§2.3#7

$$\frac{dy}{dx} - y - e^{3x} = 0$$

① $\frac{dy}{dx} - y = e^{3x}$

identify $P(x) = -1$
thus $\mu(x) = \exp\left(\int -dx\right) = e^{-x}$. ②
Multiply by the integrating factor $\mu = e^{-x}$ to find,

③ $$\underbrace{e^{-x}\frac{dy}{dx} - e^{-x}y}_{} = e^{-x}e^{3x}$$

④ $$\frac{d}{dx}\left[e^{-x}y\right] = e^{2x}$$

⑤ $$\Rightarrow \underbrace{\int \frac{d}{dx}\left[e^{-x}y\right]dx}_{\text{FTC}} = \int e^{2x}dx$$

$$e^{-x}y = \frac{1}{2}e^{2x} + C$$

$$\therefore \boxed{y = \frac{1}{2}e^{3x} + Ce^{x}}$$

Procedure (this is the "integrating factor method")

① Write eq^n in standard form $\frac{dy}{dx} + Py = Q$ if possible
② Find $\mu = \exp\left(\int P\,dx\right)$ the "integrating factor"
③ Multiply DEq^n by μ
④ Use product rule in reverse
⑤ integrate and apply initial conditions if applicable.

§ 2.3 # 9

$$\frac{dr}{d\theta} + r\tan\theta = \sec\theta$$

$$\mu = \exp\left(\int \tan\theta \, d\theta\right) = \exp\left(\int \frac{\sin\theta \, d\theta}{\cos\theta}\right) = \exp\left(\ln|\sec\theta|\right)$$

Hence $\mu = |\sec\theta|$. Multiply DEq$^{\underline{n}}$ by μ,

$$|\sec\theta|\frac{dr}{d\theta} + r|\sec\theta|\tan\theta = |\sec\theta|\sec\theta$$

$$\Rightarrow \underbrace{\sec\theta\frac{dr}{d\theta} + r\sec\theta\tan\theta}_{\frac{d}{d\theta}\left[r\sec\theta\right]} = \sec^2\theta$$

$$\frac{d}{d\theta}\left[r\sec\theta\right] = \sec^2\theta$$

Integrating $\Rightarrow r\sec\theta = \tan\theta + C$

$$\therefore \; r = \frac{\tan\theta + C}{\sec\theta} = \sin\theta + C\cos\theta$$

$$\Rightarrow \boxed{r = \sin\theta + C\cos\theta}$$

§ 2.3 # 11

$$(t + y + 1)\,dt - dy = 0$$

$$\frac{dy}{dt} = t + y + 1 \quad \Rightarrow \quad \frac{dy}{dt} - y = t + 1$$

Then $\mu = \exp\left(\int -dt\right) = e^{-t} \;\therefore\; e^{-t}\frac{dy}{dt} - ye^{-t} = (t+1)e^{-t}$

Observe $\int\left(te^{-t} + e^{-t}\right)dt = -2e^{-t} - te^{-t} + C$. Hence,

$$\int \frac{d}{dt}\left[e^{-t}y\right]dt = \int (t+1)e^{-t}dt$$

$$\Rightarrow e^{-t}y = -2e^{-t} - te^{-t} + C \quad \therefore \boxed{y = -2 - t + Ce^{t}}$$

§2.3 # 21

$$\cos(x)\frac{dy}{dx} + y\sin(x) = 2x\cos^2(x), \quad y(\pi/4) = \frac{-15\sqrt{2}\,\pi^2}{32}$$

$$\frac{dy}{dx} + \frac{\sin(x)}{\cos(x)}\,y = 2x\cos(x)$$

$$\mu = \exp\left(\int \frac{\sin(x)}{\cos(x)}dx\right) = \exp\left(\ln|\sec(x)|\right) = |\sec(x)|$$

$$\underbrace{|\sec(x)|\frac{dy}{dx} + |\sec(x)|\tan(x)\,y}_{\frac{d}{dx}\left[\,|\sec(x)|\,y\,\right]} = 2x\,|\sec(x)|\cos(x)$$

$$\frac{d}{dx}\left[\,|\sec(x)|\,y\,\right] = 2x\,\frac{\cos(x)}{|\cos(x)|}$$

Our sol^{n} has $y(\pi/4) = -15\sqrt{2}\,\pi^2/32$ so we'll be interested in $x = \pi/4$ which has $|\sec(x)| = \sec(x)$.

$$\int \frac{d}{dx}\left[\sec(x)\,y\right]dx = \int 2x\,dx$$

$$\sec(x)\,y = x^2 + C$$

$$y(\pi/4) = \frac{-15\sqrt{2}}{32}\pi^2 \Rightarrow \underbrace{\left[\sec(\pi/4)\right]}_{\sqrt{2}}\left[\frac{-15\sqrt{2}}{32}\pi^2\right] = \frac{\pi^2}{16} + C$$

$$\frac{-15}{16}\pi^2 = \frac{\pi^2}{16} + C$$

$$\Rightarrow -\pi^2 = C$$

Thus, $$y = \frac{x^2 - \pi^2}{\sec(x)}$$

$$\boxed{y = x^2\cos(x) - \pi^2\cos(x)}$$