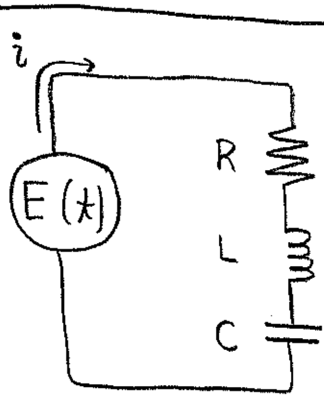
§5.7 #3 An RLC series circuit has a voltage source given by $E(t) = [10\cos(20t)]\,V$, $R = 120\,\Omega$, $L = 4H$ and $C = (2200)^{-1} F$. Find the steady state current for this circuit. What is the resonance frequency?



Kirchhoff's voltage law states

$$RI + L\frac{dI}{dt} + \frac{Q}{C} = E(t)$$

differentiate, R, L, C all constants and $I = \frac{dQ}{dt}$,

$$R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{C}I = \frac{dE}{dt}$$

Thus, omitting units,

$$4I'' + 120I' + 2200I = -200\sin(20t)$$

$$I'' + 30I' + 550I = -50\sin(20t)$$

The particular sol$^{\underline{n}}$ is the steady-state sol$^{\underline{n}}$ since $I_h \longrightarrow 0$ as $t \to \infty$ for RLC-circuits. $(R \neq 0)$

$$I_p = A\cos(20t) + B\sin(20t)$$
$$I_p' = -20A\sin(20t) + 20B\cos(20t)$$
$$I_p'' = -400I_p$$

Substitute,

$$I_p'' + 30I_p' + 550I_p = -50\sin(20t)$$

$$150\left[A\cos 20t + B\sin 20t\right] + 30\left[-20A\sin 20t + 20B\cos 20t\right] = -50\sin(20t)$$

$\cos 20t$ | $150A + 600B = 0 \longrightarrow 3A + 12B = 0 \longrightarrow A = -4B$

$\sin 20t$ | $150B - 600A = -50 \longrightarrow 3B - 12A = -1 \longrightarrow$

$$\longrightarrow 3B - 12(-4B) = -1 \Rightarrow 51B = -1 \therefore B = -1/51 \text{ and } A = 4/51$$

$$\boxed{\text{Steady State Sol}^{\underline{n}}:\quad I_\infty(t) = \frac{4}{51}\cos(20t) - \frac{1}{51}\sin(20t)}$$

continued

§5.7 #3 Continued

The resonant frequency for an RLC-circuit follows from the resonant frequency for an underdamped mass-spring system using the analogy given in Table 5.3. From §4.10,

$$\gamma_{resonance} = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} \qquad my'' + by' + ky = g(t)$$

$$\Rightarrow \gamma_{resonance} = \sqrt{\frac{1/C}{L} - \frac{R^2}{2L^2}} \qquad LI'' + RI' + \frac{1}{C}I = \frac{dE}{dt}$$

$$f_{resonance} = \frac{\gamma_{resonance}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2200}{4} - \frac{(120)^2}{32}} = \frac{1}{2\pi}\sqrt{550-450} = \frac{10}{2\pi}$$

Thus $\boxed{f_{resonance} = \frac{5}{\pi}}$

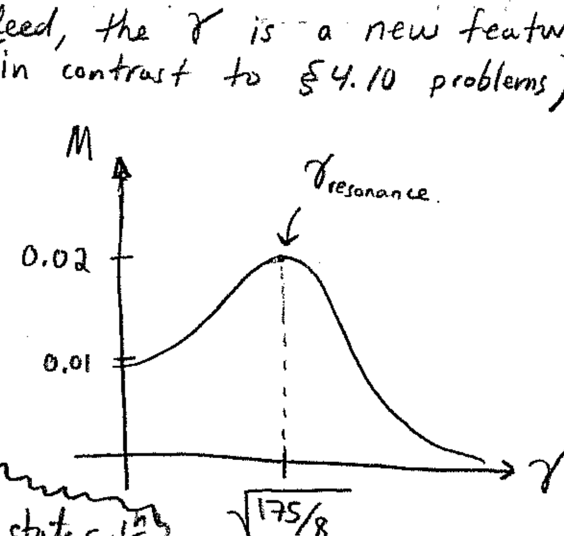
§5.7#5 Consider a series RLC-circuit with $R = 10$, $L = 4$ and $C = \frac{1}{100}$. Furthermore, suppose the voltage source $E(t) = E_0\cos(\gamma t)$ supplies energy to the circuit. Plot the frequency response curve for the circuit

Idea: the steady-state sol^{n} has form $I_p = A\sin(\gamma t + \theta)$ where $A = \dfrac{E_0}{\sqrt{R^2 + [\gamma L - 1/(\gamma C)]^2}} \equiv \gamma E_0 M(\gamma)$ (see eq^{n} 10)

- Technically it seems to me this is the response to $\frac{dE}{dt} = -\gamma E_0 \sin(\gamma t)$ is analogous to $F(t)$ in §4.10. Indeed, the γ is a new feature (in contrast to §4.10 problems)

$$\gamma M(\gamma) = \frac{1}{\sqrt{100 + [4\gamma - 100/\gamma]^2}}$$

$$M(\gamma) = \frac{1}{\sqrt{100\gamma^2 + [4\gamma^2 - 100]^2}}$$



Remark: $\gamma_{resonance}$ is the choice of γ which forces maximal amplitude in steady state sol^{n}

§5.7 #9 Complex Formalism for RLC circuits:

We can solve

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E_o e^{i\gamma t} \qquad \left(\dot{q} = \frac{dq}{dt},\ \ddot{q} = \frac{d^2q}{dt^2}\right)$$

where R, L, C, E_o, γ are real constants, however q and $e^{i\gamma t}$ are generally complex-valued functions of a real variable t.

(a.) find steady state sol^n: Guess $q_p = Ae^{i\gamma t}$ use ☆Remark☆

then $\dot{q}_p = i\gamma A e^{i\gamma t}$ and $\ddot{q}_p = -\gamma^2 A e^{i\gamma t}$. Hence,

$$L\ddot{q}_p + R\dot{q}_p + \frac{1}{C}q_p = E_o e^{i\gamma t}$$

$$\left[-A\gamma^2 L - iAR\gamma + \frac{1}{C}A\right]e^{i\gamma t} = E_o e^{i\gamma t}$$

$$A\left(\frac{1}{C} - \gamma^2 L - iR\gamma\right) = E_o$$

$$A = \frac{E_o}{1/C - \gamma^2 L - i\gamma R}$$

$$\therefore \quad \boxed{q_p(t) = \left(\frac{E_o}{1/C - \gamma^2 L - i\gamma R}\right)e^{i\gamma t}}$$

☆Remark:☆ working with functions which map $\mathbb{R}$ to $\mathbb{C}$ is a special case of the calculus of "space-curves".

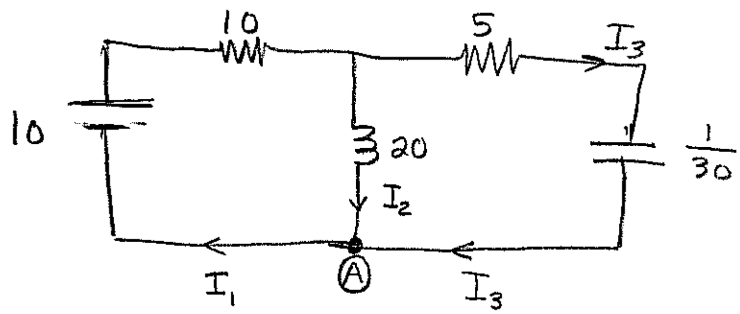
$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle$ we differentiate component-wise. Consider $f(t) = u(t) + iv(t)$ we define $\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt}$. It is interesting (and beautiful) that the chain-rule still works for the complex-exponential. Notice

$$\boxed{\begin{aligned}\frac{d}{dt}\left(e^{i\gamma t}\right) &= \frac{d}{dt}\left(\cos(\gamma t) + i\sin(\gamma t)\right)\\ &= -\gamma\sin\gamma t + i\gamma\cos\gamma t\\ &= i\gamma(\cos\gamma t + i\sin\gamma t)\\ &= i\gamma e^{i\gamma t}.\end{aligned}}$$

§5.7 #9 comment continued

In complex-variables we will study mappings from $\mathbb{C} \to \mathbb{C}$. Those require much more comment. Personally, I find it remarkable how nicely calculus transfers to complex-valued functions of a real variable. By the way, if $\lambda = \alpha + i\beta$ then $e^{i\lambda t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$ and it is still true that $\frac{d}{dt}[e^{i\lambda t}] = i\lambda e^{i\lambda t}$. //

§5.7 #11 Assume all initial currents are zero. Solve for $t > 0$,



$I_1 = I_2 + I_3$: charge conservation at A

$V = IR$ and $V = q/C$

Kirchhoff's voltage law applied to each branch gives following:

$$\boxed{\begin{aligned} 10 &= 10I_1 + 20\frac{dI_2}{dt} && \text{: left loop} \\ 20\frac{dI_2}{dt} &= 5I_3 + 30q_3 && \text{: right loop} \\ 10 &= 10I_1 + 5I_3 + 30q_3 && \text{: big loop} \end{aligned}}$$

} system of differential equations.

I have not taught how to solve these. We cover systems of differential equations in math 321 since the best way to solve them is via linear algebra. You can follow the ad-hoc procedure of Example 2 to solve these if you wish. I include this problem primarily to point out this deficiency in our course. Sorry, you'll have to wait till math 321.