§6.2 #1 $y''' + 2y'' - 8y' = 0$

Characteristic $\lambda^3 + 2\lambda^2 - 8\lambda = \lambda(\lambda^2 + 2\lambda - 8) = \lambda(\lambda+4)(\lambda-2) = 0$

Third order in λ gives 3-sol^{n}'s $\lambda_1 = 0,\ \lambda_2 = -4,\ \lambda_3 = 2$

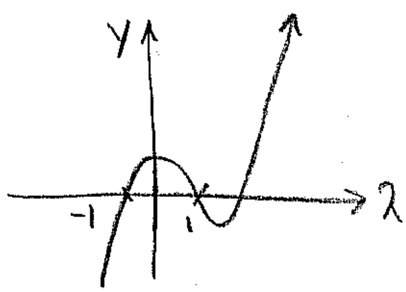
Which then gives 3-fund. sol^{n}'s $y_1 = 1,\ y_2 = e^{-4x},\ y_3 = e^{2x}$

Which then assembles the general sol^{n} $\boxed{y = c_1 + c_2 e^{-4x} + c_3 e^{2x}}$

§6.2 #2 $y''' - 3y'' - y' + 3y = 0$

Char. Eq^{n}: $\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$ (How to factor a cubic ! ?).

For most textbook problems there is always some silly natural number root, but looking towards non-textbook problems we can use the following scheme, graph it, find a root then either find the rest graphically or factor out the linear factor corresponding to the root.



$y = \lambda^3 - 3\lambda^2 - \lambda + 3 = f(\lambda)$

apparently $\lambda = 1$ is a zero $\Rightarrow f(\lambda)$ has a $(\lambda - 1)$ factor,

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = (\lambda-1)(\lambda^2 + b\lambda + c)$$
$$= \lambda^3 + \lambda^2(b-1) + \lambda(c-b) - c$$

Can see by comparing that $b = -2$ and $c = -3$.

Hence $\lambda^3 - 3\lambda^2 - \lambda + 3 = (\lambda-1)(\lambda^2 - 2\lambda - 3)$

The more algebraically adept among you might just see how to factor this cubic, but my method here allows for messier quadratics and also some quadratics do not have roots so the pure graphical method fails. Pragmatically, we could also use the root finder or polysolve option on most graphical calculators.

- In general factoring polynomials of order greater than 2 is quite a challenge. This one is relatively harmless (rational roots!)

$$\lambda^2 - 3\lambda^2 - \lambda + 3 = (\lambda-1)(\lambda^2 - 2\lambda - 3) = (\lambda-1)(\lambda-3)(\lambda+1) = 0$$

Thus $\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = -1$ hence

$$\boxed{y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{-x}}$$

Remark: another quick factoring tool is long division of polynomials

§6.2#9 $u''' - 9u'' + 27u' - 27u$

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = 0$$

$$(\lambda - 3)^3 = 0 \quad \therefore \quad \lambda_1 = \lambda_2 = \lambda_3 = 3.$$

$$\boxed{u = c_1 e^{3t} + c_2 t e^{3t} + c_3 t^2 e^{3t}}$$

§6.2#14 $y'''' + 2y''' + 10y'' + 18y' + 9y = 0$

$$\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = 0$$

Hint: $y = \sin(3x)$ is a sol$^{\underline{n}}$. This means there is at least two imaginary roots namely $\lambda = \pm 3i$. That means we can factor out (λ^2+9) corresponding to the zeros $\pm 3i$. The whole thing is 4$^{\underline{th}}$ order so if we factor out a quadratic then the rest of the thing must be a quadratic but I'll use algebra to figure out which quadratic it must be. Since λ^4 has a 1 coefficient we need not worry about "a" in $a\lambda^2 + b\lambda + c$, b & c will suffice,

$$\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = (\lambda^2+9)(\lambda^2 + b\lambda + c)$$

$$= \lambda^4 + \lambda^3(b) + \lambda^2(c+9) + \lambda 9b + 9c$$

gotta find b & c to match them up.

Gives overdetermined system of linear eq$^{\underline{n}}$'s for b & c

$$\left.\begin{array}{lll} 2 = b & \Rightarrow & b=2 \\ 10 = c+9 & \Rightarrow & c=1 \\ 18 = 9b & \Rightarrow & b=2 \\ 9 = 9c & \Rightarrow & c=1 \end{array}\right\}$$

it's ok to have extra eq$^{\underline{n}}$'s so long as they are consistent, these are.

We find then $\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = (\lambda^2+9)(\lambda^2 + 2\lambda + 1)$

Then $\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = (\lambda^2+9)(\lambda+1)^2$ yielding zeros

$$\lambda_{1,2} = \pm 3i \quad , \quad \lambda_3 = -1 = \lambda_4$$

Hence

$$\boxed{y = c_1 \cos 3x + c_2 \sin 3x + c_3 e^{-x} + c_4 x e^{-x}}$$

Remark: long division is another way. (Sorry so messy... email me if you'd like to see more.)

$$\begin{array}{r} \lambda^2 + 2\lambda + 1 \\ \lambda^2+9 \,\overline{\big)\, \lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9} \\ \underline{\lambda^4 + 9\lambda^2 } \\ 2\lambda^3 + \lambda^2 + 18\lambda + 9 \end{array}
\qquad
\begin{array}{r} 2\lambda^3 + \lambda^2 + 18\lambda + 9 \\ \underline{2\lambda^3 + 18\lambda } \\ \lambda^2 + 9 \\ \underline{\lambda^2 + 9} \end{array}$$

§6.2 #15 Here D denotes the operator $\frac{d}{dx}$ on $Y(x)$.

$$\underbrace{(D-1)^2(D+3)(D^2+2D+5)^2}_{7^{th} \text{ order polynomial in the operator } D\text{, multiplication is composition of operators.}}[Y] = 0 \quad (*)$$

$(D-1)^2[Y] = 0 \quad \Rightarrow \quad Y_1 = e^x$ or $Y_2 = xe^x$

This is straightforward to verify,

$$(D-1)[Y_1](x) = \left(\frac{d}{dx} - 1\right)e^x = e^x - e^x = 0.$$

$$(D-1)[Y_2](x) = \left(\frac{d}{dx} - 1\right)xe^x = e^x + xe^x - xe^x = e^x$$

$$(D-1)^2[Y_2] = \Big(\underset{\text{Composition}}{(D-1)\circ(D-1)}\Big)[Y_2] = (D-1)(D-1)[Y_2]$$
$$= (D-1)[Y_1]$$
$$= 0$$

- It's important to intepret $(D-1)(D-1)$ as composition of operators. Anyway using similar logic and noting

$\lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$ implies,

$(D^2+2D+5)^2[Y] = 0$ has

$$\left.\begin{aligned} Y_3 &= e^{-x}\cos 2x \\ Y_4 &= e^{-x}\sin 2x \end{aligned}\right\} \text{killed by } [D^2+2D+5]$$

$$\left.\begin{aligned} Y_5 &= xe^{-x}\cos 2x \\ Y_6 &= xe^{-x}\sin 2x \end{aligned}\right\} \text{killed by } [D^2+2D+5]^2$$

And of course $(D+3)[Y] = 0$ has $Y_7 = e^{-3x}$ as a sol$^{\underline{n}}$.

In total we have an eq$^{\underline{n}}$ which is a linear $7^{\underline{th}}$ order ODE, notice if Y_i is a sol$^{\underline{n}}$ to any of the factors it's a sol$^{\underline{n}}$ to the whole composition of operators, just one them needs to kill it. So $Y_1, Y_2, \dots, Y_7$ are all sol$^{\underline{n}}$'s to $(*)$

$$\boxed{Y = c_1e^x + c_2xe^x + c_3e^{-x}\cos 2x + c_4e^{-x}\sin 2x + c_5e^{-x}x\cos 2x + c_6xe^{-x}\sin 2x + c_7e^{-3x}}$$

<u>Remark</u>: A linear ODE of the form $L[Y] = 0$ where L is a polynomial $P(D)$ where $D = \frac{d}{dx}$ has char. eq$^{\underline{n}}$ $P(\lambda)$, same algebra!

§6.2 #35 Vibrating Beam. For constants $\underbrace{E, I \text{ and } k}_{\text{all positive}}$ solve

$$EI\,\frac{d^4y}{dx^4} - ky = 0$$

$$EI\,\lambda^4 - k = 0$$

$$\lambda^4 = k/EI \quad \therefore \quad \lambda^2 = \pm\sqrt{k/EI}$$

Thus $\lambda_1 = \lambda_2 = \sqrt{\sqrt{k/EI}}$ and $\lambda_{3,4} = \pm i\sqrt{\sqrt{k/EI}}$ which yields

$$\boxed{y = c_1\cosh(\gamma x) + c_2\sinh(\gamma x) + c_3\cos\gamma x + c_4\sin\gamma x}$$

where $\gamma = (k/EI)^{1/4}$. When we have a repeated real root hyperbolic sine/cosine are nice sol^{n}'s to use

$$\cosh(\gamma x) = \tfrac{1}{2}(e^{\gamma x} + e^{-\gamma x})$$

$$\sinh(\gamma x) = \tfrac{1}{2}(e^{\gamma x} - e^{-\gamma x})$$

$$\underbrace{c_1\cosh(\gamma x) + c_2\sinh(\gamma x) = \tfrac{1}{2}(c_1+c_2)\,e^{\gamma x} + \tfrac{1}{2}(c_1-c_2)\,e^{-\gamma x}}_{\text{different ways to express the same general sol}^{n}.}$$

Incidentally, we can always complete square and use either cos, sine or cosh, sinh
For example, $y'' + 6y' + 5y = 0$

$$\lambda^2 + 6\lambda + 5 = 0$$

$$(\lambda+3)^2 - 4 = 0$$

$$y = b_1e^{-3x}\cosh(2x) + b_2e^{-3x}\sinh(2x)$$

__exercise:__ convince yourself this is same as $\underline{y = c_1e^{5x} + c_2e^{x}}$.