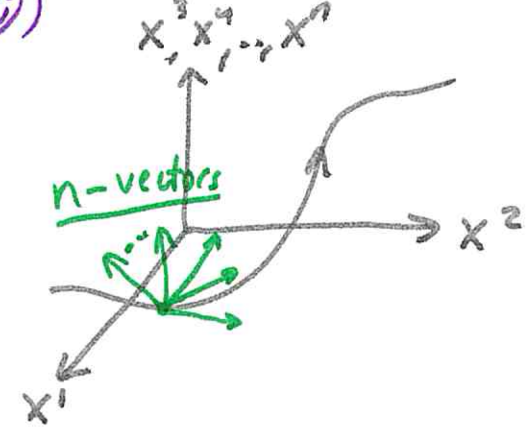
LECTURE 10: FRENET CURVES IN $\mathbb{R}^n$

- Following Wolfgang Kühnel's Chapter 2 of Differential Geometry: Curves - Surfaces - Manifolds
- Can skip to 11 w/o loss of story for $\mathbb{R}^3$ (but, don't ☺)



$$E_1, E_2, \dots, E_n \in \mathfrak{X}(\mathbb{R}^n)$$

$$E_i \cdot E_j = \delta_{ij}$$

$$E_i = \sum_{j=1}^{n} A_{ij} U_j \qquad A \in O(n) \quad (A^T A = I)$$

$\det(A) = 1$ (+)-oriented $\qquad \det(A) = -1$ (−)-oriented

Algebraic Backbone: if given $E_1, E_2, \dots, E_{n-1}$ orthonormal then you may construct $E_n = \sum_{j=1}^{n} \det[E_1 | E_2 | \cdots | E_{n-1} | U_j] U_j$

Claim: $E_1, \dots, E_n$ forms a positively oriented frame.

$\boxed{1 \le i \le n-1}$ $\qquad A_{ij} = E_i \cdot U_j$

$$E_n \cdot E_i = \sum_{j=1}^{n} \det[E_1 | E_2 | \cdots | E_{n-1} | U_j] \underbrace{(U_j \cdot E_i)}_{A_{ij}}$$

$$= \det\Big[E_1 | E_2 | \cdots | E_{n-1} | \underbrace{\sum_{j=1}^{n} A_{ij} U_j}_{E_i}\Big] = 0.$$

$$E_n \in \{E_1, \dots, E_{n-1}\}^{\perp}$$

you can demonstrate $E_n \cdot E_n = 1$.

$\det(E_1 | \cdots | E_n) = 1$.

Def^n/ Frenet Curve in $\mathbb{R}^n$

$\alpha : I \longrightarrow \mathbb{R}^n$ smooth parametrized curve, with <u>arclength</u> parametrization <u>and</u> $\alpha', \alpha'', \dots, \alpha^{(n-1)}$ are LI (evaluate at $t \in I$ and we have LI in each $T_{\alpha(t)}\mathbb{R}^n$)

<u>$t = s$</u>

We say $E_1, E_2, \dots, E_n$ is a <u>Frenet</u> <u>frame</u> if the following 3 conditions hold:

(i.) $E_1, E_2, \dots, E_n$ are orthonormal and (+)-oriented

(ii.) for each $k = 1, 2, \dots, n-1$ we have $\operatorname{span}\{E_1, \dots, E_k\} = \operatorname{span}\{\alpha', \dots, \alpha^{(k)}\}$

(iii.) $\underbrace{\alpha^{(k)} \cdot E_k > 0}$ for $k = 1, \dots, n-1$

we define $\kappa_i = E_i' \cdot E_{i+1} \geq 0$ for $i = 1, 2, \dots, n-2$.

<u>Ex</u>/ $\alpha(t) = (t, t^2, t^3, t^4)$

$\alpha'(t) = v_1 + 2t\,v_2 + 3t^2 v_3 + 4t^3 v_4$

$\alpha''(t) = 2v_2 + 6t\,v_3 + 12t^2 v_4$

$\alpha'''(t) = 6v_3 + 24t\,v_4$

(this is <u>not</u> arclength parametrized, but, if we did reparametrize it then I claim this gives Frenet Curve in $\mathbb{R}^4$)

<u>$n = 3$</u>/

$\alpha' \neq 0$

$\alpha'' \neq 0$

$\therefore$ nonlinear regular curves in $\mathbb{R}^3$ are Frenet curves

This is a natural extension of T, N, B.