§4.9 #9) A 2kg mass is attached to a spring with stiffness 40 N/m. The damping coefficient is $8\sqrt{5}$ sec/m. If the mass is pulled 10 cm = 0.1 m to the right of equilibrium and given $V_0 = 2$ m/s at $t=0$ then what is the maximum displacement from equilibrium that it will attain?

Omitting units,

$$m = 2, \quad k = 40, \quad b = 8\sqrt{5}, \quad y(0) = 0.1, \quad \frac{dy}{dt}(0) = 2.$$

We must solve,

$$2y'' + 8\sqrt{5}\,y' + 40y = 0$$

$$y'' + 4\sqrt{5}\,y' + 20y = 0$$

$$\lambda^2 + 4\sqrt{5}\,\lambda + 20 = 0$$

$$(\lambda + 2\sqrt{5})^2 + 20 - (2\sqrt{5})^2 = 0$$

$\therefore\ \lambda = -2\sqrt{5}$ twice, <u>critically damped</u>.

$$y = c_1 e^{-2\sqrt{5}t} + c_2 t e^{-2\sqrt{5}t}$$

$$y' = -2\sqrt{5}\,c_1 e^{-2\sqrt{5}t} - 2\sqrt{5}\,c_2 t e^{-2\sqrt{5}t} + c_2 e^{-2\sqrt{5}t}$$

$y(0) = \underline{0.1 = c_1}$.

$y'(0) = 2 = -2\sqrt{5}\,c_1 + c_2 \quad \therefore \ \underline{c_2 = 2 + 0.2\sqrt{5}}$.

Hence $\boxed{y(t) = 0.1e^{-2\sqrt{5}t} + 2.447\,t e^{-2\sqrt{5}t}}$

Maximum displacement found via min/max theory from calc. I.

$$\frac{dy}{dt} = -0.2\sqrt{5}\,e^{-2\sqrt{5}t} + 2.447e^{-2\sqrt{5}t} - t\,2\sqrt{5}\,(2.447)e^{-2\sqrt{5}t} = 0$$

$$\Rightarrow -0.2\sqrt{5} + 2.447 - 2\sqrt{5}\,(2.447)\,t = 0 \quad \text{for critical \#.}$$

Hence, $t = \dfrac{-2.447 + 0.2\sqrt{5}}{-2\sqrt{5}\,(2.447)} = \underline{0.183}$

Notice $\frac{dy}{dt}(0) > 0$ while $\frac{dy}{dt}(1000) < 0 \ \therefore\ \underline{y(0.183) = 0.242} \Rightarrow \boxed{24.2\text{ cm}}$

§4.9 #11 A 1kg-mass is attached to a spring with stiffness 100 N/m. The damping constant 0.2 sec/m. If the mass is pushed rightward from equilibrium position with velocity of 1 m/s <u>when</u> will it attain its maximum displacement to the right

Omitting units,

$$m=1,\quad k=100,\quad b=0.2,\quad y(0)=0,\quad y'(0)=1$$

We wish to solve,

$$y''+0.2y'+100y=0$$

$$\lambda^2+0.2\lambda+100=0$$

$$(\lambda+0.1)^2+99.99=0 \Rightarrow \underline{\lambda=-0.1\pm 9.9995i}$$

Hence $y=c_1e^{-0.1t}\cos(9.9995t)+c_2e^{-0.1t}\sin(9.9995t)$

$y(0)=0 \Rightarrow c_1=0 \therefore y=c_2e^{-0.1t}\sin(9.9995t)$

Thus $y'(t)=\left[-0.1c_2\sin(9.9995t)+9.9995c_2\cos(9.9995t)\right]e^{-0.1t}$

Yielding $y'(0)=9.9995c_2=1 \therefore \underline{c_2=\frac{1}{9.9995}=0.10005}$

The eq$^{\underline{n}}$ of motion: $\underline{\underline{y(t)=0.10005\,e^{-0.1t}\sin(9.9995t)}}$

We'll use min/max theory from calculus I to find critical time

$$\frac{dy}{dt}=\underbrace{\left[-0.010005\sin(9.9995t)+(0.10005)(9.9995)\cos(9.9995t)\right]}_{\text{must be zero for } \frac{dy}{dt}=0}\underbrace{e^{-0.1t}}_{\text{non zero}}$$

$$\tan(9.9995t)=\frac{(0.10005)(9.9995)}{0.010005}=99.995$$

$$t=\frac{1}{9.9995}\tan^{-1}(99.995)=0.1561 \quad \left(1^{\text{st}} \text{ of } \infty\text{ly many critical \#s}\right)$$

Observe $y'(0)=1>0$ and $\underbrace{y'(0.16)<0}_{\text{short calculation}} \therefore \boxed{t=0.1561\text{ sec}}$ yields local max for y.

(Future local maxima are smaller due to the $e^{-0.1t}$ factor)