§10.6#1 Find formal solⁿ of the Wave Eqⁿ $\boxed{\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}}$ — $(*)$
for $0 < x < 1$, $t > 0$ with $u(0,t) = u(1,t) = 0$, $t > 0$ (BC's)
and $u(x,0) = x(1-x)$, $\frac{\partial u}{\partial t}(x,0) = \sin(7\pi x)$ for $0<x<1$ (IC's)

Let $u(x,t) = X(x)T(t)$ then $\frac{\partial^2 u}{\partial t^2} = XT''$ while $\frac{\partial^2 u}{\partial x^2} = X''T$.
Hence $XT'' = X''T$ divide by XT to obtain $\frac{X''}{X} = \frac{T''}{T}$.
Notice X''/X is function of x alone while T''/T is fnct. of t alone
thus $\exists\, K$ constant such that $\frac{X''}{X} = K = \frac{T''}{T}$. Consequently,
if the solⁿ of $(*)$ is separable with $u = XT$ then

$$\boxed{\begin{array}{l} X'' - KX = 0 \\ T'' - KT = 0 \end{array}} \quad (**)$$

As usual we'll find $(**)$ has general solⁿ's compatible with the BC's iff we make additional assumptions about the specific form of K. Let's work it out,

$\underline{K=0}$ $X'' = 0 \Rightarrow X(x) = A + Bx$ and $u(0,t) = T(t)A = 0 \Rightarrow \underline{A=0}$
while $u(1,t) = T(t)[A+B] = 0 \Rightarrow \underline{B=0}$. Thus only trivial solⁿ.

$\underline{K>0}$ Let $K = \beta^2 > 0$ then $X'' - \beta^2 X = 0$ has solⁿ's $X = A\cosh\beta x + B\sinh\beta x$
and $u(0,t) = T(t)A = 0 \Rightarrow \underline{A=0}$, and $u(1,t) = B\sinh\beta = 0 \Rightarrow \underline{B=0}$.
Again this case for K yields only trivial solⁿs.

$\underline{K<0}$ Let $K = -\beta^2$ for $\beta \in \mathbb{R}$, $X'' + \beta^2 X = 0$ has solⁿ's
$X(x) = A\cos\beta x + B\sin\beta x$. Note $u(0,t) = T(t)A = 0 \Rightarrow \underline{A=0}$.
However, $u(1,t) = T(t)B\sin\beta = 0 \Rightarrow \underline{\sin\beta = 0}$.
Without loss of generality assume $\beta > 0$ thus we find solⁿ's
for $\sin\beta = 0$ of the form $\beta = \pi, 2\pi, \dots$ that is $\beta = n\pi$
for some $n \in \mathbb{N}$. Hence, $K = -n^2\pi^2$ for $n \in \mathbb{N}$
with solⁿ's of $X'' - KX = 0$ have form

$$\underline{X_n(x) = C_n \sin(n\pi x) \quad \text{for } n \in \mathbb{N}}$$

family of BC-satisfying general solⁿ's of $(**)$.

§10.6 #1 Continued

The eq^{n}'s (**) come in a pair. For each $n \in \mathbb{N}$ we face the problem of finding $T_n(t)$ to pair $X_n(x)$,

$$T'' + n^2\pi^2 T = 0 \longrightarrow \underline{T_n(t) = \bar{A}_n \cos(n\pi t) + \bar{B}_n \sin(n\pi t)}.$$

We can absorb c_n of X_n into the $\bar{A}_n$, $\bar{B}_n$ to get

$$\boxed{u_n(x,t) = A_n \cos(n\pi t)\sin(n\pi x) + B_n \sin(n\pi t)\sin(n\pi x)}$$

These give general sol^{n}'s to (*) subject to the BC's. To fit the initial conditions we'll sum all the u_n's together and then once the IC's are also expanded as Fourier series we can compare and read off the sol^{n}. We propose,

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos(n\pi t)\sin(n\pi x) + B_n \sin(n\pi t)\sin(n\pi x)\right]$$

A short calculation reveals

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = x - x^2 \quad , \quad 0 < x < 1 \quad ☆$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} n\pi B_n \sin(n\pi x) = \sin(7\pi x)$$

To satisfy the IC's a particular choice of A_n, B_n must be made. The B_n are easy since clearly $\underline{\text{only}}$ $B_7 \neq 0$ moreover, $7\pi B_7 = 1 \quad \therefore \boxed{B_7 = 1/7\pi}$. To find the values of A_n we need to find the Fourier sine series of $x - x^2$,

$$x - x^2 = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \text{where we calculate}$$

$$a_n = 2\int_0^1 (x - x^2)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 x\sin(n\pi x)dx - 2\int_0^1 x^2 \sin(n\pi x)\,dx \longrightarrow \boxed{\begin{array}{ll} u = x^2 & du = 2x\,dx \\ dv = \sin n\pi x\,dx & V = -\frac{\cos n\pi x}{n\pi} \end{array}}$$

$$= \underbrace{2\int_0^1 x\sin(n\pi x)dx}_{\xi_1} - 2\left[\underbrace{-\frac{x^2\cos n\pi x}{n\pi}\Big|_0^1}_{\xi_2} + \int_0^1 \frac{2x\cos n\pi x}{n\pi}\,dx\right]$$

$$u = \frac{2x}{n\pi}, \quad du = \frac{2d}{n\pi}$$

$$dV = \cos n\pi x$$

$$V = \frac{\sin n\pi x}{n\pi}$$

$$= \xi_1 - 2\xi_2 - 2\left[\cancel{\frac{2x\sin n\pi x}{n^2\pi^2}\Big|_0^1}^{\,0} - \frac{2}{n^2\pi^2}\int_0^1 \sin n\pi x\,dx\right]$$

$$= \xi_1 + 2\left[\frac{\cos n\pi}{n\pi}\right] - 2\left[\frac{-2}{n^2\pi^2}\left[-\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi}\right]\right]$$

§10.6 # 1 Continued

$$\xi_1 = 2\int_0^1 x\sin(n\pi x)\,dx \qquad \begin{array}{ll} u = x & du = dx \\ dV = \sin(n\pi x)dx & V = \dfrac{-\cos n\pi x}{n\pi} \end{array}$$

$$= -\frac{2x\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}\,dx$$

$$= \frac{-2\cos n\pi}{n\pi} + \frac{\sin n\pi}{(n\pi)^2}\Big|_0^1$$

$$= \frac{-2\cos n\pi}{n\pi}$$

Returning to a_n,

$$a_n = \cancel{\frac{-2\cos n\pi}{n\pi}} + \cancel{\frac{2\cos n\pi}{n\pi}} + \frac{4}{n^3\pi^3}\left[1 - \cos n\pi\right]$$

$$a_n = \frac{4}{n^3\pi^3}\left[1-(-1)^n\right] = \begin{cases} 0 & \text{if } n = 2k \text{ for } k\in\mathbb{N}. \\ \dfrac{8}{(n\pi)^3} & \text{if } n = 2k-1 \text{ for } k\in\mathbb{N}. \end{cases}$$

We find,

$$x - x^2 = \sum_{k=1}^{\infty} \frac{8}{(2k-1)^3\pi^3}\sin((2k-1)\pi x)$$

Return to ☆ to see $A_n = a_n$ thus $A_n = 0$ for n even whereas $A_{2k-1} = \dfrac{8}{(2k-1)^3\pi^3}$ for $k\in\mathbb{N}$. Our proposed sol$^{\underline{n}}$ $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ fit to the IC's is:

$$\boxed{u(x,t) = \frac{1}{7\pi}\sin(7\pi t)\sin(7\pi x) + \sum_{k=1}^{\infty}\frac{8}{(2k-1)^3\pi^3}\cos[(2k-1)\pi t]\sin[(2k-1)\pi x]}$$

<u>Remark</u>: only 3 pgs., not too bad.

§10.6 #13 Find sol^{n} for wave eq^{n} $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ on $-\infty < x < \infty$, $t > 0$ given that $u(x,0) = 0$ while $\frac{\partial u}{\partial t}(x,0) = \cos(x)$.

We use d'Alembert's sol^{n} $u(x,t) = A(x+\alpha t) + B(x-\alpha t)$ where A, B are unknown functions. Let's follow Ex. 2 on pg. 657 except we'll put in our formulas for $f(x) = 0$ and $g(x) = \cos(x)$,

$$u(x,0) = A(x) + B(x) = 0 \implies \underline{A(x) = -B(x) \quad \forall x \in \mathbb{R}}.$$

$$\frac{\partial u}{\partial t}(x,0) = \underbrace{\alpha A'(x) - \alpha B'(x)} = \cos(x)$$

$$-2\alpha B'(x) = \cos(x) \implies \boxed{\begin{aligned} B(x) &= \frac{\sin(x)}{-2\alpha} + c \\ A(x) &= \frac{\sin(x)}{2\alpha} - c \end{aligned}}$$

cancel out in $u(x,t)$

Now replace x with $x \pm \alpha t$,

$$\boxed{u(x,t) = \frac{1}{2\alpha}\left[\sin(x+\alpha t) - \sin(x-\alpha t)\right]}$$

§10.6#15 Find sol^{n} for wave eq^{n} $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ on $-\infty < x < \infty$, $t > 0$ given that $u(x,0) = x$ and $\frac{\partial u}{\partial t}(x,0) = x$

Again use d'Alembert's sol^{n} $u(x,t) = A(x+\alpha t) + B(x-\alpha t)$. We seek to determine the functions A, B. Apply initial cond,

$$u(x,0) = A(x) + B(x) = x \implies \underline{B(x) = x - A(x)}$$

$$\frac{\partial u}{\partial t}(x,0) = \alpha A'(x) - \alpha B'(x) = x$$

$$\alpha A'(x) - \alpha[1 - A'(x)] = x$$

$$2\alpha \frac{dA}{dx} = x + \alpha \implies \frac{dA}{dx} = \frac{x}{2\alpha} + \frac{\alpha}{2\alpha}$$

$$\implies A(x) = \frac{1}{4\alpha}x^2 + \frac{1}{2}x + c$$

$$\implies B(x) = \frac{1}{2}x - \frac{1}{4\alpha}x^2 - c$$

cancel again

Replace x with $x \pm \alpha t$ as d'Alembert's sol^{n} requires,

$$u(x,t) = \frac{1}{4\alpha}(x+\alpha t)^2 + \frac{1}{2}(x+\alpha t) + \frac{1}{2}(x-\alpha t) - \frac{1}{4\alpha}(x-\alpha t)^2$$

$$= \frac{1}{4\alpha}\left[\cancel{x^2} + 2x\alpha t + \cancel{\alpha^2 t^2}\right] + x - \frac{1}{4\alpha}\left[\cancel{x^2} - 2\alpha x t + \cancel{\alpha^2 t^2}\right] = \boxed{x + xt}$$

Remark: d'Alembert's sol^{n} is quite interesting. Seems like a large departure from our usual Fourier series approach yet we'll find the same sol^{n}'s in certain cases. Note #13 had

$$u(x,t) = \frac{1}{2\alpha}\left[\sin(x+\alpha t) - \sin(x-\alpha t)\right]$$

$$= \frac{1}{2\alpha}\left[\cancel{\sin(x)\cos(\alpha t)} + \cos(x)\sin(\alpha t) - \cancel{\sin(x)\cos(\alpha t)} + \cos x \sin \alpha t\right]$$

$$= \underbrace{\frac{1}{\alpha}\cos(x)\sin(\alpha t)}$$

this is a sol^{n} we would find (I think) for $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ with $u(x,0) = 0$ $0 < x < \pi$ and $\frac{\partial u}{\partial t}(x,0) = \cos(x)$ for $0 < x < \pi$

So terms like $\cos(kx)\sin(Vt)$ can be seen as two travelling waves interefering to produce a standing wave (658–659 also discuss this idea)