§7.3 #5 Hard part first,

$$\mathcal{L}\{2t^2e^{-t}\} = 2\,\mathcal{L}\{t^2e^{-t}\} \underset{Th^m(3)}{=} 2F(s+1) = 2\frac{2!}{(s+1)^3} = \frac{4}{(s+1)^3}$$

Now use what we just learned, plus table 7.1 to get the next 2 terms,

$$\mathcal{L}\{2t^2e^{-t} - t + \cos 4t\} = \boxed{\frac{4}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2+16}}$$

§7.3#6 Use Th^m(3) that says $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ where $\mathcal{L}\{f\}(s) = F(s)$

$$\mathcal{L}\{e^{3t}t^2\}(s) = F_1(s-3) = \frac{2}{(s-3)^3} \qquad \left(\text{Used } \mathcal{L}\{t^2\}(s) = 2/s^3\right.$$

$$\mathcal{L}\{e^{-2t}\sin 2t\}(s) = F_2(s+2) = \frac{2}{(s+2)^2+4} \qquad \left.\text{used } \mathcal{L}\{\sin 2t\}(s) = \frac{2}{s^2+4}\right)$$

Thus $\mathcal{L}\{e^{3t}t^2 + e^{-2t}\sin 2t\} = \boxed{\frac{2}{(s-3)^3} + \frac{2}{(s+2)^2+4}}$

§7.3 #12 $\sin 3t\cos 3t$ can be converted into a sum of sines and cosines by the right trig identities, or use the complex exponentials

$$\sin(3t) = \frac{1}{2i}\left(e^{3it} - e^{-3it}\right) \quad \text{and} \quad \cos 3t = \frac{1}{2}\left(e^{3it} + e^{-3it}\right)$$

Just a particular application of $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ and $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$. These followed naturally from $e^{i\theta} = \cos\theta + i\sin\theta$

Any way lets get to work,

$$\begin{aligned}\sin(3t)\cos(3t) &= \frac{1}{2i}\left(e^{3it} - e^{-3it}\right)\frac{1}{2}\left(e^{3it} + e^{-3it}\right)\\ &= \frac{1}{4i}\left(e^{6it} + 1 - 1 - e^{-6it}\right)\\ &= \frac{1}{2}\frac{1}{2i}\left(e^{6it} - e^{-6it}\right)\\ &= \frac{1}{2}\sin(6t)\end{aligned}$$

we derived $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ for $\theta = 3t$

Now it's easy $\mathcal{L}\{\sin(3t)\cos(3t)\}(s) = \mathcal{L}\{\frac{1}{2}\sin 6t\}(s) = \frac{1}{2}\,\frac{6}{s^2+36} = \boxed{\frac{3}{s^2+36}}$

§7.3#15

$$\begin{aligned}\cos^3 t &= \frac{1}{8}\left(e^{it} + e^{-it}\right)\left(e^{it} + e^{-it}\right)\left(e^{it} + e^{-it}\right)\\ &= \frac{1}{8}\left(e^{2it} + 2 + e^{-2it}\right)\left(e^{it} + e^{-it}\right)\\ &= \frac{1}{8}\left(e^{3it} + e^{it} + 2e^{it} + 2e^{-it} + e^{-it} + e^{-3it}\right)\\ &= \frac{1}{4}\,\frac{1}{2}\left(e^{3it} + e^{-3it}\right) + \frac{3}{4}\,\frac{1}{2}\left(e^{it} + e^{-it}\right)\\ &= \frac{1}{4}\cos(3t) + \frac{3}{4}\cos t.\end{aligned}$$

(Check it with TI-89 tCollect (cos(x)^3))

Then $\mathcal{L}\{\cos^3 t\} = \boxed{\frac{1}{4}\left(\frac{s}{s^2+9}\right) + \frac{3}{4}\left(\frac{s}{s^2+1}\right)}$

§7.3 #22 Starting with $\mathcal{L}\{1\}(s) = \frac{1}{s}$ use the theorem
$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)$ to derive the f-la $\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}$

We take $f(t) = 1$ in the theorem then calculate

$$\begin{aligned}
\mathcal{L}\{t^n\}(s) = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) &= (-1)^n \frac{d^{n-1}}{ds^{n-1}}\left(\frac{-1}{s^2}\right) \\
&= (-1)^n \frac{d^{n-2}}{ds^{n-2}}\left(\frac{(-1)(-2)}{s^3}\right) \\
&= (-1)^n (-1)^2 \frac{d^{n-2}}{ds^{n-2}}\left(\frac{2\cdot 1}{s^3}\right) \\
&= (-1)^n (-1)^3 \frac{d^{n-3}}{ds^{n-3}}\left(\frac{3\cdot 2\cdot 1}{s^4}\right) \\
&= (-1)^n (-1)^n \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{s^{n+1}} \\
&= \boxed{\frac{n!}{s^{n+1}} = \mathcal{L}\{t^n\}(s)}
\end{aligned}$$

(Sorry this one's out of order)

§7.3 #11

$$\begin{aligned}
\mathcal{L}\{\cosh(bt)\}(s) &= \mathcal{L}\left\{\tfrac{1}{2}\left(e^{bt} + e^{-bt}\right)\right\}(s) \\
&= \tfrac{1}{2}\mathcal{L}\{e^{bt}\}(s) + \tfrac{1}{2}\mathcal{L}\{e^{-bt}\}(s) \\
&= \tfrac{1}{2}\left(\frac{1}{s-b}\right) + \tfrac{1}{2}\left(\frac{1}{s+b}\right) \\
&= \tfrac{1}{2}\left[\frac{s+b+s-b}{(s+b)(s-b)}\right] \\
&= \frac{s}{s^2-b^2}
\end{aligned}$$

Remark: You can check $\mathcal{L}\{\sinh(bt)\}(s) = \frac{b}{s^2-b^2}$.
In contrast,

$$\mathcal{L}\{\cos(bt)\}(s) = \frac{s}{s^2+b^2} \quad \text{and} \quad \mathcal{L}\{\sin(bt)\}(s) = \frac{b}{s^2+b^2}.$$

If we use sinh & cosh then we'll see more symmetry in later calculations.