§7.5 #1 Solve $Y'' - 2Y' + 5Y = 0$ subject to $Y(0) = 2$, $Y'(0) = 4$

$$\left.\begin{aligned} \mathcal{L}\{Y''\} &= s^2\bar{Y} - 2s - 4 \\ \mathcal{L}\{Y'\} &= s\bar{Y} - 2 \end{aligned}\right\} \text{ from Table 7.2 using our intial conditions.}$$

Take the Laplace transform of the DEq^{n} to obtain,

$$s^2\bar{Y} - 2s - 4 - 2(s\bar{Y} - 2) + 5\bar{Y} = 0$$

$$\bar{Y}(s^2 - 2s + 5) = 2s + 4 - 4$$

$$\bar{Y} = \frac{2s}{s^2 - 2s + 5} = \frac{2s}{(s-1)^2 + 4} = \frac{2(s-1) + 2}{(s-1)^2 + 4}$$

$$\mathcal{L}^{-1}\{\bar{Y}\}(t) = \mathcal{L}^{-1}\left\{2\frac{s-1}{(s-1)^2+4} + \frac{2}{(s-1)^2+4}\right\}(t)$$

$$= \boxed{2e^t\cos(2t) + e^t\sin(2t) = Y}$$

Remark: A good check on the algebra is, that if you do it correctly you ought to see the characteristic eq^{n} $as^2 + bs + c$ appear as the coefficient of $\bar{Y}$. In the above we saw $s^2 - 2s + 5$ appear.

§7.5 #4

$$\mathcal{L}\{Y'' + 6Y' + 5Y\} = \mathcal{L}\{12e^t\} \quad ; \quad Y(0) = -1 \text{ \& } Y'(0) = 7$$

$$[s^2\bar{Y} + sY(0) - Y'(0)] + 6[s\bar{Y} - Y(0)] + 5\bar{Y} = \frac{12}{s-1}$$

$$(s^2 + 6s + 5)\bar{Y} - s - 7 + 6 = \frac{12}{s-1}$$

$$\bar{Y} = \frac{1}{s^2 + 6s + 5}\left[\frac{12}{s-1} - s - 1\right]$$

$$= \frac{1}{(s+5)(s+1)}\left[\frac{12 - (s-1)(s+1)}{s-1}\right] = \frac{A}{s+5} + \frac{B}{s+1} + \frac{C}{s-1}$$

Need to do partial fractions as indicated, need to find A, B \& C,

§7.5#4 Continued Completing partial fractions algebra yields,

$$Y(s) = \frac{12-(s-1)(s+1)}{(s-1)(s+5)(s+1)} = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s+5}$$

Then the inverse transform reveals

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = \boxed{e^{t} - e^{-t} - e^{-5t} = y(t)}$$

Remark: we know another method to solve this type of problem. I think the other method is easier because partial fractions is often time consuming to complete correctly. It is still nice to have another way to do problems we can already do with undetermined coefficients.

§7.5 #5 $w'' + w = t^2 + 2$ subject to $w(0) = 1$ and $w'(0) = -1$

$$s^2 W - s + 1 + W = \mathcal{L}\{t^2 + 2\} = \frac{2}{s^3} + \frac{2}{s}$$

$$(s^2+1)W = \frac{2}{s^3} + \frac{2}{s} + s - 1 = \frac{1}{s^3}(2 + 2s^2 + s^4 - s^3)$$

$$W = \frac{2 + 2s^2 - s^3 + s^4}{s^3(s^2+1)} \underset{(*)}{=} \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{2}{s^3}$$

non-obvious step (*) use partial fractions to prove it

$$\mathcal{L}^{-1}\{W\}(t) = \boxed{w(t) = \cos(t) - \sin(t) + t^2}$$

Remark: The partial fractions algebra I've omitted here is straightforward but tedious, you should try it, but you should also find a method to check your work.

- I like to use the TI-89's "expand" function. technology should be used to check your work at this stage in the game.

§7.5 #10 Solve $Y'' - 4Y = 4t - 8e^{-2t}$ with $Y(0) = 0$, $Y'(0) = 5$

$$s^2 \bar{Y} - 5 - 4\bar{Y} = \frac{4}{s^2} - \frac{8}{s+2}$$

$$(s^2-4)\bar{Y} = \frac{4}{s^2} - \frac{8}{s+2} + 5$$

$$\bar{Y} = \frac{1}{s^2-4}\left(\frac{4}{s^2} - \frac{8}{s+2} + 5\right)$$ making a common denominator would lead to a denom. of $(s-2)(s+2)^2 s^2$ hence the partial fractal decomposition is a good idea.

$$= \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s} + \frac{D}{s^2} + \frac{E}{(s+2)^2}$$

In contrast to #4 or #5 I have not bothered to make the common denominator explicitly. I suspect in retrospect this will save labor, multiply by the denom. to get, noting $(s-2)(s+2) = s^2-4$ and that just cancells from outset,

$$\frac{\cancel{s^2-4}}{\cancel{s^2-4}}\left[\frac{4}{s^2}(s+2)s^2 - \frac{8}{s+2}(s+2)s^2 + 5(s+2)s^2\right] =$$

$$= 4(s+2) - 8s^2 + 5(s+2)s^2$$

$$= (s+2)[4+5s^2] - 8s^2$$

$$= A(s+2)^2 s^2 + B(s-2)(s+2)s^2 + C(s-2)(s+2)^2 s + D(s+2)^2(s-2) + E(s-2)s^2$$

Now put in some nice choices for values of s to obtain eq^{n}'s for determining A, B, C, D, E,

$\underline{s=0}$ $\quad 8 = -8D \Rightarrow D = -1$

$\underline{s=2}$ $\quad 4[4+20] - 32 = 64 = 64A \Rightarrow A = 1$

$\underline{s=-2}$ $\quad -32 = -4(4)E \Rightarrow E = 2$

$\underline{s=1}$ $\quad 19 = 9A - 3B - 9C - 9D - E = 9 - 3B - 9C + 9 - 2 = 16 - 3B - 9C = 19$

$\underline{s=3}$ $\quad 5[4+45] - 72 = 173 = 75C + 45B + 218 \Rightarrow 75C + 45B = -45$

$$-3B - 9C = 3$$
$$+\left(3B + \tfrac{75}{15}C = \tfrac{-45}{15} = -3\right)$$
$$\left(\frac{75}{15} - 9\right)C = 0 \Rightarrow C = 0 \Rightarrow B = -1$$

We found after some tedious calculation that

$$\mathcal{Y}(s) = \frac{1}{s-2} - \frac{1}{s+2} - \frac{1}{s^2} + \frac{2}{(s+2)^2}$$

Now we can take the inverse Laplace transform with ease,

$$Y(t) = e^{2t} - e^{-2t} - t + 2te^{-2t}$$
$$= \boxed{2\sinh(2t) - t + 2te^{-2t} = Y}$$

<u>Remark:</u> the partial fractions algebra was not easy here, it seems to me that the repeated factor s^2 was the main source of the difficulty. It is wise to check your work with technology on such problems. However, you should complete a # of these problems to build your skill.

§7.5#35 Solve $y'' + 3ty' - 6y = 1$ with $y(0) = 0$, $y'(0) = 0$

$$\mathcal{L}\{y''\}(s) + 3\mathcal{L}\{ty'\}(s) - 6\mathcal{L}\{y\}(s) = \mathcal{L}\{1\}(s)$$

$$s^2\mathcal{Y} - sy(0) - y'(0) + 3\left[-1\frac{d}{ds}\left[\mathcal{L}\{y'\}(s)\right]\right] - 6\mathcal{Y} = 1/s$$

$$s^2\mathcal{Y} - 3\frac{d}{ds}\left[s\mathcal{Y} - y(0)\right] - 6\mathcal{Y} = 1/s$$

$$s^2\mathcal{Y} - 3\mathcal{Y} - 3s\frac{d\mathcal{Y}}{ds} - 6\mathcal{Y} = 1/s$$

: <u>Remark:</u> the $\frac{d\mathcal{Y}}{ds}$ is an unusual feature. It comes from the <u>variable coefficient</u> of $3t$ in the given DEq$^{\underline{n}}$

$$(s^2 - 9)\mathcal{Y} - 3s\frac{d\mathcal{Y}}{ds} = \frac{1}{s}$$

$$\frac{d\mathcal{Y}}{ds} - \left(\frac{s^2-9}{3s}\right)\mathcal{Y} = \frac{-1}{3s^2}$$ (linear 1$^{\underline{st}}$ order, solve via $\int$-factors)

$$\mu = \exp\left(\int\left(\frac{-s^2}{3s} + \frac{9}{3s}\right)ds\right) = \exp\left(\frac{-s^2}{6} + 3\ln|s|\right) = s^3 e^{-s^2/6}$$ hence,

$$\underbrace{s^3 e^{-s^2/6}\mathcal{Y}' - \left(\frac{s}{3} - \frac{3}{s}\right)s^3 e^{-s^2/6}\mathcal{Y}} = \frac{-1}{3s^2}s^3 e^{-s^2/6}$$

$$\int\frac{d}{ds}\left[s^3 e^{-s^2/6}\mathcal{Y}\right]ds = \int -\frac{1}{3}se^{-s^2/6}ds = e^{-s^2/6}$$

$$\Rightarrow s^3 e^{-s^2/6}\mathcal{Y} = e^{-s^2/6} \Rightarrow \mathcal{Y} = \frac{1}{s^3} \text{ and } y = \mathcal{L}^{-1}\{\mathcal{Y}\} \therefore \boxed{y(t) = \frac{t^2}{2}}$$

§7.5 #14 $y'' + y = t \qquad y(\pi) = 0 \;\;\&\;\; y'(\pi) = 0$

$$s^2 Y + Y = \mathcal{L}\{t\}(s) = \frac{1}{s^2}$$

$$Y(s^2+1) = \frac{1}{s^2} \Rightarrow Y = \frac{1}{s^2(s^2+1)}$$

Need to break it up via partial fractions algebra,

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2 \qquad (*)$$

Let's use complex arithmetic for a change. Over $\mathbb{C}$ a polynomial of order n always has n-roots. Here the roots are 0, i and $-i$.

$\underline{s=0|}$ $\boxed{1 = B}$

$\underline{s=i|}$ $1 = (Ci+D)i^2 = -iC - D \qquad \underline{Eq^n (1)}$

$\underline{s=-i|}$ $1 = (-Ci+D)(-i)^2 = iC - D \qquad Eq^n (2)$

Then $Eq^n(1) + Eq^n(2) \Rightarrow 2 = -2D \Rightarrow \boxed{D=-1}$

$\qquad Eq^n(1) - Eq^n(2) \Rightarrow 0 = -2iC \Rightarrow \boxed{C=0}$

Unforunately we don't have enough data to find A yet (the repeated root zero is the source of this problem, if we had four distinct roots then we'd get 4 easy eq^n's from evaluating $(*)$ at those roots) We evaluate $(*)$ at $s=1$ to get another eq^n;

$$1 = 2A + 2B + C + D = 2A + 2 - 1 = 2A + 1 \;\therefore\; A = 0$$

$$y = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\}$$

$$= \boxed{t - \sin(t) = y(t)}$$

Remark: Find the inverse transform is not usually easy!