§8.4 #7 Find power series sol^{n} to $y' + 2(x-1)y = 0$ centered at $x_o = 1$.

One method is to use $y = \sum_{n=0}^{\infty} c_n (x-1)^n$ and follow same method as in §8.3, but notationally this would be tiresome since $(x-1)$ - factors would be everywhere. I'll follow the idea of the text and substitute

$$t = x-1 \quad \text{and} \quad Y(t) = y(t+1)$$

this makes $Y'(t) = y'(t+1)\frac{d}{dt}(t+1) = y'(t+1)$. Hence,

$$y'(t+1) + 2(x-1)y(t+1) = 0$$

$$\hookrightarrow Y'(t) + 2t\,Y(t) = 0$$

Suppose $Y(t) = \sum_{n=0}^{\infty} c_n t^n \longrightarrow Y'(t) = \sum_{n=0}^{\infty} n c_n t^{n-1}$

$$\therefore \quad Y'(t) + 2t\,Y(t) = \sum_{n=0}^{\infty} n c_n t^{n-1} + \sum_{n=0}^{\infty} 2c_n t^{n+1}$$

$$= \sum_{k=0}^{\infty} (k+1) c_{k+1} t^k + \sum_{j=1}^{\infty} 2c_{j-1} t^j$$

$$= \sum_{m=1}^{\infty} \left[(m+1) c_{m+1} + 2c_{m-1} \right] t^m + c_1 = 0$$

We find,

$$c_1 = 0, \quad 2c_2 + 2c_0 = 0, \quad 3c_3 + 2c_1 = 0, \quad 4c_4 + 2c_2 = 0, \ \dots$$

It follows $c_1 = c_3 = c_5 = \dots = 0$ and $c_2 = -c_0$, $c_4 = \frac{-2}{4}c_2 = \frac{1}{2}c_0$ and $6c_6 + 2c_4 = 0 \Rightarrow c_6 = \frac{-1}{3}\left(\frac{1}{2}\right)c_0$, $8c_8 + 2c_6 = 0 \Rightarrow c_8 = \frac{-1}{4}\left(\frac{-1}{3}\right)\left(\frac{1}{2}\right)c_0$ etc...

generally $Y(t) = c_0\left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} t^{2n}\right)$. To convert back to y, x

note that $y(x) = Y(x-1)$,

$$\boxed{y(x) = c_0\left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}(x-1)^{2n}\right) = c_0\left[1 - (x-1)^2 + \frac{1}{2}(x-1)^4 - \frac{1}{6}(x-1)^6 + \cdots\right.}$$

$\boxed{\S 8.4 \# 11 \quad x^2 y'' - y' + y = 0, \text{ find sol}^{n} \text{ centered at } x_o = 2}$ PH-99

Let $t = x-2$ and $Y(t) = y(t+2)$ then $Y'(t) = y'(t+2)$ and $Y''(t) = y''(t+2)$. Evaluate DEq^{n} at $t+2$ to see

$$(t+2)^2\, y''(t+2) - y'(t+2) + y(t+2) = 0$$

$$(t+2)^2\, Y'' - Y' + Y = 0$$

Solve this via Maclaurin series expansion.

Let $Y(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4$

$\Rightarrow \quad Y'(t) = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + \cdots$

$\Rightarrow \quad Y''(t) = 2c_2 + 6c_3 t + 12 c_4 t^2 + \cdots$

Substitute into DEq^{n},

$$(t^2 + 4t + 4)\left[2c_2 + 6c_3 t + 12 c_4 t^2 + 20 c_5 t^3 + \cdots\right]$$
$$- c_1 - 2c_2 t - 3c_3 t^2 - 4c_4 t^3 + \cdots +$$
$$+ c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \cdots = 0$$

$$[8c_2 - c_1 + c_0] + t[8c_2 + 24c_3 - 2c_2 + c_1] +$$
$$+ t^2[2c_2 + 24c_3 + 48c_4 - 3c_3 + c_2] + t^3[6c_3 + 48c_4 + 80c_5 - 4c_4 + c_3] = 0$$

$8c_2 = c_1 - c_0 \Rightarrow \underline{c_2 = \frac{1}{8}(c_1 - c_0)}$.

$24c_3 = -6c_2 - c_1 = \frac{-3}{4}(c_1 - c_0) - c_1 = \frac{3}{4}c_0 - \frac{7}{4}c_1 \Rightarrow \underline{c_3 = \frac{1}{32}c_0 - \frac{7}{96}c_1}$.

$\left.\begin{array}{l} 48c_4 = -21c_3 - 3c_2 \\ 80c_5 = -44c_4 - 7c_3 \end{array}\right\}$ don't need these if I only want 4 terms

$$Y(t) = c_0 + c_1 t + \frac{1}{8}(c_1 - c_0)t^2 + \left(\frac{1}{32}c_0 - \frac{7}{96}c_1\right)t^3 + \cdots$$
$$= c_0\left(1 - \frac{1}{8}t^2 + \frac{1}{32}t^3 + \cdots\right) + c_1\left(t + \frac{1}{8}t^2 - \frac{7}{96}t^3 + \cdots\right)$$

Therefore, switching back to y, x where $t = x-2$,

$$\boxed{y(x) = c_0\left(1 - \frac{1}{8}(x-2)^2 + \frac{1}{32}(x-2)^3 + \cdots\right) + c_1\left((x-2) + \frac{1}{8}(x-2)^2 - \frac{7}{96}(x-2)^3 + \cdots\right)}$$

§8.4#13 Find first four nontrivial terms in power series sol^{n} centered about zero for $x' + (\sin t)x = 0$, $x(0) = 1$

Let $x = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$ denote the sol^{n} we wish to determine. The initial condition $x(0) = 1 \Rightarrow \underline{c_0 = 1}$.
Recall $\sin t = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 + \cdots$ thus,

$$\left[c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3\right] + \left[t - \frac{1}{6}t^3 + \frac{1}{120}t^5\right]\left[c_0 + c_1 t + c_2 t^2 + \cdots\right] = 0$$

$$c_1 + t\left[2c_2 + c_0\right] + t^2\left[3c_3 + c_1\right] + t^3\left[4c_4 - \frac{c_0}{6} + c_2\right] + \cdots = 0$$

Thus,

$$c_1 = 0$$
$$c_2 = \frac{-c_0}{2} = \frac{-1}{2}.$$
$$c_3 = \frac{-c_1}{3} = 0$$
$$c_4 = \frac{1}{4}\left[\frac{c_0}{6} - c_2\right] = \frac{1}{4}\left[\frac{1}{6} + \frac{1}{2}\right] = \frac{1}{4}\left[\frac{2}{3}\right] = \frac{1}{6} = c_4$$

Oops, I only have enough terms for three terms.

$$\boxed{x(t) = 1 - \frac{1}{2}t^2 + \frac{1}{6}t^4 + \cdots}$$

I'll let you find the 4^{th}, same principles just keep a few more terms.

§8.4#17 Solve $y'' - (\sin(x))y = 0$ with $y(\pi) = 1$, $y'(\pi) = 0$

Initial conditions at π suggest centering our proposed sol^{n} at π to keep calculation at a minimum,

$$y = c_0 + c_1(x-\pi) + c_2(x-\pi)^2 + c_3(x-\pi)^3 + \cdots$$
$$y' = c_1 + 2c_2(x-\pi) + 3c_3(x-\pi)^2 + \cdots$$
$$y'' = 2c_2 + 6c_3(x-\pi) + \cdots$$

Note $y(\pi) = 1 \Rightarrow \underline{c_0 = 1}$.
Moreover $y'(\pi) = 0 \Rightarrow \underline{c_1 = 0}$.

Continued

§8.4#17 continued

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$$\begin{aligned}\sin(x) &= \sin(x-\pi+\pi)\\ &= \sin(x-\pi)\underbrace{\cos(\pi)}_{-1} + \underbrace{\sin(\pi)}_{0}\cos(x-\pi)\\ &= -\sin(x-\pi)\\ &= -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \cdots\end{aligned}$$

Substitute what we've learned into $y'' - \sin(x)\,y = 0$, $\overbrace{c_0 = 1,\ c_1 = 0}^{\text{remember.}}$

$$2c_2 + 6c_3(x-\pi) + 12c_4(x-\pi)^2 + \left[(x-\pi) - \tfrac{1}{6}(x-\pi)^3 + \cdots\right]\left[1 + c_2(x-\pi)^2 + \cdots\right] = 0$$

$$2c_2 + (x-\pi)\left[6c_3 + 1\right] + (x-\pi)^2\left[12c_4\right] + (x-\pi)^3\left[20c_5 - \tfrac{1}{6} + c_2\right] + \cdots = 0$$

We find,

$$\begin{aligned}2c_2 &= 0 &&\Rightarrow \underline{c_2 = 0}.\\ 6c_3 + 1 &= 0 &&\Rightarrow \underline{c_3 = -1/6}.\\ 12c_4 &= 0 &&\Rightarrow \underline{c_4 = 0}.\\ 20c_5 - \tfrac{1}{6} + c_2 &= 0 &&\Rightarrow \underline{c_5 = \tfrac{1}{120}}.\end{aligned}$$

Thus,

$$\boxed{y = 1 - \frac{1}{6}(x-\pi)^3 + \frac{1}{120}(x-\pi)^5 + \cdots}$$

Remark: in previous problems I switched to Y and t and solved the problem via a Maclaurin series then switched back at the end. I figured it would be good to show the Y, t trick is not absolutely necessary. Conceptually the problem in $(x-\pi)$ is more transparent (in my opinion).

$\boxed{\S 8.4 \# 21}$ Find first few terms in Maclaurin series solution to the nonhomogeneous DEq^{n}; $y' - xy = \sin(x)$

Let $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$ thus,

$$c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 - c_0 x - c_1 x^2 - c_2 x^3 = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots$$

$$c_1 + x[2c_2 - c_0] + x^2[3c_3 - c_1] + x^3[4c_4 - c_2] + \cdots = x - \frac{1}{6}x^3 + \cdots$$

Equating coefficients,

$$c_1 = 0$$

$$2c_2 - c_0 = 1 \Rightarrow \underline{c_2 = \tfrac{1}{2}c_0 + \tfrac{1}{2}}.$$

$$3c_3 - c_1 = 0 \Rightarrow c_3 = c_1/3 = 0.$$

$$4c_4 - c_2 = \frac{-1}{6} \Rightarrow c_4 = \frac{1}{4}c_2 - \frac{1}{24}$$

$$\Rightarrow c_4 = \frac{1}{4}\left(\frac{1}{2}c_0 + \frac{1}{2}\right) - \frac{1}{24}$$

$$\Rightarrow \underline{c_4 = \tfrac{1}{8}c_0 + \tfrac{1}{12}}.$$

Thus, $y = c_0 + \left(\frac{1}{2}c_0 + \frac{1}{2}\right)x^2 + \left(\frac{1}{8}c_0 + \frac{1}{12}\right)x^4 + \cdots$

$$\therefore \boxed{y = \underbrace{c_0\left(1 + \tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 + \cdots\right)}_{y_h} + \underbrace{\tfrac{1}{2}x^2 + \tfrac{1}{12}x^4}_{y_p} + \cdots}$$

Remark: solving nonhomogeneous problems are essentially the same difficulty as the homogeneous case in this context. The particular sol^{n} will appear as a series without arbitrary coefficients.

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§8.4#31 Spring which weakens with age is described by

$$mx'' + bx' + ke^{-\eta t}x = 0$$

essentially the spring constant $ke^{-\eta t}$ decreases with time assuming $\eta > 0$. Assume $m=1$, $k=1$, $b=0$ but leave η as fixed but unknown. Find a few terms and analyze the $\eta = 0$ case. Assume $x(0)=1$, $x'(0)=0$

Solve $x'' + e^{-\eta t}x = 0$. Let $x = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$

Notice $e^{-\eta t} = 1 - \eta t + \frac{1}{2}\eta^2 t^2 - \frac{1}{6}\eta^3 t^3 + \cdots$. We substitute our proposed sol^n to find conditions on c_n,

$$2c_2 + 6c_3 t + 12c_4 t^2 + 20c_5 t^3 + \cdots$$
$$+ \left(1 - \eta t + \tfrac{1}{2}\eta^2 t^2 - \tfrac{1}{6}\eta^3 t^3 + \cdots\right)\left(c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots\right) = 0$$

Notice that $x(0) = 1 \Rightarrow \underline{c_0 = 1}$. Also $x'(0) = 0 \Rightarrow \underline{c_1 = 0}$.

$$[2c_2 + c_0] + t[6c_3 - c_0\eta + c_1] + t^2\left[12c_4 + \tfrac{1}{2}\eta^2 c_0 - \eta c_1 + c_2\right] +$$
$$+ t^3\left[20c_5 + c_3 - \eta c_2 + \tfrac{1}{2}\eta^2 c_1\right] + \cdots = 0$$

We find, using $c_0 = 1$, $c_1 = 0$ that,

$2c_2 + 1 = 0 \longrightarrow \underline{c_2 = -1/2}$.

$6c_3 - \eta = 0 \longrightarrow \underline{c_3 = \eta/6}$.

$12c_4 + \frac{1}{2}\eta^2 + c_2 = 0 \rightarrow c_4 = \frac{1}{12}\left(-c_2 - \frac{1}{2}\eta^2\right) = \frac{1}{24} - \frac{\eta^2}{24}$

$20c_5 + c_3 - \eta c_2 = 0 \longrightarrow c_5 = \frac{1}{20}\left(-c_3 + \eta c_2\right) = \frac{-\eta}{120} - \frac{\eta}{40}$

Consequently, $\Rightarrow c_5 = \frac{-4\eta}{120} = \underline{-\eta/30 = c_5}$.

$$x = 1 - \tfrac{1}{2}t^2 + \tfrac{1}{6}\eta t^3 + \tfrac{1}{24}(1-\eta^2)t^4 - \tfrac{1}{30}\eta t^5 + \cdots$$

$$\Rightarrow \boxed{x = \cos t + \tfrac{1}{6}\eta t^3 - \tfrac{1}{24}\eta^2 t^4 - \tfrac{1}{30}\eta t^5 + \cdots}$$

Clearly $\eta \longrightarrow 0$ gives $x \longrightarrow \cos t$.