Separable Equations : § 2.2

§ 2.2 # 9

$$\underbrace{\frac{dy}{dx} = \frac{1-x^2}{y^2}}_{(*)} \implies y^2\,dy = (1-x^2)\,dx$$

$$\implies \int y^2\,dy = \int (1-x^2)\,dx$$

$$\implies \underbrace{\boxed{\frac{1}{3}y^3 = x - \frac{x^3}{3} + C}}_{\text{implicit sol}^{\underline{n}} \text{ for } *}$$

§ 2.2 # 11

$$\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2} \implies \frac{dy}{\sec^2 y} = \frac{dx}{1+x^2}$$

$$\implies \int \cos^2 y\,dy = \int \frac{dx}{1+x^2}$$

$$\implies \int \frac{1}{2}\left(1+\cos(2y)\right)dy = \int \frac{dx}{1+x^2}$$

$$\implies \underbrace{\boxed{\frac{1}{2}y + \frac{1}{4}\sin(2y) = \tan^{-1}(x) + C}}_{\text{this is an implicit sol}^{\underline{n}} \text{ to the DEq}^{\underline{n}}.}$$

§ 2.2 # 15

$$y^{-1}\,dy + y e^{\cos(x)}\sin(x)\,dx = 0$$

$$\implies \int \frac{dy}{y^2} = -\int e^{\cos(x)}\sin(x)\,dx$$

$$\implies -\frac{1}{y} = e^{\cos(x)} + C$$

$$\implies \boxed{y = \frac{-1}{e^{\cos(x)} + C}}$$

explicit sol$^{\underline{n}}$ since I actually solved for y.

Remark: it's not always possible to solve for y

§2.2 #21)

$$\frac{dY}{d\theta} = Y \sin\theta \quad \text{subject to initial condition } y(\pi) = -3.$$

$$\Rightarrow \int \frac{dY}{Y} = \int \sin\theta \, d\theta \quad \Rightarrow \quad \ln|Y| = -\cos\theta + C$$

Note $y(\pi) = -3 \Rightarrow \ln|-3| = -\cos(\pi) + C$

$$\Rightarrow \ln(3) = C + 1$$

$$\Rightarrow C = \ln(3) - 1$$

$$\Rightarrow \ln|Y| = -\cos\theta + \ln(3) - 1$$

$$\Rightarrow |Y| = e^{-\cos\theta + \ln(3) - 1}$$

$$= e^{-\cos\theta} e^{\ln(3)} e^{-1}$$

$$= 3e^{-\cos\theta - 1}$$

$$\Rightarrow Y = \pm 3e^{-\cos\theta - 1}$$

But, $y(\pi) = -3 \therefore$ we must choose $(-)$ and

$$\boxed{Y = -3e^{-\cos\theta - 1}}$$

§2.2 #25)

$$\frac{dy}{dx} = x^2(1+y) \quad \text{with } y(0) = 3$$

$$\int \frac{dy}{1+y} = \int x^2 dx$$

$$\ln|1+y| = \frac{1}{3}x^3 + C \Rightarrow |1+y| = e^{\frac{1}{3}x^3 + C} = e^{\frac{1}{3}x^3} e^{C}$$

$$\Rightarrow y + 1 = \pm e^C e^{\frac{1}{3}x^3}$$

(for convenience denote $\pm e^C = k$) $\Rightarrow y = ke^{x^3/3} - 1$

$$y(0) = 3 = k - 1 \therefore k = 4 \Rightarrow \boxed{y = 4e^{x^3/3} - 1}$$

§ 2.2 # 35 Blood plasma is stored at 40° F. Before it is used it must be warmed to 90° F. When the plasma is placed in an oven of 120° F it takes 45 minutes for the plasma to warm to 90° F. Assume Newton's Law of Cooling applies, how long will it take to warm if the oven is set at 100° F ?

Newton's Law of Cooling states that $\exists\, k$ such that

$$\frac{dT}{dt} = k(T - R)$$

(change in temp. is proportional to difference from room-temp.)

where T is temperature of blood at time t and R is the background temperature. We assume k is independent of R (there are real-world examples where this assumption fails)

$$\frac{dT}{T-R} = k\,dt$$

$$\ln|T-R| = kt + c_1$$

$$|T-R| = \exp(kt + c_1) = e^{c_1} e^{kt}$$

$$T = R \pm e^{c_1} e^{kt}$$

$$\therefore\ T(t) = R + C_2 e^{kt}$$

Note, $T(0) = 40 = R + C_2 \ \therefore\ C_2 = 40 - R$. If $R = 120$ then $C_2 = 40 - 120 = -80$ and $T(t) = 120 - 80e^{kt}$ moreover, $T(45) = 90 = 120 - 80e^{45k}$

$$\frac{-30}{-80} = e^{45k} \Rightarrow \frac{1}{45}\ln(3/8) = k$$

Thus, $R = 100 \Rightarrow T(t) = 100 - 80\exp\left(\frac{1}{45}\ln(3/8)\, t\right)$

Set $T(t) = 90$ and solve for t to find $t = 82.2$.