§4.5 #1 I refuse to use y_1 and y_2 for particular sol^{t}s since I wish to reserve that notation for fundamental sol^{n}s to the homogeneous case. Instead, suppose that

$$y_{p_1} = \cos t \quad \text{solves} \quad y'' - y' + y = \sin t$$
$$y_{p_2} = e^{2t}/3 \quad \text{solves} \quad y'' - y' + y = e^{2t}$$

Given these two facts above, solve the following by superposition,

(a.) $y'' - y' + y = 5\sin t$ is solved by $\boxed{y = 5y_{p_1} = 5\cos t}$

Let me prove it,

$$(5\cos t)'' - (5\cos t)' + 5\cos t = -\cancel{5\cos t} + 5\sin t + \cancel{5\cos t}$$
$$= 5\sin t \checkmark$$

(b.) $y'' - y' + y = \sin t - 3e^{2t}$. Let $y = Ay_{p_1} + By_{p_2}$

$$(Ay_{p_1} + By_{p_2})'' - (Ay_{p_1} + By_{p_2})' + Ay_{p_1} + By_{p_2} = \sin t - 3e^{2t}$$
$$A(y_{p_1}'' - y_{p_1}' + y_{p_1}) + B(y_{p_2}'' - y_{p_2}' + y_{p_2}) = \sin t - 3e^{2t}$$
$$A\sin t + Be^{2t} = \sin t - 3e^{2t}$$
$$\Rightarrow A = 1, \quad B = -3 \quad \therefore \boxed{y = \cos t - e^{2t}}$$

(c.) Want $A\sin t + Be^{2t} = 4\sin t + 18e^{2t}$

$$\Rightarrow A = 4, \quad B = 18 \quad \therefore \boxed{y = 4\cos t - 6e^{2t}}$$

Remark: the sol^{n}s given in a, b, c are not general sol^{n}s. They correspond to the choice $c_1 = 0$ and $c_2 = 0$ for the $y_h = c_1 e^{t/2}\cos\left(\frac{\sqrt{3}t}{2}\right) + c_2 e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)$

I noted, $y'' - y' + y = 0$ has $\lambda^2 - \lambda + 1 = 0$

$$(\lambda - 1/2)^2 + \frac{3}{4} = 0$$
$$\therefore \lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

§4.5#3 $y'' - y = t$ has $y_p = -t$. Find general sol$^{\underline{n}}$

$\lambda^2 - 1 = 0 \Rightarrow (\lambda+1)(\lambda-1) = 0 \therefore \lambda_1 = -1, \lambda_2 = 1$

Hence $y_h = c_1 e^{-t} + c_2 e^{t}$. Consequently

$$\boxed{y = c_1 e^{-t} + c_2 e^{t} - t}$$

§4.5#17 Find general sol$^{\underline{n}}$ to $y'' - y = -11t + 1$

Again $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow y_h = c_1 e^{-t} + c_2 e^{t}$.

Suppose $y_p = At + B$ (I know this via experience a.k.a §6.3 understood well.)

$y_p' = A$

$y_p'' = 0$

Substitute into DEq$^{\underline{n}}$,

$$y_p'' - y_p = -11t + 1$$

$$-At - B = -11t + 1$$

Hence $A = 11$ and $B = -1$. We find the general sol$^{\underline{n}}$,

$$\boxed{y = c_1 e^{-t} + c_2 e^{t} + 11t - 1}$$

§4.5#18 $y'' - 2y' - 3y = 3t^2 - 5$

As usual find homogeneous sol$^{\underline{n}}$ to begin $\lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$
Hence $\lambda_1 = 3$ and $\lambda_2 = -1$ $\therefore$ $y_h = c_1 \exp(3t) + c_2 \exp(-t)$.

Assemble y_p from $3t^2 - 5$ $\Rightarrow$

$$y_p = At^2 + Bt + C$$
$$y_p' = 2At + B$$
$$y_p'' = 2A$$

Now substitute y_p into the DEq$^{\underline{n}}$ to determine $A, B \,\&\, C$,

$$y_p'' - 2y_p' - 3y_p = 3t^2 - 5$$

$$2A - 2(2At + B) - 3(At^2 + Bt + C) = 3t^2 - 5$$

$$[2A - 2B - 3C] + t[-4A - 3B] + t^2[-3A] = [-5] + t[0] + t^2[3]$$

Hence comparing coefficients,

$$3 = -3A \longrightarrow A = -1$$
$$0 = -4A - 3B \longrightarrow 3B = 4 \longrightarrow B = 4/3$$
$$-5 = 2A - 2B - 3C \longrightarrow 3C = 2A - 2B + 5 \Rightarrow C = \tfrac{1}{3}\left(-2 - \tfrac{8}{3} + 5\right) = \tfrac{1}{9}$$

Thus $\boxed{y = y_h + y_p = c_1 \exp(3t) + c_2 \exp(-t) - t^2 + \tfrac{4}{3}t + \tfrac{1}{9}}$ (general sol$^{\underline{n}}$)

§4.5#22 $y'' + 6y' + 10y = 10x^4 + 24x^3 + 2x^2 - 12x + 18 \equiv g(x)$

Auxillary sol$^{\underline{n}}$ $\lambda^2 + 6\lambda + 10 = 0 \Rightarrow \lambda = \frac{-6 \pm \sqrt{36-40}}{2} = -3 \pm i \Rightarrow y_h = e^{-3x}(c_1 \cos(x) + c_2 \sin(x)$

Guess:

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx + E$$
$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D$$
$$y_p'' = 12Ax^2 + 6Bx + 2C$$

Hence substituting into $y_p'' + 6y_p' + 10y_p = 10x^4 + 24x^3 + 2x^2 - 12x + 18$,

$$g(x) = (12Ax^2 + 6Bx + 2C) + 6(4Ax^3 + 3Bx^2 + 2Cx + D) + 10(Ax^4 + Bx^3 + Cx^2 + Dx + E)$$

(*)
$$= x^4[10A] + x^3[10B + 24A] + x^2[10C + 18B + 12A] + x[10D + 12C + 6B] + [10E + 6D + 2C$$
$$= g(x) = x^4[10] + x^3[24] + x^2[2] + x[-12] + [18]$$

§4.5 #22 Continued, compare coefficients of (#) work from x^4 down,

$$10 = 10A \longrightarrow A = 1$$
$$24 = 24A + 10B \longrightarrow B = 0$$
$$2 = 10C + 18B + 12A \rightarrow C = -1$$
$$-12 = 10D + 12C + 6B \rightarrow D = 0$$
$$18 = 10E + 6D + 2C \rightarrow E = 2$$

In each level I use what I learned in the last. Notice this would be much harder if you went from constants to x^4 instead.

Hence $\boxed{y = y_h + y_p = e^{-3x}(c_1 \cos x + c_2 \sin x) + x^4 - x^2 + 2}$

§4.5 #23 $y' - y = 1$ with $y(0) = 0$, find sol^{n}. We could use the integrating factor method, but the following is easier,

$$\lambda - 1 = 0 \Rightarrow \lambda = 1 \Rightarrow y_h = c_1 \exp(t)$$, no overlap with 1.

Hence, $y_p = A$ then $y_p' = 0$ so $y_p' - y_p = -A = 1 \therefore \underline{A = -1}$

Thus $y = c_1 \exp(t) - 1$. (the general sol^{n}, btw it is the general sol^{n} you must fit to intial data because fitting the homogeneous sol^{n} alone would be nonsense.)

$$y(0) = 0 = c_1 - 1 \therefore c_1 = 1 \therefore \boxed{y = \exp(t) - 1}$$

§4.5 #26 $y'' + 9y = 27 \Rightarrow y_h = c_1 \cos 3x + c_2 \sin(3x) \Rightarrow$ no overlap.

Guess $y_p = A$ then $y_p' = y_p'' = 0$ thus $9A = 27 \Rightarrow A = 3$.

Giving general sol^{n} $\underline{y = c_1 \cos(3x) + c_2 \sin(3x) + 3}$

givens $\begin{cases} y(0) = 4 = c_1 + 3 \Rightarrow c_1 = 1 \\ y'(0) = 6 = 3c_2 \Rightarrow c_2 = 2 \end{cases}$

Thus, $\boxed{y = \cos(3x) + 2\sin(3x) + 3}$

<u>Remark</u>: adding intial conditions is straight forward, all we need to do is to use the data to fit c_1 and c_2 (or in general for a n^{th}-order problem $c_1, c_2, \dots, c_n$).