BASIC THEORY OF LINEAR DIFFERENTIAL EQUATIONS

Th$^{\underline{m}}$(1) [EXISTENCE & UNIQUENESS] Suppose $P_1, P_2, \ldots, P_n, g$ are continuous functions on an interval (a,b) which contains the point x_0. Then, for any choice of initial values $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ there exists a unique sol$^{\underline{n}}$ Y on the whole interval (a,b) to the initial value problem

$$y^{(n)}(x) + P_1(x)\,y^{(n-1)}(x) + P_2(x)\,y^{(n-2)}(x) + \cdots + P_n(x)\,y(x) = g(x)$$

$$\text{with } y(x_0) = \gamma_0,\; y'(x_0) = \gamma_1, \ldots,\; y^{(n-1)}(x_0) = \gamma_n$$

- this Th$^{\underline{m}}$ is from §6.1. It says most of the problems we solve in this course have unique sol$^{\underline{n}}$'s.

§6.1#1 Find largest interval for which Th$^{\underline{m}}$(1) gaurantees a unique sol$^{\underline{n}}$,

$$\underbrace{xy''' - 3y' + e^x y = x^2 - 1}, \quad y(-2) = 1,\; y'(-2) = 0,\; y''(-2) = 2$$

$$\downarrow$$

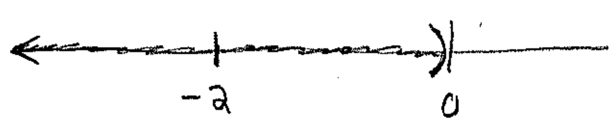
$$y''' - \frac{3}{x}y' + \frac{e^x}{x}y = \frac{x^2-1}{x}$$

(need coefficient of one in front of highest derivative term y''')

In the notation of Th$^{\underline{m}}$(1),

$$P_1(x) = {-3}/x, \quad P_3(x) = e^x/x, \quad g(x) = \frac{x^2-1}{x}$$

Clearly the coefficient functions are discontinuous at $x = 0$ thus



$\boxed{\text{thus } (-\infty, 0) \text{ is the largest "interval"}}$

§6.1#5 Same as #1 but for

$$x\sqrt{x+1}\,y''' - y' + xy = 0 \quad \text{with } y(1/2) = y'(1/2) = -1,\; y''(1/2) = 1$$

$$y''' - \frac{1}{x\sqrt{x+1}}y' + \frac{x}{x\sqrt{x+1}}y = 0$$

Maximal intersection of domains of coefficient functions is $(-1,0) \cup (0,\infty)$

Then $1/2 \in (0,\infty)$ $\therefore$ $\boxed{\text{largest interval is } (0,\infty) \text{ for Th}^{\underline{m}}\text{(1) to apply an yield unique sol}^{\underline{n}}}$

$\boxed{\S 6.1 \# 7}$ Consider $\{e^{3x}, e^{5x}, e^{-x}\}$. Are these functions linearly independent on $(-\infty, \infty)$?

Notice these form the sol^{n} set for $(D-3)(D-5)(D+1)[y] = 0$ thus we can apply Th^{m} (3) and test for LI via the Wronskian,

$$W[e^{3x}, e^{5x}, e^{-x}](x) = \det\begin{bmatrix} e^{3x} & e^{5x} & e^{-x} \\ 3e^{3x} & 5e^{5x} & -e^{-x} \\ 9e^{3x} & 25e^{5x} & e^{-x} \end{bmatrix}$$

$$= e^{3x}\left(5e^{5x}e^{-x} + 25e^{-x}e^{5x}\right)$$

$$- e^{5x}\left(3e^{3x}e^{-x} + 9e^{-x}e^{3x}\right)$$

$$+ e^{-x}\left(75e^{3x}e^{5x} - 45e^{5x}e^{3x}\right)$$

$$= e^{7x}(30 - 12 + 30)$$

$$= 48e^{7x} \neq 0 \quad \forall x \in (-\infty, \infty)$$

$\therefore \{e^{3x}, e^{5x}, e^{-x}\}$ is a LI set of functions on $\mathbb{R}$.

$\boxed{\S 6.1 \# 9}$ Consider $\{\sin^2 x, \cos^2 x, 1\}$ on $(-\infty, \infty)$. Is this set linearly dependent?

Notice $\sin^2 x + \cos^2 x = 1$ thus $\exists$ $c_1 \sin^2 x + c_2 \cos^2 x + c_3 \cdot 1 = 0$ such that $c_1 = 1$, $c_2 = 1$, $c_3 = -1$ (not all zero). Hence this set of functions is linearly dependent

$\boxed{\S 6.1 \# 17}$ Does $\{x, x^2, x^3\}$ form a fundamental sol^{n} set for $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$? (for $x > 0$)

Notice $x^3 (x)''' - 3x^2 (x)'' + 6x(x)' - 6x = 6x - 6x = 0$ thus $y = x$ is sol^{n}.
Notice $x^3 (x^2)''' - 3x^2 (x^2)'' + 6x(x^2)' - 6x^2 = -6x^2 + 12x^2 - 6x^2 = 0$ thus $y = x^2$ is sol^{n}.
Notice $x^3 (x^3)''' - 3x^2 (x^3)'' + 6x(x^3)' - 6x^3 = 6x^3 - 18x^3 + 18x^3 - 6x^3 = 0$ thus $y = x^3$ is sol^{n}.
Thus we can use Th^{m} (3) to check LI via the Wronskian,

$$W[x, x^2, x^3](x) = \det\begin{bmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{bmatrix} = x(12x^2 - 6x^2) - x^2(6x) + x^3(2) = 2x^3$$

Thus $W[x, x^2, x^3](x) = 2x^3 \neq 0$ for $x > 0$ $\therefore$ $\boxed{\{x, x^2, x^3\} \text{ is fundamental solution set}}$

§6.1 # 23 Let $L[y] \equiv y''' + y' + xy$. Also let $y_1(x) \equiv \sin(x)$ and $y_2(x) \equiv x$. Calculate $L[y_1]$ and $L[y_2]$ and then use superposition to solve (a.) $L[y] = 2x\sin(x) - x^2 - 1$ and then (b.) $L[y] = 4x^2 + 4 - 6x\sin(x)$.

We need to observe three identities,

① $$\begin{aligned} L[c_1 f + c_2 g] &= (c_1 f + c_2 g)''' + (c_1 f + c_2 g)' + x(c_1 f + c_2 g) \\ &= c_1[f''' + f' + xf] + c_2[g''' + g' + xg] \\ &= c_1 L[f] + c_2 L[g] \end{aligned}$$

$\forall c_1, c_2 \in \mathbb{R}$ and $f, g \in C^\infty(\mathbb{R})$, $\therefore$ L is <u>linear operator</u>.

② $$\begin{aligned} L[\sin(x)](x) &= (\sin(x))''' + (\sin(x))' + x\sin(x) \\ &= -\cos(x) + \cos(x) + x\sin(x) \\ &= \underline{x\cos(x) = L[y_1](x)}. \end{aligned}$$

③ $$\begin{aligned} L[x](x) &= x''' + x' + xx \\ &= 0 + 1 + x^2 \\ &= \underline{1 + x^2 = L[y_2](x)}. \end{aligned}$$

(a.) We want L to output $2x\sin(x)$ and $-1(x^2+1)$ clearly $\boxed{y = 2y_1 - y_2}$ will do nicely. $L[2y_1 - y_2] = 2x\sin(x) - (x^2+1)$.

(b.) We want to output $4(x^2+1) - 6(x\sin(x))$. Note $\boxed{y = 4y_2 - 6y_1}$ works, $L[4y_2 - 6y_1] = 4(x^2+1) - 6(x\sin(x))$.

[this problem is a little strange. Usually one is faced with the terms like $2x\sin x - x^2 - 1$ to start with, then our job is to find y_1 and y_2 that match them.]

<u>Remark</u>: The Wronskian played a prominent role in verifying the linear dependence / independence of sol^{n} sets. It is a natural curiousity to wonder if the Wronskian has anything to say about sets of functions which are <u>not</u> sol^{n}'s to a common DEq^{n}. The text claims it's not interesting. However, the text offers no proof of this claim...