Exact Eq$^{\underline{n}}$'s: § 2.4

§ 2.4 # 11

$$\underbrace{(\cos x \cos y + 2x)}_{M} dx - (\underbrace{\sin x \sin y + 2y}_{N}) dy = 0 \quad , \quad \underbrace{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

thus it's exact.

We wish to find $F(x, y)$ with $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

Note $\frac{\partial F}{\partial x} = \cos(x)\cos(y) + 2x \Rightarrow F = \sin(x)\cos(y) + x^2 + C_1(y)$.

and $\frac{\partial F}{\partial y} = -\sin(x)\sin(y) - 2y \Rightarrow F = \sin(x)\cos(y) - y^2 + C_2(x)$.

Comparing we conclude $\boxed{F(x,y) = \sin(x)\cos(y) + x^2 - y^2 = k}$ is the sol$^{\underline{n}}$. See Remark on PH-8 for more on why.

§ 2.4 # 13 Assume $y > 0$ for this problem

$$\underbrace{(t/y)}_{M} dy + \underbrace{(1 + \ln(y))}_{N} dt = 0$$

$$\frac{\partial M}{\partial t} = \frac{1}{y} = \frac{\partial N}{\partial y} \quad \therefore \text{ this DEq}^{\underline{n}} \text{ is } \underline{\text{exact}}.$$

Hence, $\exists\, F(y, t)$ such that $\frac{\partial F}{\partial y} = M$ and $\frac{\partial F}{\partial t} = N$

$$\frac{\partial F}{\partial y} = \frac{t}{y} \Rightarrow F = t\ln(y) + c_1(t)$$

$$\frac{\partial F}{\partial t} = 1 + \ln(y) \Rightarrow \ln(y) + \frac{\partial c_1}{\partial t} = 1 + \ln(y)$$

$$\Rightarrow \frac{\partial c_1}{\partial t} = 1$$

$$\Rightarrow \underline{c_1 = t + \text{constant}.}$$

Thus the given DEq$^{\underline{n}}$ can be restated as $dF = 0$ for $F(y, t) = t\ln(y) + t$. Moreover, $F(y, t) = k$ solves $dF = 0$ thus our sol$^{\underline{n}}$ is

$$\boxed{t\ln(y) + t = k}$$

<u>Remark</u>: exact eq$^{\underline{n}}$'s are solved by level curves.

Remark: If $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ where $\mathbb{R}^2$ has x, y coordinates then the total differential

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

is used as a shorthand for expressing certain identities. For example, divide by dt to get

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

This is the chain-rule applied to $F(x(t), y(t))$.

When I say $F(x,y) = k$ solves $dF = 0$ this should be understood parametrically. Take any parametrization of the level-k curve $F(x(t), y(t)) = k$ and differentiate w.r.t the parameter; $\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = \frac{d}{dt}(k) = 0$

Multiply by dt to reveal $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$.

We usually begin with $M dx + N dy = 0$ then find $F(x,y)$ such that $M = F_x$ and $N = F_y$. I hope you can see why $F(x,y) = k$ is a sol$^{\underline{n}}$ to $M dx + N dy = 0$ in view of the comments above. Notice that it is often the case dx, dy alone are simply shorthand for a more precise mathematically honest parametrically stated eq$^{\underline{s}}$.

Remember,

$$\int \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

(same sort of idea)

§2.4 #17

$$\underbrace{\left(\frac{1}{y}\right)}_{M}dx - \underbrace{\left(3y - \frac{x}{y^2}\right)}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2} \quad \& \quad \frac{\partial N}{\partial x} = \frac{1}{y^2} \quad \therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

this is not an exact eq$^{\underline{n}}$.

§2.4 #29 Consider the eq$^{\underline{n}}$ $\boxed{(y^2 + 2xy)dx - x^2 dy = 0}$ —(*)
(a.) Show it is inexact (b.) multiply by $1/y^2$ and show resulting DEq$^{\underline{n}}$ is exact. (c.) Solve original DEq$^{\underline{n}}$ via sol^n to DEq$^{\underline{n}}$ from part (b). Finally (d.) were any sol$^{\underline{n}}$'s lost ?

(a.) $M = y^2 + 2xy$ and $N = -x^2$ clearly $\frac{\partial M}{\partial y} = 2y + 2x \neq \frac{\partial N}{\partial x}$
thus the (*) DEq$^{\underline{n}}$ is not exact.

(b.) $$\boxed{\underbrace{\frac{1}{y^2}(y^2 + 2xy)}_{P}dx - \underbrace{\frac{x^2}{y^2}}_{Q}dy = \frac{1}{y^2}(0) = 0.} \text{—(**)}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left[1 + \frac{2x}{y}\right] = \frac{-2x}{y^2} \quad \& \quad \frac{\partial Q}{\partial x} = \frac{-2x}{y^2} \quad \therefore \text{ (**) is exact.}$$

Observe $F(x,y) = x + x^2/y$ has $\frac{\partial F}{\partial x} = 1 + \frac{2x}{y}$ & $\frac{\partial F}{\partial y} = \frac{-x^2}{y^2}$
hence (**) has sol$^{\underline{n}}$'s $\underline{F(x,y) = x + x^2/y = k}$.

(c.) $\frac{M}{y^2}dx + \frac{N}{y^2}dy = 0 \iff Mdx + Ndy = 0, \; y \neq 0$
thus $\boxed{x + x^2/y = k \text{ are sol}^{\underline{n}}\text{'s to (*) for } y \neq 0}$

(d.) $y = 0$ solves (*) but is lost in (**)'s sol$^{\underline{n}}$ set.

Remark: I have used three methods to find F in these sol$^{\underline{n}}$'s:
- integrate twice & compare (§2.4#11 on PH-7)
- integrate once & substitute (§2.4#13 on PH-7)
- guess. (part (b.) in §2.4#29 on this page)