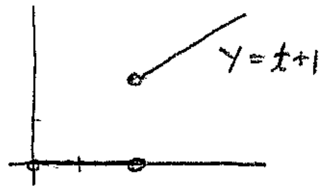
§7.6 #6

$$g(t) = \begin{cases} 0 & 0 < t < 2 \\ t+1 & 2 < t \end{cases} \leftarrow \text{turn this on at } t > 2.$$




$u(t-2)$ has what we want, it is zero for $t<2$ and it is one for $t>2$ thus $\boxed{g(t) = (t+1)u(t-2)}$

continued at bottom of page

§7.6 #5

$$g(t) = \begin{cases} 0 & 0<t<1 \\ 2 & 1<t<2 \\ 1 & 2<t<3 \\ 3 & 3<t \end{cases}$$

$$g(t) = \underset{\substack{\uparrow \\ \text{turns on the 2 at } t=1}}{2u(t-1)} + \underset{\substack{\uparrow \\ \text{turns off the 2 and turns on the 1 at } t=2}}{(1-2)u(t-2)} + \underset{\substack{\uparrow \\ \text{turns off the 1 turns on the 3 at } t=3}}{(3-1)u(t-3)}$$

$$\boxed{g(t) = 2u(t-1) - u(t-2) + 2u(t-3)}$$

Now compute the Laplace transform using Th$^{m}$ (8) which says $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$.

$$\begin{aligned} \mathcal{L}\{g(t)\}(s) &= 2\mathcal{L}\{u(t-1)\}(s) - \mathcal{L}\{u(t-2)\}(s) + 2\mathcal{L}\{u(t-3)\}(s) \\ &= \frac{2}{s}e^{-s} - \frac{1}{s}e^{-2s} + \frac{2}{s}e^{-3s} \qquad \left(\text{used } \mathcal{L}\{1\}(s) = \frac{1}{s}\right) \\ &= \boxed{\frac{1}{s}\left(2e^{-s} - e^{-2s} + 2e^{-3s}\right) = G(s)} \end{aligned}$$

§7.6 #6 Continued

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \mathcal{L}\{(t+1)u(t-2)\}(s) \\ &= \mathcal{L}\{((t-2)+3)u(t-2)\}(s) \\ &= e^{-as}F(s) \quad \text{where } f(t) = t+3 \ \& \ a=2. \\ &= \boxed{e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right)} \end{aligned}$$

§7.6 #9 $g(t) = \begin{cases} 0 & t<1 \\ t-1 & 1<t<2 \\ 3-t & 2<t<3 \\ 0 & 3<t \end{cases}$ } just fit lines to the graph

$$g(t) = (t-1)u(t-1) + [(3-t)-(t-1)]u(t-2) + [0-(3-t)]u(t-3)$$

$$\mathcal{L}\{g(t)\}(s) = \mathcal{L}\{(t-1)u(t-1)\} + \mathcal{L}\{(4-2t)u(t-2)\} + \mathcal{L}\{(t-3)u(t-3)\}$$

$$= \frac{1}{s^2}e^{-s} - \mathcal{L}\{2(t-2)u(t-2)\} + \frac{1}{s^2}e^{-3s}$$

$$= \frac{1}{s^2}e^{-s} - \frac{2}{s^2}e^{-2s} + \frac{1}{s^2}e^{-3s} = \boxed{\frac{1}{s^2}\left(e^{-s} - 2e^{-2s} + e^{-3s}\right)}$$

§7.6 #12 Let $G(s) = e^{-3s}/s^2$ then recall Th^m (8) says that

$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)$. Here $F(s) = 1/s^2 \Rightarrow f(t) = t$.

$$\mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s^2}\right\} = f(t-3)\,u(t-3) = \boxed{(t-3)\,u(t-3)}$$

§7.6 #13

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s+2}\right\}(t) = e^{-2\tau}\Big|_{\tau=t-2} u(t-2) = e^{-2(t-2)}u(t-2)$$

$$\mathcal{L}^{-1}\left\{e^{-4s}\frac{1}{s+2}\right\}(t) = e^{-2(t-4)}u(t-4)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-2s}-3e^{-4s}}{s+2}\right\}(t) = \boxed{e^{-2(t-2)}u(t-2) - 3e^{-2(t-4)}u(t-4)}$$

§7.6 #16 Note $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t) = \frac{1}{2}\sin(2t)$ thus

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{s^2+4}\right\}(t) = \boxed{\frac{1}{2}\sin(2(t-1))\,u(t-1)}$$

§7.6 #34

$$y'' + 4y' + 4y = u(t-\pi) - u(t-2\pi) \equiv g(t) \quad \text{with } y(0)=0, \; y'(0)=0$$

Transform the eq$^{\underline{n}}$ to the "frequency domain",

$$s^2 Y + 4sY + 4Y = \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s} \quad \text{(used intial conditions here)}$$

$$Y(s) = \frac{1}{s^2+4s+4}\left(\frac{1}{s}\right)\left\{e^{-\pi s} - e^{-2\pi s}\right\}$$

$$= \frac{1}{(s+2)^2}\frac{1}{s}\left\{e^{-\pi s} - e^{-2\pi s}\right\}$$

partial fractions.

$$= \left(\frac{-1}{4(s+2)} - \frac{1}{2(s+2)^2} + \frac{1}{4s}\right)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

Now to solve for $y$ we must take the inverse Laplace transform. Use $\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)\,u(t-a)$

$$\mathcal{L}^{-1}\left\{\frac{-1}{4(s+2)} - \frac{1}{2(s+2)^2} + \frac{1}{4s}\right\}(t) = \frac{-1}{4}e^{-2t} - \frac{1}{2}te^{-2t} + \frac{1}{4} = f(t).$$

Thus

$$\mathcal{L}^{-1}\{Y\}(t) = f(t-\pi)\,u(t-\pi) - f(t-2\pi)\,u(t-2\pi)$$

$$= \boxed{\left(\frac{-1}{4}e^{-2(t-\pi)} - \frac{1}{2}(t-\pi)e^{-2(t-\pi)} + \frac{1}{4}\right)u(t-\pi) - \left(\frac{-1}{4}e^{-2(t-2\pi)} - \frac{1}{2}(t-2\pi)e^{-2(t-2\pi)} + \frac{1}{4}\right)u(t-2\pi)}$$

<u>Comment</u>: Solving such a problem w/o Laplace transforms is probably even more trouble. This type of eq$^{\underline{n}}$ is quite important to Electrical Engineering. However, not just that, in fact any application with piecewise defined inputs tend to be more naturally treated by the Laplace transform. The later sections in this chapter will show even more convincing proof that the Laplace transform is truly a powerful tool.

**§7.6 # 39** Solve the initial value problem,

$$y'' + 5y' + 6y = g(t) = \begin{cases} 0 & 0 \le t < 1 \\ t & 1 < t \le 5 \\ 1 & 5 < t \end{cases}$$

with $y(0) = 0$ and $y'(0) = 2$,

Observe that $g(t) = t[u(t-1) - u(t-5)] + 1u(t-5)$.

Thus,

$$y'' + 5y' + 6y = tu(t-1) + (1-t)u(t-5)$$

$$s^2 Y - sy(0) - y'(0) + 5sY - 5y(0) + 6Y = \mathcal{L}\{tu(t-1) + (1-t)u(t-5)\}(s$$

$$(s^2 + 5s + 6)Y - 2 = e^{-s}\mathcal{L}\{t+1\}(s) + e^{-5s}\mathcal{L}\{1-(t+5)\}(s)$$

$$= e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + e^{-5s}\left(\frac{-4}{s} - \frac{1}{s^2}\right)$$

$$Y(s) = \frac{2}{s^2+5s+6} + e^{-s}\left[\left(\frac{1}{s^2+5s+6}\right)\left(\frac{1}{s^2} + \frac{1}{s}\right)\right] + e^{-5s}\left[\left(\frac{1}{s^2+5s+6}\right)\left(\frac{-1}{s^2} - \frac{4}{s}\right)\right]$$

$$= \frac{2}{(s+2)(s+3)} + e^{-s}\left[\frac{1+s}{(s^2+5s+6)s^2}\right] + e^{-5s}\left[\frac{-1-4s}{(s^2+5s+6)s^2}\right]$$

partial fractions 1/2 pg. of work at least

$$= \frac{2}{s+2} - \frac{2}{s+3} + e^{-s}\left[\frac{2/9}{s+3} - \frac{1/4}{s+2} + \frac{1/36}{s} + \frac{1/6}{s^2}\right] + e^{-5s}\left[\frac{-11/9}{s+3} + \frac{7/4}{s+2} - \frac{-19/36}{s} + \frac{-1/6}{s^2}\right]$$

Take inverse transform,

$$\boxed{\begin{aligned} y(t) = {} & 2e^{-2t} - 2e^{-3t} + u(t-1)\left[\frac{2}{9}e^{-3(t-1)} - \frac{1}{4}e^{-2(t-1)} + \frac{1}{36} + \frac{t-1}{6}\right] \\ & + u(t-5)\left[\frac{-11}{9}e^{-3(t-5)} + \frac{7}{4}e^{-2(t-5)} - \frac{19}{36} - \frac{t-5}{6}\right] \end{aligned}}$$

§7.6 #59 Let $G_3(t-1) = u(t-1) - u(t-1-3)$. Solve

$$y'' - y = G_3(t-1) \quad \text{with } y(0) = 0 \ \& \ y'(0) = 2.$$

$$s^2 Y - 2 - Y = \mathcal{L}\{u(t-1) - u(t-4)\}$$

$$(s^2-1)Y = 2 + \frac{1}{s}e^{-s} - \frac{1}{s}e^{-4s}$$

$$Y = \frac{2}{s^2-1} + e^{-s}\left[\frac{1}{s(s^2-1)}\right] - e^{-4s}\left[\frac{1}{s(s^2-1)}\right]$$

after some algebra,

$$Y = \frac{1}{s-1} - \frac{1}{s+1} + e^{-s}\left[\frac{1/2}{s+1} + \frac{1/2}{s-1} - \frac{1}{s}\right] - e^{-4s}\left[\frac{1/2}{s+1} + \frac{1/2}{s-1} - \frac{1}{s}\right]$$

Now take inverse transform,

$$\boxed{y(t) = e^{t} - e^{-t} + u(t-1)\left[\tfrac{1}{2}\left(e^{-(t-1)} + e^{t-1}\right) - 1\right] - u(t-4)\left[\tfrac{1}{2}\left(e^{-(t-4)} + e^{t-4}\right) - 1\right]}$$

Also known as,

$$\boxed{y(t) = 2\sinh t + [\cosh(t-1) - 1]\,u(t-1) - [\cosh(t-4) - 1]\,u(t-4)}$$