§6.3 # 9 $y''' - 3y'' + 3y' - y = e^x \quad (*)$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$$

Thus the associated homog. eqᵗ has form $(D-1)^3[y] = 0$. Note e^x is anihilated by $(D-1)$. Thus every solⁿ of $(*)$ must satisfy

$$(D-1)^3(D-1)[y] = (D-1)^4[y] = 0 \quad (**)$$

The general solⁿ to $(**)$ is $\left((\lambda-1)^4 = 0 \Rightarrow \lambda_1 = 1 = \lambda_2 = \lambda_3 = \lambda_4\right)$

$$y = \underbrace{c_1 e^x + c_2 x e^x + c_3 x^2 e^x}_{y_h} + \underbrace{c_4 x^3 e^x}_{y_p}$$

- This method is neat in that it translates the problem into a corresponding homogeneous problem. We already knew to guess $y_p = Ax^3e^x$ since we'd have observed to begin that $y_h = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$ thus the naive guess $y_p(\text{naive}) = Ae^x$ overlaps with y_1 then $y_p(\text{less naive}) = xAe^x$ overlaps with y_2 then $y_p(\text{even less naive}) = x^2Ae^x$ still overlaps with $y_3 = x^2e^x$, finally we'd have "guessed" that $y_p = Ax^3e^x$. Notice how the Anihilator Method has discovered the x^3 through alternate reasoning. We view the anihilator method as one proof of our previous method, of course the fact it works is a proof in itself.

$$y_p = Ax^3e^x$$
$$y_p' = A(3x^2 + x^3)e^x$$
$$y_p'' = A(6x + 6x^2 + x^3)e^x$$
$$y_p''' = A(6 + 18x + 9x^2 + x^3)e^x$$

$$y_p''' - 3y_p'' + 3y_p' - y_p = e^x$$

$$A(6 + 18x + 9x^2 + x^3)e^x + -3A(6x + 6x^2 + x^3)e^x + +3A(3x^2 + x^3)e^x + -Ax^3e^x = e^x$$

$$x^3\underbrace{(A - 3A + 3A - A)}_{0} + x^2\underbrace{(9A - 18A + 9A)}_{0} + x\underbrace{(18A - 18A)}_{0} + 6A = 1$$

$$\Rightarrow A = 1/6$$

Thus $\boxed{y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + \frac{1}{6}x^3 e^x}$

§6.3 #12 find an operator A which anihilates $f(x) = 3x^2 - 6x + 1$.

Easy use D^3, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(3x^2 - 6x + 1\right)\right)\right) = 0 \therefore A[f] = 0$.

§6.3 #14 try $A = D - 5$.

§6.3 #17 $x^2 \underbrace{e^{-x} \sin 2x}$

sol^n from $\lambda = -1 \pm 2i$
which has corresponding $(\lambda + 1 - 2i)(\lambda + 1 + 2i) = \lambda^2 + 2\lambda + 5$ char eg
which has corresponding operator $D^2 + 2D + 5$ that is,

$$(D^2 + 2D + 5)[e^{-x} \sin 2x] = 0.$$

The x^2 suggests this is the sol^n to a D Eq^n with not just one $(\lambda^2 + 2\lambda + 5)$ factor in the characteristic eg^n. To get upto $x^2 e^{-x} \sin 2x$ we'd need to have already

$$\underbrace{\underbrace{\underbrace{e^{-x}(\sin 2x),\ e^{-x}\cos 2x}_{(D^2+2D+5)[Y]=0},\ xe^{-x}\sin 2x,\ xe^{-x}\cos 2x}_{(D^2+2D+5)^2[Y]=0},\ x^2e^{-x}\sin 2x,\ x^2e^{-x}\cos 2x}_{(D^2+2D+5)^3[Y]=0}$$

Identify that $x^2 e^{-x} \sin 2x$ is one of the fundamental sol^n's to $(D^2 + D + 5)^3[Y] = 0$. So clearly $\boxed{A = (D^2 + 2D + 5)^3}$

Remark: I'm just trying to validate the books list on pg. 359. You don't need to do all the writing I'm doing here. You may just remember (i) $\longrightarrow$ (iv) of p. 359 to help find Anihilators for suitable $f(x)$.

Remark: In the sol^{n}'s that follow I make no attempt to justify my choice of annihilator. I have already described ideas justifying this list on p. 359. I now illustrate the method without extraneous commentary, ..

§6.3 #21 Find the form of y_p via the annihilator

$$u'' - 5u' + 6u = \cos(2x) + 1$$

$$(D^2 - 5D + 6)[u](x) = \underbrace{\cos(2x) + 1}_{\text{annihilated by } A = (D^2+4)D}$$

Now operate by A,

$$(D^2+4)D(D^2-5D+6)[u] = (D^2+4)D[\cos(2x)+1]$$

$$(D^2+4)D(D-3)(D-2)[u] = 0$$

Has sol^{n}, $u = \underbrace{c_1\cos(2x) + c_2\sin(2x) + c_3}_{u_p} + \underbrace{c_4e^{3x} + c_5e^{2x}}_{u_h}$

$$\therefore \quad \boxed{u_p = A\cos(2x) + B\sin(2x) + C}$$

Remark: to complete problem there is much more to do. The annihilator method only helps us find the general form of the particular sol^{n}.

§6.3 #23 Find form of y_p,

$$y'' - 5y' + 6y = e^{3x} - x^2$$

$$(D-3)(D-2)[y] = e^{3x} - x^2$$

Note $(D-3)[e^{3x}] = 0$ and $D^3[x^2] = 0$, hence $A = D^3(D-3)$ works nicely. Operate on DEq^{n} by A on both sides,

$$D^3(D-3)^2(D-2)[y] = D^3(D-3)[e^{3x} - x^2] = 0$$

$$\Rightarrow \quad y = \underbrace{c_1 + c_2x + c_3x^2 + c_4xe^{3x}}_{y_p} + \underbrace{c_5e^{3x} + c_6e^{2x}}_{y_h}$$

$$\therefore \quad \boxed{y_p = A + Bx + Cx^2 + Dxe^{3x}}$$

(the homogeneous sol^{n} always uses terms w/o x, x^2 as much as possible, I knew xe^{3x} belongs with y_p)

§6.3 #25 Find form of y_p via annihilator method,

$$\underbrace{y'' - 6y' + 9y}_{(D-3)^2[y]} = \underbrace{\sin(2x) + x}_{\text{use } A = (D^2+4)D^2}$$

$$(D^2+4)D^2(D-3)^2[y] = (D^2+4)D^2[\sin(2x)+x] = 0.$$

$$y = \underbrace{c_1\cos 2x + c_2\sin 2x + c_3 + c_4 x}_{y_p} + \underbrace{c_5 e^{3x} + c_6 x e^{3x}}_{y_h}$$

$$\therefore \quad \boxed{y_p = A\cos 2x + B\sin 2x + C + Dx}$$

§6.3 #27 Find form of y_p via annihilator method,

$$\underbrace{y'' + 2y' + 2y}_{[D^2+2D+2][y]} = \underbrace{e^{-x}\cos x + x^2}_{\text{annihilated by } A = D^3((D+1)^2+1)}$$

$$\Rightarrow [(D+1)^2+1][y] = e^{-x}\cos x + x^2$$

Operate on both sides by A,

$$D^3[(D+1)^2+1][(D+1)^2+1][y] = D^3((D+1)^2+1)[e^{-x}\cos x + x^2] = 0$$

$$\Rightarrow y = \underbrace{c_1 + c_2 x + c_3 x^2 + c_4 x e^{-x}\cos(x) + c_5 x e^{-x}\sin x}_{y_p} + \underbrace{c_6 e^{-x}\cos x + c_7 e^{-x}\sin x}_{y_h}$$

$$\therefore \quad \boxed{y_p = A + Bx + Cx^2 + Dxe^{-x}\cos x + Exe^{-x}\sin x}$$