lambda_n = \pm i\sqrt{-n^2-K} = \pm i\gamma_n \quad \text{for } \gamma_n \in \mathbb{R}$$

Hence $Y(y) = A\cos(\gamma_n y) + B\sin(\gamma_n y)$

$$Y(0) = A\cos(0) + B\sin(0) \Rightarrow A = 0$$
$$Y(\pi) = B\sin(\gamma_n \pi) = 0 \Rightarrow \gamma_n \pi = m\pi \quad \text{for } m \in \mathbb{N}$$

Thus $\sqrt{-n^2-K} = m$ for some $m \in \mathbb{N}$. Solving for K reveals $\underline{K = -m^2 - n^2}$ thus

$$Y_{mn}(y) = a_{mn}\sin(my) \quad \text{VII}$$

For each n indexing sol^n's to II we find a whole family of sol^n's to III. Finally we look at I, we know $K = -m^2 - n^2$ thus $T' = -(m^2+n^2)T$ hence $T(t) = \bar{a}_{mn} e^{-(m^2+n^2)t}$
In total we find the following family of sol^n's for the given BC's

$$u_{mn}(x,y,t) = a_{mn}\cos(nx)\sin(my)\, e^{(-m^2-n^2)t} \quad \text{VIII}$$

§10.5 #15 Continued:

The sol^{n}'s given in VIII. satisfy the BC's but the initial conditions often require an infinite sum of the $U_{mn}(x, y, 0)$. Fortunately, our initial condition is

$$u(x, y, 0) = \cos 6x \sin 4y - 3\cos x \sin 11y$$

thus the general sol^{n}

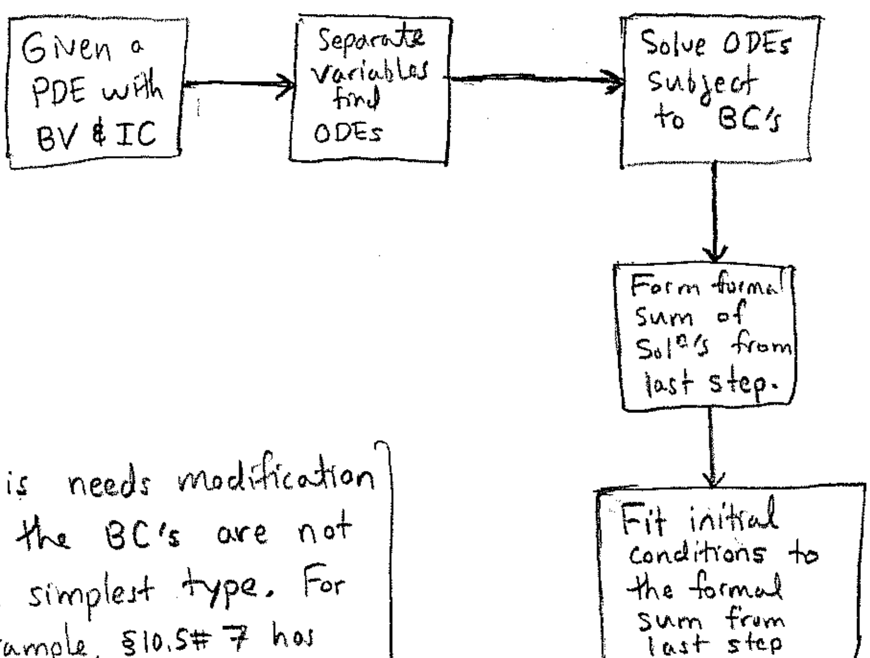
$$u(x, y, t) = \sum_{m,n=1}^{\infty} a_{mn} \cos(nx) \sin(my)\, e^{-(m^2+n^2)t}$$

has only $a_{4,6} = 1$ and $a_{11,1} = -3$ non trivial,

$$\boxed{u(x, y, t) = \cos(6x)\sin(4y)\, e^{-52t} - 3\cos(x)\sin(11y)\, e^{-122t}}$$

The function $f(x, y) = \cos 6x \sin 4y - 3\cos x \sin 11y$ is an example of a "double Fourier series" of an extremely simple type. If $f(x, y) = xy$ or most anything else we'd have to do integrations detailed in eq^{n}'s (54)–(55).

General Idea for solving PDEs with BC's and an IC.



This needs modification if the BC's are not the simplest type. For example, §10.5 #7 has extra steps due to the mismatched BC's.