§7.2 #1 Let $f(t)=t$ calculate the Laplace transform of f; $\mathcal{L}\{f\}(s)$,

$$\mathcal{L}\{f\}(s) \equiv F(s) = \int_0^\infty e^{-st}t\,dt$$

$$= \lim_{N\to\infty}\int_0^N e^{-st}t\,dt$$

So calculate the integral and then finish it,

$$\int \underbrace{t}_{u}\,\underbrace{e^{-st}dt}_{dV} = \underbrace{-\tfrac{1}{s}te^{-st} - \int -\tfrac{1}{s}e^{-st}dt}_{\text{Int. by Parts.}} = \tfrac{1}{s}\left(-te^{-st} - \tfrac{1}{s}e^{-st}\right)$$

So returning to the Laplace transform,

$$F(s) = \lim_{N\to\infty}\left(\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right)\Bigg|_{0=t}^{N=t}$$

$$= \lim_{N\to\infty}\left(\underbrace{\cancel{\frac{-Ne^{-sN}}{s}}}_{0} - \underbrace{\cancel{\frac{e^{-sN}}{s^2}}}_{0} + \frac{1}{s^2}\right) = \boxed{\frac{1}{s^2}}$$

assuming $s>0$

We noticed $\lim_{N\to\infty}(Ne^{-sN}) = \lim_{N\to\infty}\left(\frac{N}{e^{sN}}\right) \stackrel{\text{L}}{=} \lim_{N\to\infty}\left(\frac{1}{se^{sN}}\right) = 0$ to see why the $-Ne^{-sN}/s$ term vanished.

§7.2 #4 Find Laplace transform of $f(t)=te^{3t}$

$$F(s) = \lim_{N\to\infty}\int_0^N e^{-st}te^{3t}\,dt$$

$$= \lim_{N\to\infty}\int_0^N te^{t(3-s)}\,dt$$

$$= \lim_{N\to\infty}\left(\frac{1}{3-s}te^{t(3-s)}\Big|_0^N - \int_0^N \frac{1}{3-s}e^{t(3-s)}\,dt\right)$$

I.B.P
$u=t$
$dV = e^{t(3-s)}dt$

$$= \lim_{N\to\infty}\left(\cancel{\frac{N}{3-s}e^{N(3-s)}}^{\,0} - \frac{1}{(3-s)^2}\left[\cancel{e^{N(3-s)}}^{\,0} - 1\right]\right)$$

$$= \frac{1}{(3-s)^2} = \boxed{\frac{1}{(s-3)^2} = \mathcal{L}\{te^{3t}\}(s)}$$

assuming $s>3$ to insure the integral converges.

§7.2 # 11

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\pi \sin(t) e^{-st} dt + \cancel{\int_\pi^\infty 0 \cdot e^{-st} dt} \to 0$$

(*)

$$= \left(\frac{-\cos t}{s^2+1} - \frac{s \sin t}{s^2+1} \right) e^{-st} \Big|_{0=t}^{\pi=t}$$

$$= \frac{-\cos\pi}{s^2+1} e^{-s\pi} + \frac{\cos(0)}{s^2+1}$$

$$= \boxed{\frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1}}$$

(no restriction on s here)

The integral used at (*) is derived by IBP,

$$I = \int \underbrace{\sin t}_{u}\ \underbrace{e^{at}\, dt}_{dV} = \frac{1}{a} \sin t\, e^{at} - \int \frac{1}{a} e^{at} \cos t\, dt$$

$$= \frac{1}{a} e^{at} \sin t - \frac{1}{a}\left(\frac{1}{a} \cos t\, e^{at} + \int \frac{1}{a} e^{at} \sin t\, dt \right)$$

$$= e^{at}\left(\frac{1}{a} \sin t - \frac{1}{a^2} \cos t \right) - \frac{1}{a^2} \underbrace{\int e^{at} \sin t\, dt}_{I}$$

$I \leftarrow$ full circle so solve for I.

$$I\left(1 + \frac{1}{a^2}\right) = e^{at}\left(\frac{\sin t}{a} - \frac{\cos t}{a^2} \right)$$

$$\Rightarrow \quad I = e^{at} \frac{1}{1 + 1/a^2} \left(\frac{\sin t}{a} - \frac{\cos t}{a^2} \right) = e^{at}\left(\frac{a \sin t}{a^2+1} - \frac{\cos t}{a^2+1} \right)$$

§7.2 #17

$$\mathcal{L}\{ e^{3t} \sin 6t - t^3 + e^t \} = \mathcal{L}\{e^{3t} \sin 6t\} - \mathcal{L}\{t^3\} + \mathcal{L}\{e^t\}$$

$$= \boxed{\frac{6}{(s-3)^2 + 36} - \frac{6}{s^4} + \frac{1}{s-1}}$$

using Table on p. 384

§7.2 #18

$$\mathcal{L}\{ t^4 - t^2 - t + \sin\sqrt{2}\, t \} = \mathcal{L}\{t^4\} - \mathcal{L}\{t^2\} - \mathcal{L}\{t\} + \mathcal{L}\{\sin\sqrt{2}\, t\}$$

$$= \boxed{\frac{24}{s^5} - \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2+2}}$$