LAPLACE'S EQUATION: (§10.7)

<u>Remark</u>: the technique for solving $\nabla^2 u = 0$ is not terribly different from what we've already done. Laplace's Eq$^{\underline{n}}$ refers to many different eq$^{\underline{n}}$'s since $\nabla^2 = \frac{\partial^2}{\partial x^2}$ or $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ or $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ depending on the context. The operator ∇^2 is called the "Laplacian", the text denotes $\nabla^2 = \Delta$. In non-Cartesian coordinates the Laplacian's formula is complicated since $\nabla^2 u = \nabla \cdot \nabla u = \left(e_1 \frac{\partial}{\partial u_1} + e_2 \frac{\partial}{\partial u_2} + e_3 \frac{\partial}{\partial u_3}\right) \cdot \left(e_1 \frac{\partial u}{\partial u_1} + e_2 \frac{\partial u}{\partial u_2} + e_3 \frac{\partial u}{\partial u_3}\right)$ and e_1, e_2, e_3 are the unit-vectors in directions of increasing u_1, u_2, u_3 (in particular $e_1 = \frac{\nabla u_1}{|\nabla u_1|}$, $e_2 = \frac{\nabla u_2}{|\nabla u_2|}$, $e_3 = \frac{\nabla u_3}{|\nabla u_3|}$) these u_i - coordinate vectors depend on position. For example, it turns out that the Laplacian in polar coordinates is,

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

there are many ways to derive this.

<u>Electrostatics</u>: Gauss' Law states $\nabla \cdot \vec{E} = \rho/\epsilon_0$ where $\vec{E}$ is the electric field and $\rho = \frac{dQ}{dV}$ the charge density. The electric potential φ satisfies $\vec{E} = -\nabla\varphi$ thus

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla\varphi) = -\nabla^2\varphi = \rho/\epsilon_0$$

The eq$^{\underline{n}}$ $\boxed{\nabla^2 \varphi = -\rho/\epsilon_0}$ is called Poisson's Eq$^{\underline{n}}$. If there is no charge present then $\rho = 0$ thus we find Laplace's Eq$^{\underline{n}}$ $\boxed{\nabla^2 \varphi = 0}$. Similar arguments can be made for Newtonian Gravity where $\vec{F} = -\nabla U$ and in the absense of mass we expect $\underline{\nabla^2 U = 0}$.

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§10.7 #1) Find formal sol$^{\underline{n}}$ for $\boxed{\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0}$ —(*)
for $\partial u/\partial x\,(0,y) = \partial u/\partial x\,(\pi, y) = 0$ for $0 \le y \le 1$ (BC's)
and $u(x,0) = 4\cos 6x + \cos 7x$ and $u(x,1) = 0$ for $0 \le x \le \pi$

Let $u(x,y) = X(x)\,Y(y)$ then $\frac{\partial^2 u}{\partial x^2} = X''Y$ and $\frac{\partial^2 u}{\partial y^2} = XY''$ thus (*) gives

$$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = \frac{-Y''}{Y} = K$$ since the $\frac{X''}{X}$ is

a function of x whereas $-Y''/Y$ is only a function of y. Apply the BC's $u_x(0,y) = u_x(\pi,y) = 0$ for $0 \le y \le 1$,

$$\frac{X''}{X} = K \rightarrow X'' - KX = 0$$

$K=0 \rightarrow$ $X(x) = Ax + B$ apply BC's to find $u_x(0,y) = A\,Y(y) = 0$ for $0 \le y \le 1 \Rightarrow A = 0$, Consequently $\underline{X(x) = B}$ in this case.

$K > 0$ $\rightarrow$ Let $K = \alpha^2$ then $X = A\cosh\alpha x + B\sinh\alpha x$
$u_x(0,y) = X'(0)\,Y(y) = \alpha B\cosh(0) = 0 \Rightarrow B = 0$
$u_x(\pi,y) = X'(\pi)\,Y(y) = \alpha A\sinh(\pi\alpha) = 0 \Rightarrow A = 0$
$\therefore u(x,y) = 0$ for $K > 0$ case

$K<0$ $\rightarrow$ Let $K = -\beta^2$ for some $\beta \in \mathbb{R}$ with $\beta > 0$. Sol$^{\underline{n}}$ is of form $\underline{X(x) = A\cos\beta x + B\sin\beta x}$.
Note, $u_x(0,y) = X'(0)\,Y(y) = \beta B\cos(0)\,Y(y) = 0 \Rightarrow \underline{B = 0}$.
Likewise $u_x(\pi,y) = X'(\pi)\,Y(y) = -\beta A\sin\beta\pi = 0 \Rightarrow \beta\pi = n\pi$ for some $n \in \mathbb{N}$ $(\beta = n)$.
Hence $K \le 0$ yields interesting sol$^{\underline{n}}$'s,

$$\underline{X_n(x) = c_n\cos(nx)}.$$ ($n = 0$ corresponds to $K = 0$ constant sol$^{\underline{n}}$)

Now consider $Y'' + KY = 0$ in the $K \le 0$ case, if $K = 0$ then $X(x) = B$ thus $u(x,t) = B\,Y(y)$ and $u_x = 0$ so the BC's yield no conditions on $Y(y)$. Unless I'm missing something it appears $\underline{Y_0(y) = cy + D}$. Let us continue to $K < 0$ case where $K = -\beta^2$ thus $Y'' - n^2 Y = 0$

$$\underline{Y_n(y) = a_n\cosh(ny) + b_n\sinh(ny)}$$

We've determined $Y_n(y) = a_n \cosh(ny) + b_n \sinh(ny)$ for $n \geq 1$, it follows,

$$U_0(x,y) = Cy + D$$
$$U_n(x,y) = a_n \cos(nx)\cosh(ny) + b_n \cos(nx)\sinh(ny)$$

We form our general (formal) sol^{n} as a sum over all possible BC-satisfying solutions,

$$\boxed{U(x,y) = Cy + D + \sum_{n=1}^{\infty}\left[a_n \cos(nx)\cosh(ny) + b_n \cos(nx)\sinh(ny)\right]} \quad (**)$$

We apply the other Bound. Conditions to (**) to determine the values for C, D, a_n, b_n,

$$U(x,0) = 4\cos 6x + \cos 7x$$
$$= C(0) + D + \sum_{n=1}^{\infty} a_n \cos nx + \underbrace{b_n \cos nx \sinh(0)}_{\text{vanishes due to } \sinh(0)=0 \text{, no info on } b_n \text{ here.}}$$

We deduce that

$$D = 0, \quad a_n = 0 \text{ for } n \neq 6, 7$$
$$\underline{a_6 = 4} \text{ and } \underline{a_7 = 1}.$$

Next, consider the remaining BC,

$$U(x,1) = C + 4\cos(6x)\cosh(6) + \cos(7x)\cosh(7) + \sum_{n=1}^{\infty} b_n \cos(nx)\sinh(n) = 0$$

We deduce we need $C = 0$, $b_n = 0$ for $n \neq 6, 7$ and

$$4\cosh(6) + b_6 \sinh(6) = 0 \longrightarrow b_6 = -4\left(\frac{\cosh 6}{\sinh 6}\right)$$
$$\cosh(7) + b_7 \sinh(7) = 0 \longrightarrow b_7 = -\left(\frac{\cosh 7}{\sinh 7}\right)$$

Collecting together the nontrivial terms,

$$U(x,y) = 4\cos 6x \cosh 6y + \cos 7x \cosh 7y - \frac{4\cos 6x \sinh 6y}{\tanh 6} - \frac{\cos 7x \sinh 7y}{\tanh 7}$$

$$= 4\cos 6x\left[\cosh(6y) - \frac{\sinh(6y)\cosh(6)}{\sinh(6)}\right] + \cos 7x\left[\cosh 7y - \frac{\sinh 7y \cosh 7}{\sinh 7}\right]$$

$$= \frac{4\cos 6x}{\sinh 6}\left[\cosh(6y)\sinh(6) - \sinh 6y \cosh 6\right] + \frac{\cos 7x}{\sinh 7}\left[\cosh 7y \sinh 7 - \sinh 7y \cosh 7\right]$$

$$\boxed{U(x,y) = -\frac{4\cos 6x}{\sinh 6}\sinh(6y-6) - \frac{\cos 7x}{\sinh 7}\sinh(7y-7)}$$

See Remark

Remark: Since $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$ and $\cosh\gamma = \frac{1}{2}(e^{\gamma}+e^{-\gamma})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta})$ and $\sinh(\gamma) = \frac{1}{2}(e^{\gamma}-e^{-\gamma})$ there are simple formulas connecting hyperbolic and ordinary trig functions, $\cosh(i\theta) = \cos\theta$ and $\sinh(i\theta) = i\sin(\theta)$ which are equivalent to $\boxed{\cosh(\theta) = \cos(i\theta) \text{ and } \sinh\theta = -i\sin(i\theta)}$

For example, $-i\sin(i\theta) = \frac{-i}{2i}\left(e^{i(i\theta)} - e^{-i(i\theta)}\right) = \frac{-1}{2}\left(e^{-\theta}-e^{\theta}\right) = \sinh\theta.$

This correspondance allows us to convert trig-identities to hyperbolic trig-identities and vice-versa, consider then

$$\begin{aligned}
\sinh(A+B) &= -i\sin(iA+iB) \\
&= -i\left[\sin iA \cos iB + \sin iB \cos iA\right] \\
&= -i\sin iA \cos iB - i\sin iB \cos iA \\
&= \sinh(A)\cosh(B) + \sinh(B)\cosh(A).
\end{aligned}$$

Of course this can be proved w/o the correspondance, but this is more fun. Anyhow, it's clear that

$$\begin{aligned}
\sinh(6y-6) &= \sinh(6y)\cosh(-6) + \sinh(-6)\cosh(6y) \\
&= \underline{\cosh 6 \sinh(6y) - \sinh(6)\cosh(6y)}.
\end{aligned}$$

We used that

$$-(\cosh 6 \sinh 6y - \sinh 6 \cosh 6y) = -\sinh(6y-6)$$

on the last step of the last page. Similiar remarks go for the $n=7$ terms.

$\boxed{\S 10.7 \# 3}$ Find formal sol^{n} for $\boxed{u_{xx} + u_{yy} = 0}$ — $(*)$ subject to the BC's $u(0,y) = u(\pi, y) = 0$ for $0 \leq y \leq \pi$ and the BC's $u(x,0) = f(x)$, $u(x,\pi) = 0$ for $0 \leq x \leq \pi$

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Suppose $u(x,y) = X(x)Y(y)$ then $u_{xx} + u_{yy} = X''Y + XY'' = 0$

Hence $\dfrac{X''}{X} = \dfrac{-Y''}{Y} = K$ some constant since LHS and RHS are independent. Consider then $X'' - KX = 0$ for various K,

$\underline{K=0}$ $X'' = 0$, $X(x) = Ax + B$ and $u(0,y) = BY(y) = 0 \Rightarrow \underline{B=0}$.
whereas $u(\pi, y) = A\pi Y(y) = 0 \Rightarrow \underline{A=0}$. No nontrivial sol^{n}.

$\underline{K>0}$ Let $K = \gamma^2$ for $\gamma \in \mathbb{R}$ then $X(x) = A\cosh \gamma x + B\sinh \gamma x$
and $u(0,y) = AY(y) = 0 \Rightarrow \underline{A=0}$, $u(\pi,y) = B\sinh \pi Y(y) = 0$
thus $\underline{B=0}$. It follows there are no nontrivial sol^{n}'s here.

$\underline{K<0}$ Let $K = -\beta^2$ for $\beta > 0$ and $\beta \in \mathbb{R}$. Note
$X(x) = A\cos\beta x + B\sin\beta x$ in this case and our BC's
give $u(0,y) = AY(y) = 0 \Rightarrow \underline{A=0}$. Whereas
$u(\pi, y) = B\sin(\beta\pi) Y(y) = 0 \Rightarrow \sin(\beta\pi) = 0$
$\Rightarrow \underline{\beta\pi = n\pi \text{ for } n \in \mathbb{N}}$.

Hence $K = -n^2$ for $n \in \mathbb{N}$ with $\underline{X_n(x) = C_n \sin(nx)}$.

We turn to the sol^{n} of Y in the only interesting case $K = -n^2$,
$$Y'' + KY = Y'' - n^2 Y = 0 \Rightarrow Y_n(y) = A_n \cosh ny + B_n \sinh(ny)$$

In total, after we merge $C_n A_n = a_n$ and $C_n B_n = b_n$ we have
$$\underline{u_n(x,y) = a_n \sin(nx)\cosh(ny) + b_n \sin(nx)\sinh(ny)}.$$

This family of general sol^{n}'s satisfy the 1st two BC's but we also wish to fit $u(x,0) = f(x)$ and $u(x,\pi) = 0$.
The 2nd of these is fairly easy to interpret

$$u_n(x,\pi) = a_n \sin nx \cosh n\pi + b_n \sin nx \sinh n\pi = 0$$

$$\Rightarrow \underline{\underline{b_n = \left(\frac{-\cosh n\pi}{\sinh n\pi}\right) a_n}}.$$

§ 10.7 #3 Continued:

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To satisfy $u(x,0) = f(x)$ for $0 \leq x \leq \pi$ it seems clear we'll need an sum of the $u_n(x,y)$ sol^{n}'s. We propose,

$$\boxed{u(x,y) = \sum_{n=1}^{\infty}{}' \; a_n \sin(nx)\cosh(ny) + b_n \sin(nx)\sinh(ny)} \quad (**)$$

Suppose $f(x)$ has Fourier sine expansion $f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$, apply $**$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx)\cosh(0) + \cancel{b_n \sin(nx)\underbrace{\sinh(0)}_{0}} = \sum_{n=1}^{\infty} c_n \sin(nx)$$

Evidently $a_n = c_n$ for $n = 1, 2, 3, \dots$ thus we have found the sol^{n} if we can find the Fourier coeff. for $f(x) = \sum_{n=1}^{\infty}{}' c_n \sin(nx)$ on $0 \leq x \leq \pi$, fortunately we know how to do that in general,

$$\underline{c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = a_n}.$$

Then we can attempt to simplify the answer,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin(nx)\left[\cosh(ny) - \frac{\cosh n\pi}{\sinh n\pi}\sinh(ny)\right]$$

$$= \sum_{n=1}^{\infty} \frac{a_n \sin(nx)}{\sinh(n\pi)}\left[\cosh(ny)\sinh(n\pi) - \cosh(n\pi)\sinh(ny)\right]$$

this trick again.

$$= \sum_{n=1}^{\infty} \frac{a_n \sin nx}{\sinh(-n\pi)} \sinh(ny - n\pi)$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{a_n}{\sinh(-n\pi)} \sin(nx)\sinh(n(y-\pi))}$$

Where $\underline{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx}$.

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§10.7 #7 Solve the Dirichlet BVP on disk $0 \le r < 2$ where $-\pi \le \theta \le \pi$ and $\boxed{\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0}$ —(*)
subject to the BC $u(2,\theta) = f(\theta) = |\theta| = \sqrt{\theta^2}$ on $-\pi \le \theta \le \pi$

$u(r,\theta) = R(r)T(\theta) \Rightarrow u_{rr} = R''T$, $u_r = R'T$, $u_{\theta\theta} = RT''$.
Substituting $u = RT$ into (*) yields

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0 \Rightarrow \frac{r^2R'' + rR'}{R} = \frac{-T''}{T} = K$$

We are faced with solving $r^2R'' + rR' - KR = 0$ and $T'' + KT = 0$.
Since $T'' + KT$ is easier let's start with it. Notice geometry requires $T(\pi) = T(-\pi)$ since θ and $\theta + 2\pi$ point along the same ray eminating from the origin. We imediately rule out $K < 0$ since cosh, sinh are not cyclic.
If $K \ge 0$ then $T'' = 0 \Rightarrow T(\theta) = A_0\theta + B_0 \Rightarrow \underline{T(\theta) = B_0}$.
If $K > 0$ then $K = \beta^2 \Rightarrow T(\theta) = A\cos\beta\theta + B\sin\beta\theta$. Notice that $T(\theta) = T(\theta + 2\pi) \Rightarrow \beta = 1, 2, 3, \ldots$ since:

$$\begin{aligned} T(\theta + 2\pi) &= A\cos(n\theta + n\cdot 2\pi) + B\sin(n\theta + n\cdot 2\pi) \\ &= A\cos n\theta + B\sin n\theta \\ &= T(\theta). \end{aligned}$$

and $T_0(\theta) = A_0$

We find $K = n^2$ for $n \in \mathbb{N}$ and $T_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$

Now turn to solving $r^2R_n'' + rR_n' - n^2R_n = 0$ (Cauchy - Euler Problem)
Suppose $R_n(r) = r^\lambda$ then $R_n' = \lambda r^{\lambda-1}$ and $R_n'' = \lambda(\lambda-1)r^{\lambda-2}$ hence we find

$$\begin{aligned} r^2R_n'' + rR_n - n^2R_n &= [\lambda(\lambda-1) + \lambda - n^2]r^\lambda = 0 \\ &\Rightarrow \lambda^2 - n^2 = (\lambda+n)(\lambda-n) = 0 \quad \therefore \underline{\lambda_1 = n,\ \lambda_2 = -n} \end{aligned}$$

Thus $R_n(r) = c_nr^n + d_n/r^n$ but we want expansion including $r = 0$ so we let $d_n = 0$ and keep only λ_1 - type terms

$$\underline{R_n(r) = c_nr^n}$$

($R_0(r) = c_0 + c_1\ln(r)$ but $c_1 = 0$ since $r = 0$ is in our domain)

§10.7 #7 Continued

We found $u_n(r,\theta) = C_n r^n (A_n \cos n\theta + B_n \sin n\theta)$

let $C_n = 1$ since A_n, B_n are sufficiently arbitrary.

We propose a formal sol$^{\underline{n}}$,

$$\boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)} \quad (**)$$

We need to fit (**) to the given BC's, notice

$$u(2,\theta) = \sum_{n=1}^{\infty} 2^n (A_n \cos n\theta + B_n \sin n\theta) = |\theta|$$

We need to expand $|\theta|$ into its Fourier expansion on $-\pi \leq \theta \leq \pi$,

denote $|\theta| = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)]$ we can calculate

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\theta| \, d\theta = \frac{2}{\pi}\int_0^{\pi} |\theta| d\theta = \frac{2}{\pi}\frac{\theta^2}{2}\Big|_0^{\pi} = \underline{\pi = a_0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\theta| \cos(n\theta) d\theta = \frac{2}{\pi}\int_0^{\pi} \theta \cos n\theta \, d\theta = \frac{2}{\pi}\left[\frac{\theta \sin n\theta}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin(n\theta) d\theta\right.$$

$$= \frac{2}{\pi n^2} \cos(n\theta)\Big|_0^{\pi}$$

$$= \frac{2}{\pi n^2}[\cos n\pi - 1]$$

$$= \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$= \frac{-4}{\pi n^2} \text{ if } n = \text{odd \#}, \text{ zero if } n \text{ even.}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|\theta| \sin(n\theta)}_{\text{odd fnct.}} d\theta = 0$$

§10.7 #7 Continued

To summarize,

$$a_0 = \pi \ , \quad a_{2k-1} = \frac{-4}{\pi(2k-1)^2} \ , \quad a_{2k} = 0 \quad \text{for } k \in \mathbb{N}.$$

Compare with $u(2,\theta) = A_0 + \sum_{n=1}^{\infty} 2^n (A_n \cos n\theta + B_n \sin n\theta)$

$$|\theta| = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{-4}{\pi(2k-1)^2} \cos[(2k-1)\theta]$$

we find, for $k \in \mathbb{N}$,

$$A_0 = \frac{\pi}{2} \ , \quad B_n = 0, \quad 2^{2k-1} A_{2k-1} = \frac{-4}{\pi(2k-1)^2} \ , \quad A_{2k} = 0$$

We can simplify the A_{2k-1} formula, $4 = 2^2$

$$A_{2k-1} = \frac{-1}{\pi(2k-1)^2 \, 2^{2k-3}}$$

Thus,

$$\boxed{u(r,\theta) = \frac{\pi}{2} - \sum_{k=1}^{\infty} r^{2k-1} \left(\frac{1}{\pi(2k-1)^2 \, 2^{2k-3}} \right) \cos[(2k-1)\theta]}$$

Remark: notice the reason we had to solve a Cauchy-Euler problem is the use of polar coordinates.

§10.7 #9 Solve the Neumann boundary value problem for a disk:

$$\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$ for $0 \le r < a$ and $\frac{\partial u}{\partial r}(a,\theta) = f(\theta)$, $-\pi \le \theta \le \pi$

The set-up is much like #7. We suppose $u(r,\theta) = R(r)T(\theta)$. It follows $\frac{r^2R'' + rR'}{R} = \frac{-T''}{T} = K$ and geometry of θ forces us to have $K = n^2$ for $n = 0,1,2,3,\dots$ and

$$\underline{T_0(\theta) = A_0} \quad \text{while} \quad \underline{T_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)}.$$

Then we turn to the $r^2R'' + rR' - n^2R = 0$ to find $\lambda(\lambda-1)+\lambda-n^2 = 0$.
$\underline{n=0}$ $\Rightarrow$ $R_0(r) = c_1 + c_2\ln(r)$ but $r=0$ has $\ln(r)$ blow-up $\Rightarrow c_2 = 0$
Hence $R_0(r) = c_1$. If $\underline{n\in\mathbb{N}}$ then $\lambda = \pm n$ hence $R_n(r) = c_nr^n + d_n/r^n$ and again we need to set $d_n = 0$ to avoid $1/r^n$ blowing up at $r=0$. Consequently,

$$\underline{R_0(r) = C_0} \quad \text{and} \quad \underline{R_n(r) = C_nr^n}.$$

Consequently, our formal, general sol^n has the form

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$$

Apply the boundary condition,

$$\frac{\partial u}{\partial r}(a,\theta) = \sum_{n=1}^{\infty} na^{n-1}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right] = f(\theta)$$

Compare against Fourier exp. $f(\theta) = \frac{\bar{a}_0}{2} + \sum_{n=1}^{\infty}\bar{a}_n\cos(n\theta) + \bar{b}_n\sin(n\theta)$
we see that $na^{n-1}A_n = \bar{a}_n$ and $na^{n-1}B_n = \bar{b}_n$ thus

$$A_n = \left(\frac{1}{na^{n-1}}\right)\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta \quad \text{and} \quad B_n = \left(\frac{1}{na^{n-1}}\right)\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta$$

Hence,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left\{r^n\left[\frac{1}{n\pi a^{n-1}}\int_{-\pi}^{\pi} f(\bar\theta)\cos(n\bar\theta)\,d\bar\theta\right]\cos(n\theta) + r^n\left[\frac{1}{n\pi a^{n-1}}\int_{-\pi}^{\pi} f(\bar\theta)\sin(n\bar\theta)\,d\bar\theta\right]\sin n\theta\right\}$$

$$\boxed{u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left[\left(\frac{a}{n\pi}\int_{-\pi}^{\pi} f(u)\cos(nu)\,du\right)\cos(n\theta) + \left(\frac{a}{n\pi}\int_{-\pi}^{\pi} f(u)\sin(nu)\,du\right)\sin n\theta\right]}$$

§10.7 #11 Solve the Dirichlet problem for an annulus,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad , \quad 1 < r < 2 \quad , \quad -\pi \le \theta \le \pi$$

where $u(1,\theta) = \sin 4\theta - \cos\theta$, $u(2,\theta) = \sin\theta$ on $-\pi \le \theta \le \pi$

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We suppose $u_n(r,\theta) = R(r)\,T(\theta)$. Thus $\frac{r^2R'' + rR'}{R} = \frac{-T''}{T} = K$.

Due to geometry of θ we need $K = n^2$ for $n \in \mathbb{N}$ or $K = 0$.

$\underline{T_0(\theta) = c_0}$. and $\underline{T_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \text{ for } n \in \mathbb{N}}$.

As usual for R we face a Cauchy – Euler problem,

$$r^2R'' + rR' - n^2R = 0 \longrightarrow \lambda(\lambda-1) + \lambda - n^2 = 0$$

$$\lambda^2 - n^2 = 0 \rightarrow \underline{\lambda = \pm n}$$

In $n=0$ case we get double root sol$^{\underline{n}}$ $\underline{R_0(r) = c_0 + d_0\ln(r)}$.

for $n \in \mathbb{N}$ we get $\underline{R_n(r) = c_n r^n + d_n/r^n}$. Previously we set $d_0, d_n = 0$ because those terms blow up at $r = 0$.

This time $1 < r < 2$ so $r = 0$ doesn't trouble us.

$$u(r,\theta) = c_0 + d_0\ln(r) + \sum_{n=1}^{\infty}\left[A_n r^n\cos n\theta + B_n r^{-n}\cos n\theta + C_n r^n \sin n\theta + D_n r^{-n}\sin n\theta\right.$$

We need to determine $c_0, d_0, A_n, B_n, C_n, D_n$ which fit the BC's.

$$u(1,\theta) = c_0 + \sum_{n=1}^{\infty}\left[(A_n + B_n)\cos n\theta + (C_n + D_n)\sin n\theta\right] = \sin 4\theta - \cos\theta$$

$$u(2,\theta) = c_0 + d_0\ln(2) + \sum_{n=1}^{\infty}\left[\left(A_n 2^n + \frac{1}{2^n}B_n\right)\cos n\theta + \left(2^n C_n + \frac{1}{2^n}D_n\right)\sin n\theta\right] = \sin\theta$$

Happily we do not need to find Fourier coeff. for the BC's since they're already Fourier "polynomials" in a manner of speaking, compare coefficients of matching functions,

$\underbrace{c_0 = 0}_{u(1,\theta)}$, $\underbrace{d_0 = 0}_{u(2,\theta)}$,

$$\left.\begin{array}{l} A_n = -B_n \text{ for } n \neq 1, \quad \underline{A_1 + B_1 = -1}. \\ C_n = -D_n \text{ for } n \neq 4, \quad \underline{C_4 + D_4 = 1}. \end{array}\right\} u(1,\theta)$$

$$\left.\begin{array}{l} A_n 2^n + \frac{1}{2^n}B_n = 0 \\ 2^n C_n + D_n/2^n = 0 \text{ for } n \neq 1, \quad \underline{2C_1 + D_1/2 = 1} \end{array}\right\} u(2,\theta)$$

§10.7 #11 Continued

Let's solve the eq^n's the BC's gave,

$n=1$) $A_1 + B_1 = -1$ and $2A_1 + B_1/2 = 0$

$B_1 = -1 - A_1 \Rightarrow 2A_1 + \frac{1}{2}(-1-A_1) = \frac{3}{2}A_1 - \frac{1}{2} = 0 \to \underline{A_1 = \frac{1}{3}}$.

$\underline{B_1 = \frac{-4}{3}}$.

$2C_1 + D_1/2 = 1$, $C_1 = -D_1$

$\to -2D_1 + D_1/2 = 1 \to -3D_1 = 2 \therefore \underline{D_1 = -2/3, \; C_1 = 2/3}$.

$n=4$) $C_4 + D_4 = 1$, $2^4 C_4 + D_4/2^4 = 0$

$C_4 = 1 - D_4 \Rightarrow 16(1-D_4) + D_4/16 = 0$

$256 - 256 D_4 + D_4 = 0 \to \underline{D_4 = \frac{256}{255}}$.

$\underline{C_4 = \frac{-1}{255}}$.

$A_4 = -B_4$ and $16A_4 + \frac{1}{16}B_4 = 0$

$\to -16B_4 - \frac{1}{16}B_4 = 0 \Rightarrow \underline{B_4 = 0, \; A_4 = 0}$.

$n \neq 1,4$) $A_n = -B_n$, $A_n 2^n + B_n/2^n = 0 \Rightarrow \left(-2^n + \frac{1}{2^n}\right)B_n = 0$

$\Rightarrow \underline{A_n = B_n = 0}$.

Likewise $C_n = -D_n$ and $2^n C_n + D_n/2^n = 0$

$\Rightarrow \underline{C_n = D_n = 0}$.

To summarize, we found

$$\boxed{u(r,\theta) = \frac{1}{3}\left(r - \frac{4}{r}\right)\cos\theta + \frac{2}{3}\left(r - \frac{1}{r}\right)\sin\theta + \frac{1}{255}\left(-r^4 + \frac{256}{r^4}\right)\sin(4\theta)}$$